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LINEAR INDEPENDENCE OF THE TRANSLATES OF AN EXPONENTIAL BOX SPLINE

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ABSTRACT. Complex exponential box splines are introduced and the space of entire functions spanned by their integer translates is investigated. The information obtained allows us to establish a necessary and sufficient condition for local linear independence of the integer translates of a complex exponential box spline. It is interesting to note that the condition known for polynomial box splines is necessary here but not sufficient.

1. Introduction. Let ϕ be a compactly supported distribution in $\mathcal{D}' := \mathcal{D}'(\mathbf{R}^s)$ the space of all Schwartz's s-dimensional distributions, and let $\mathbf{Z}_h^s$ be $\{\boldsymbol{\alpha} \in \mathbf{R}^s \mid h^{-1}\boldsymbol{\alpha} \in \mathbf{Z}^s\}$, where h is a fixed positive scaling parameter. For $c \in \mathbf{C}^{\mathbf{Z}_h^s}$ denote

$$\phi *_h c = \sum_{\boldsymbol{\alpha}\in\mathbf{Z}_h^s} \phi(\cdot - \boldsymbol{\alpha})c_{\boldsymbol{\alpha}}. \tag{1.1}$$

Let $\mathbf{S}_h(\phi)$ be the range of $\phi *_h$. It is well known that significant properties of $\mathbf{S}_h(\phi)$ may be characterized by $\pi_h(\phi)$: the set of polynomials in $\mathbf{S}_h(\phi)$. For $s = 1$ this observation was already made by Schoenberg [**20**]. In [**22**] the authors considered the multivariate situation and established conditions usually referred to as "Strang and Fix conditions" which characterize certain approximation properties of $\mathbf{S}_h(\phi)$ in terms of $\pi_h(\phi)$ and $\hat{\phi}$: the Fourier transform of ϕ. The introduction of the box splines [**2, 3**] renewed the interest in this area and since then many results concerning $\hat{\phi}, \pi_h(\phi)$ and $\mathbf{S}_h(\phi)$ have been established. We selected some of them in the references [**3, 8, 9, 11, 4, 6, 7, 1**]. See also [**10**] and the references within.

Our interest here concerns the case when ϕ is an *exponential box spline* (EB-spline). To recall its definition from [**18**], denote as usual, by $\cdot$, $|\ |$, and $\langle\ \rangle$, scalar product, cardinality and linear hull, respectively, and let

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γ be a pair $(\mathbf{x}_\gamma, \lambda_\gamma)$ where $\lambda_\gamma \in \mathbf{C}$ and $\mathbf{x}_\gamma \in \mathbf{Z}^s \backslash \{\mathbf{0}\}$. The EB-spline based on a single element, $B_h(\gamma)$, is a distribution which can be defined via its Fourier transform by

$$\hat{B}_h(\gamma)(\mathbf{x}) = \int_0^h e^{(\lambda_\gamma - i\mathbf{x}_\gamma \cdot \mathbf{x})t}\, dt. \tag{1.2}$$

When Γ contains $|\Gamma|$ elements of the above form, the EB-spline $B_h(\Gamma)$ is defined by

