ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 1, Winter 1992

ON A FAMILY OF CONVEX POLYNOMIALS

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Consider the nth partial sum of the series $e^{1+z} = \sum_{k=0}^{\infty}((1+z)^k/k!)$. Set $P_n(z) = \sum_{k=0}^{n}((1+z)^k/k!)$ and note that $P_{n-1}(z) = P_n'(z)$. We wish to show that $P_n(D)$ is convex where $D = \{|z| < 1\}$, $n \geq 1$. The proof is by induction. Clearly $P_1(D)$ is convex. Also, $P_2(z) = (5/2) + 2z + (z^2/2)$ and it is easy to see that $P_2(D)$ is convex. That is,

$$\text{Re}\left[\frac{zP_2''}{P_2'} + 1\right] = \text{Re}\left[\frac{2+2z}{2+z}\right] > 0$$

when $|z| < 1$.

Suppose it is known $P_k(D)$ is convex for $k < n$ where $n \geq 3$. Because of the convexity and the fact that all the coefficients are positive, $\text{Re}\,(P_n'(z)) = \text{Re}\,(P_{n-1}(z)) \geq P_{n-1}(-1) = 1$ so that $|P_n'(z)| \geq 1$, $|z| \leq 1$.

Thus, we have

$$\begin{aligned} zP_n''(z) + P_n'(z) &= P_{n-1}(z) + zP_{n-2}(z) \\ &= P_{n-1}(z) + z\left[P_{n-1}(z) - \frac{(1+z)^{n-1}}{(n-1)!}\right] \\ &= (1+z)P_{n-1}(z) - \frac{z(1+z)^{n-1}}{(n-1)!}. \end{aligned}$$

Since the minimum value of a harmonic function occurs on the boundary, we set $z = e^{i\theta}$ and see that

$$\begin{aligned} \text{Re}\left[1 + z - \frac{z(1+z)^{n-1}}{(n-1)!P_n'(z)}\right] &\geq 1 + \cos\theta - \frac{|1+z|^{n-1}}{(n-1)!} \\ &\geq (1+\cos\theta) - \frac{(1+\cos\theta)2^{n-2}}{(n-1)!} \\ &= (1+\cos\theta)\left(1 - \frac{2^{n-2}}{(n-1)!}\right) \\ &\geq 0 \end{aligned}$$

Received by the editors on December 17, 1988.
1980 *AMS Subject Classification.* 30C10, 30C45, 30C50.

if $n \geq 3$.

Thus, we have proved the following theorem.

Theorem 1. *Set* $C_n(z) = (P_n(z) - P_n(0))/P_n'(0)$, $n = 1, 2, \dots$ *so that*

$$C_n(z) = \sum_{k=1}^{n} \left(\left(\sum_{l=0}^{n-k} \frac{1}{l!} \bigg/ \sum_{l=0}^{n-1} \frac{1}{l!} \right) \frac{1}{k!} \right) z^k,$$
$$n = 1, 2, \dots .$$

Then, $C_n, C_n', \dots, C_n^{(n-1)}$ *all map the unit disk* $[|z| < 1]$ *onto convex domains.*

If we define $K = K_0 = \{f : f$ is analytic in D, $f(0) = 0$, $f'(0) = 1$ and $f(D)$ is convex$\}$ and $K_{n+1} = \{f \in K_n : f^{(n+1)}(D)$ is convex or $f^{(n+1)}$ is constant$\}$, then the theorem says $C_n \in K_n$ (and hence, trivially, $C_n \in K_\infty = \cap_{n=1}^{\infty} K_n$).

Observe that $\lim_{n\to\infty} C_n(z) = e^z - 1$ uniformly on compact sets. Further, the function $e^z - 1$ is conjectured to be extremal in K_∞ in the sense that for $f \in K_\infty$ it is conjectured that the MacLaurin coefficients of f satisfy $|a_k| \leq 1/k!$ and also that $1 - e^{-|z|} \leq |f(z)| \leq e^{|z|} - 1$, $|z| < 1$, [**1**] and [**2**].