$$\hat{B}_h(\Gamma) = \prod_{\gamma \in \Gamma} \hat{B}_h(\gamma), \tag{1.3}$$

which means that $B_h(\Gamma)$ is a convolution of $B_h(\gamma)$, $\gamma \in \Gamma$. A set Γ as above is termed a *defining set.* In case $\langle X_\Gamma := \{\mathbf{x}_\gamma \mid \gamma \in \Gamma\}\rangle = \mathbf{R}^s$, $B_h(\Gamma)$ defines a function $B_h(\Gamma \mid \mathbf{x})$. We review in Section 2 some relevant properties of $B_h(\Gamma)$ from [**18**]. In particular, revealing the local structure of $B_h(\Gamma|\mathbf{x})$, we show that usually $\pi_h(\Gamma) := \pi_h(B_h(\Gamma)) = \{\mathbf{0}\}$, and that the right space to be examined instead is $\mathbf{H}_h(\Gamma)$: the space of all *entire functions* contained in $\mathbf{S}_h(\Gamma) := \mathbf{S}_h(B_h(\Gamma))$. It is known, [**18**], that every function in $\mathbf{H}_h(\Gamma)$ is a finite linear combination of functions of the form $f(\mathbf{x}) = e^{\boldsymbol{\theta}\cdot\mathbf{x}}p(\mathbf{x})$, where $\boldsymbol{\theta} \in \mathbf{C}^s$ and $p(\mathbf{x}) \in \pi$, the space of all polynomials.

Given $f(\mathbf{x}) = e^{\boldsymbol{\theta}\cdot\mathbf{x}}p(\mathbf{x})$, we derive in Section 3 a formula for $\phi * f$, which is obtained by an elementary application of Fourier transform in Gelfand–Shilov's sense and is valid for all distributions of compact support. This formula is used in Section 4 for the investigation of $\mathbf{H}_h(\Gamma)$. The information obtained, combined with a fundamental result from [**11**], allows us to establish a necessary and sufficient condition for the local linear independence of translates of EB-splines. In Section 5 we follow [**6, 7**] and [**1**] and derive two recurrence formulae for the solution g of the equation

$$B_h(\Gamma) *_h g|_{\mathbf{Z}_h^s} = e^{\boldsymbol{\theta}\cdot\mathbf{x}}p(\mathbf{x}), \qquad \boldsymbol{\theta} \in \mathbf{C}^s,\ p(\mathbf{x}) \in \pi, \tag{1.4}$$

i.e., to Marsden's identity for EB-splines. The results of Section 3 are applied again in Section 6 to connect between certain results from [**3**] and [**8**].

Finally, we would like to mention several other investigations of exponential box splines completed after this paper was submitted [**19, 13, 5**]. Specifically, the question of linear independence of the integer translates of an exponential box spline was solved in [**13**] with the aid of Poisson's summation formula and in [**5**] by a direct application of the relevant difference and differential operators.

2. Some preliminaries on EB**-splines.** We select here some relevant results from [**18**] concerning EB-splines. It should be emphasized that the definition of $B_h(\Gamma)$ in [**18**] was restricted to the case $\boldsymbol{\lambda}_\Gamma := \{\lambda_\gamma \mid \gamma \in \Gamma\} \subset \mathbf{R}$. Nevertheless, the results stated here, as well as their proofs, remain unchanged under the extension $\boldsymbol{\lambda}_\Gamma \subset \mathbf{C}$. The case $\boldsymbol{\lambda}_\Gamma \subset \mathbf{R}$ will be referred to later as *real* EB-*splines*.

Let Γ be a defining set as in the introduction. The set $X_\Gamma = \{\mathbf{x}_\gamma \mid \gamma \in \Gamma\}$ is treated here as a set of vectors as well as a matrix with s rows and $|\Gamma|$ columns. When $\boldsymbol{\lambda}_\Gamma = \mathbf{0}$, $B_h(\Gamma)$ is the usual *polynomial* box spline (PB-spline). Therefore, to each set of vectors $X \subset \mathbf{Z}^s \backslash \{\mathbf{0}\}$ one can associate $\boldsymbol{\lambda} = \mathbf{0}$ to get a PB-spline denoted by $B_h(X)$. We also use $\langle \Gamma \rangle := \langle X_\Gamma \rangle$.

Proposition 2.1 [**18**, Corollary 2.2]. *Assume* $\gamma \in \Gamma$ *and* $\langle \Gamma \backslash \gamma \rangle = \mathbf{R}^s$, *then*

$$B_h(\Gamma \mid \mathbf{x}) = \int_0^h e^{\lambda_\gamma t} B_h(\Gamma \backslash \gamma \mid \mathbf{x} - t\mathbf{x}_\lambda)\, dt. \tag{2.1}$$

The above proposition is valid (in the distributional sense) even for the case $\langle \Gamma \backslash \gamma \rangle \neq \mathbf{R}^s$.