Notice that the families K_{n+1} and K_n are related as follows. If $f \in K_{n+1}$, then

$$f'(z) = 1 + 2ag(z), \qquad g \in K_n. \tag{1}$$

Further, it is proved [**3**] that if $f \in K_{n+1}$ is given by (1), then $|a| \leq 1/(2(\rho_g + \rho_{zg'}))$ where ρ_g and $\rho_{zg'}$ are the radii of the disks of maximum radius centered at 0 that are contained in the images of g and zg', respectively. Now suppose F is a subfamily of K_n and G is a subfamily of K_{n+1}. Further, suppose $g \in F$ has all its coefficients of maximum modulus in F (i.e., if $h \in F$, $|h^{(k)}(0)| \leq |g^{(k)}(0)|$ for $k = 2, 3, \dots$). In addition, if $\rho_g \leq \rho_h$ and $\rho_{zg'} \leq \rho_{zh'}$ for all $h \in F$ and if f given by (1) with $2a = 1/(\rho_g + \rho_{zg'})$ is in the family G, then clearly f has all its coefficients of maximum modulus in G. I believe this is the situation with regard to the polynomials C_n.

Set $F_n = \{f \in K_n : f \text{ is a polynomial of degree } \le n\}$. Since $F_1 = \{z\} = \{C_1\}$, C_1 is trivially extremal for all coefficient problems in F_1 as well as trivially satisfying $\rho_{C_1} = 1 \le \rho_h$ and $\rho_{zC_1'} = 1 \le \rho_{zh'}$ for all $h \in F_1$. Further, $C_2(z) = z + (1/4)z^2$ so that $C_2'(z) = 1 + 2az = 1 + 2aC_1$ where $2a = 1/(\rho_{C_1} + \rho_{zC_1'})$ (i.e., C_2 is given by (1)). In fact, it is straightforward to check that $C_{n+1}'(z) = 1 + 2aC_n(z)$ where

$$2a = \frac{1}{C_n'(-1) - C_n(-1)} = \frac{1}{\rho_{zC_n'} + \rho_{C_n}}.$$

Also, by theorem 1, $C_{n+1} \in F_{n+1}$. It remains to show $\rho_{C_n} \le \rho_h$ and $\rho_{zC_n'} \le \rho_{zh'}$ for all $h \in F_n$ in order to conclude that the coefficients of C_n have the maximum modulus among all functions in F_n. We can prove the following.

Theorem 2. *Let $n = 2, 3$ or 4 and assume $P(z) = z + \sum_{k=2}^{n} a_k z^k \in F_n$. Then*

$$(2) \qquad |a_k| \le \frac{1}{k!}\left(\sum_{l=0}^{n-k}\frac{1}{l!}\Big/\sum_{l=0}^{n-1}\frac{1}{l!}\right), \qquad 2 \le k \le n$$

with equality if and only if $P = C_n$.

Proof. We observed above that the theorem is true for $n = 2$. Note that $h \in F_2$ implies $h(z) = z + az^2$ where $|a| \le 1/4$. Therefore, $|h(z)| \ge |z| - |a||z|^2 \ge |z| - (1/4)|z| - (1/4)|z|^2 \ge 3/4 = \rho_{C_2}$ while $|zh'(z)| = |z + 2az^2| \ge |z| - 2|a||z|^2 \ge |z| - (1/2)|z|^2 \ge 1/2 = \rho_{zC_2'}$. By our remarks above, (2) now follows for $n = 3$. To show (2) holds with $n = 4$, we will show that $\rho_{C_3} \le \rho_h$ and $\rho_{zC_3'} \le \rho_{zh'}$ for all $h \in F_3$. Therefore, assume $h'(z) = 1 + 2a(z + (\alpha/4)z^2)$ where $|\alpha| \le 1$ and a is chosen so that $h(z) = z + az^2 + (a\ \alpha/6)z^3 \in F_3$. The relation $\operatorname{Re}(zh''(z)/h'(z) + 1) \ge 0$ is equivalent to $|zh''(z) + 2h'(z)| \ge |zh''(z)|$, $|z| < 1$. Thus,

$$(3) \qquad |2 + 6az + 2a\alpha z^2| \ge |a||2z + \alpha z^2|.$$

Choose z so that $|z| = 1$ and $6az + 2a\alpha z^2 < 0$. Then $2 - |a||6 + 2\alpha z| \ge |a||2 + \alpha z|$ so that

$$\begin{aligned} 2 &\ge |a|[|2 + \alpha z| + |6 + 2\alpha z|] \\ &\ge |a|[2 - |\alpha| + 6 - 2|\alpha|] \\ &= |a|(8 - 3|\alpha|). \end{aligned}$$

Thus, $2/(8-3|\alpha|) \geq |a|$.

Returning to (3), divide by 2 and use the fact that when $|h'(z)|$ is a minimum on $|z|=1$, then $zh''(z)/h'(z) < 0$ because $h'(D)$ is convex and lies in a half plane that does not contain the origin while $zh''(z)$ is an outer normal to the curve $h'(z)$, $|z| = $ constant. Therefore, choosing z so that $|z|=1$ and $|h'(z)|$ is a minimum, we have

$$\begin{aligned}\left|1+3az+a\alpha z^2\right| &= \left|1+2az+\frac{a\alpha}{2}z^2+\left(az+\frac{a\alpha}{2}z^2\right)\right| \\ &= \left|1+2az+\frac{a\alpha}{2}z^2\right|-\left|az+\frac{a\alpha}{2}z^2\right| \geq |a|\,\left|z+\frac{\alpha}{2}z^2\right|.\end{aligned}$$

Thus,

$$\rho_{zc_3'} = c_3'(-1) = \frac{2}{5} > \rho_{zh'} = \left|1+2az+\frac{a\alpha}{2}z^2\right|$$

implies

$$\frac{3}{5} < |a|\,\left|2z+az^2-\frac{\alpha z^2}{2}\right| \leq |a|\,\left|2z+az^2\right|+\frac{1}{5} < \frac{3}{5},$$

a contradiction. Further, on $|z|=1$, we have

$$\begin{aligned}\frac{2}{5} &\leq \left|z+2az^2+\frac{a\alpha}{2}z^3\right| = \left|1+2az+\frac{a\alpha}{2}z^2\right| \\ &= \left|1+az+\frac{a\alpha}{6}z^2+\left(az+\frac{a\alpha}{3}z^2\right)\right|.\end{aligned}$$

As before, choose z, $|z|=1$ so that $|1+az+(a\alpha/6)z^2|$ is a minimum, then

$$\left|1+az+\frac{a\alpha}{6}z^2+\left(az+\frac{a\alpha}{3}z^2\right)\right| = \left|1+az+\frac{a\alpha}{6}z^2\right| - |a|\,\left|1+\frac{\alpha}{3}z\right|.$$

Then,

$$\rho_{c_3} = -c_3(-1) = \frac{2}{3} > \rho_h = \left|1+az+\frac{a\alpha}{6}z^2\right|$$

implies

$$\begin{aligned}\frac{2}{3} &> \frac{2}{5}+|a|\,\left|z+\frac{\alpha}{3}z^2\right| \geq \frac{2}{5}+|a|\,\left|z+\frac{\alpha}{6}z^2+\frac{\alpha}{6}z^2\right| \\ &\geq \frac{2}{5}+|a|\,\left|z+\frac{\alpha}{6}z^2\right|-\frac{1}{15} > \frac{2}{5}+\frac{1}{3}-\frac{1}{15} > \frac{2}{3},\end{aligned}$$

a contradiction. This completes the proof. □

REFERENCES

1. S.M. Shah and S.Y. Trimble, *Entire functions with univalent derivatives*, J. Math. Analysis and App. **33** (1971), 220–229.

2. T.J. Suffridge, *Analytic functions with univalent derivatives*, Ann. Univ. Mariae Curie-Skłodowska **37** (1982/83), 143–148.

3. ———, *Convex functions with convex derivatives*, submitted.

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