Proposition 2.2 [**18**, Proposition 2.1]. $B_h(\Gamma)$ *is a compactly supported distribution of order* 0. *Its support is contained in the image of* $[0,h]^{|\Gamma|}$ *under* X_Γ.

When $B_h(\Gamma)$ is a real EB-spline (and in particular PB-spline), $B_h(\Gamma)$ is positive and its support *equals* $X_\Gamma([0,h]^{|\Gamma|})$. Later, we give an example where $\operatorname{supp} B_h(\Gamma) \neq X_\Gamma([0,h]^{|\Gamma|})$.

An important feature of $B_h(\Gamma)$ is the set of its *nodes*. In case $\langle\Gamma\rangle = \mathbf{R}^s$ and $\boldsymbol{\theta} \in \mathbf{C}^s$ denote first

$$\Gamma_{\boldsymbol{\theta}} := \{\gamma \in \Gamma \mid \mathbf{x}_\gamma \cdot \boldsymbol{\theta} = \lambda_\gamma\}. \tag{2.2}$$

Then the set of nodes $\Theta(\Gamma)$ is defined as follows

$$\Theta(\Gamma) = \{\boldsymbol{\theta} \in \mathbf{C}^s \mid \langle\Gamma_{\boldsymbol{\theta}}\rangle = \mathbf{R}^s\}. \tag{2.3}$$

When $|\Theta(\Gamma)| = 1$, Γ is termed *single-noded*. Clearly, if $B_h(\Gamma)$ is a PB-spline, then $\mathbf{0}$ is its only node. We also have

Proposition 2.3 [**18**, Theorem 3.2]. *Γ is single noded if and only if $B_h(\Gamma) = e^{\boldsymbol{\theta}\cdot\mathbf{x}} B_h(X_\Gamma)$ where $\boldsymbol{\theta}$ is the unique node of $\Theta(\Gamma)$ and $B_h(X_\Gamma)$ is the* PB-*spline based on the directions set X_Γ.*

In the following we use the notations $X_{\boldsymbol{\theta}} := X_{\Gamma_{\boldsymbol{\theta}}}$ where $\Gamma_{\boldsymbol{\theta}}$ is as in (2.2). So, for $\boldsymbol{\theta} \in \Theta(\Gamma)$, Proposition 2.3 implies

$$B_h(\Gamma_{\boldsymbol{\theta}}) = e^{\boldsymbol{\theta}\cdot\mathbf{x}} B_h(X_\theta). \tag{2.4}$$

From [**18**, Corollary 2.5] we know that $\{B_h(\Gamma| \cdot - \boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in \mathbf{Z}_h^s\}$ are infinitely differentiable in an open and dense set of $\mathbf{R}^s$. Every connected component of this set is termed a $\Gamma - h$ *cell*.

Using recurrence relations for EB-splines one has

Proposition 2.4 [**18**, Corollary 4.3]. *Assume $\langle\Gamma\rangle = \mathbf{R}^s$; then the restriction of $B_h(\Gamma \mid \mathbf{x})$ to each $\Gamma - h$ cell belongs to*

$$\tilde{\mathbf{H}}(\Gamma) := \langle\{f(\mathbf{x}) \in \mathbf{H}_h(\Gamma_{\boldsymbol{\theta}}) \mid \boldsymbol{\theta} \in \Theta(\Gamma)\}\rangle. \tag{2.5}$$

In view of (2.4) *we also have*

$$\tilde{\mathbf{H}}(\Gamma) = \langle\{e^{\boldsymbol{\theta}\cdot\mathbf{x}} p(\mathbf{x}) \mid \boldsymbol{\theta} \in \Theta(\Gamma), p(\mathbf{x}) \in \pi_h(X_{\boldsymbol{\theta}})\}\rangle. \tag{2.6}$$

The subscript h was omitted from $\tilde{\mathbf{H}}(\Gamma)$ since it is well known that $\pi_h(X_{\boldsymbol{\theta}})$ (which equals $\mathbf{H}_h(X_{\boldsymbol{\theta}})$) is independent of h. We will see later that for EB-splines $\mathbf{H}_h(\Gamma)$ is sometimes *dependent* on h.

Proposition 2.5 [**18**, Corollary 5.1]. *Assume* $\langle \Gamma \rangle = \mathbf{R}^s$ *and let* $\boldsymbol{\theta} \in \Theta(\Gamma)$. *Then*

$$B_h(\Gamma) *_h e^{\boldsymbol{\theta}\cdot\mathbf{x}}|_{\mathbf{Z}_h^s} = h^{-s}\hat{B}_h(\Gamma \mid -i\boldsymbol{\theta})e^{\boldsymbol{\theta}\cdot\mathbf{x}}. \tag{2.7}$$

3. Invariant subspaces of $\mathbf{S_h}(\phi)$ under convolution. In this section it is shown that certain subspaces of $\mathbf{S}_h(\phi)$ are preserved when convolving ϕ with a compactly supported distribution.

We use here some standard multivariate notations. For $\boldsymbol{\alpha}, \boldsymbol{\nu} \in \mathbf{R}^s$, $\boldsymbol{\alpha} \le \boldsymbol{\nu}$ means $\alpha_j \le \nu_j$, $j = 1, \dots, s$, whereas $\boldsymbol{\alpha} < \boldsymbol{\nu}$ means that at least one of the inequalities above is strict. $\mathbf{Z}_+^s := \{\boldsymbol{\alpha} \in \mathbf{Z}^s \mid \boldsymbol{\alpha} \ge \mathbf{0}\}$. For $\boldsymbol{\alpha}, \boldsymbol{\nu} \in \mathbf{Z}_+^s$ and $\boldsymbol{\alpha} \ge \boldsymbol{\nu}$ denote $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_s, \boldsymbol{\alpha}! = \alpha_1! \cdots \alpha_s!$ and $\binom{\boldsymbol{\alpha}}{\boldsymbol{\nu}} = \boldsymbol{\alpha}!/(\boldsymbol{\nu}!(\boldsymbol{\alpha}-\boldsymbol{\nu})!)$. Otherwise $|\boldsymbol{\alpha}| = \boldsymbol{\alpha}! = \binom{\boldsymbol{\alpha}}{\boldsymbol{\nu}} = 0$. The partial derivative $\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1}\cdots\partial x_s^{\alpha_s}} f(\cdot)$ is denoted by $f^{(\boldsymbol{\alpha})}(\cdot)$ or sometimes $D^{\boldsymbol{\alpha}} f$ and $\mathbf{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} \cdots x_s^{\alpha_s}$. Finally, $\hat{\mathcal{D}}'$ denotes the range of $\mathcal{D}'$ under the Fourier transform and $\delta(\cdot)$ is the Dirac distribution.

The following proposition is a slight modification of [**1**, Proposition 2.1].

Proposition 3.1. *If* $f(\mathbf{x}) = e^{\boldsymbol{\theta}\cdot\mathbf{x}} p(\mathbf{x}) \in \mathbf{S}_h(\phi)$, *where* $\boldsymbol{\theta} \in \mathbf{C}^s$ *and* $p(\mathbf{x}) \in \pi$, *then for every* $\boldsymbol{\alpha} \in \mathbf{Z}_+^s$ *we have* $e^{\boldsymbol{\theta}\cdot\mathbf{x}} p^{(\boldsymbol{\alpha})}(\mathbf{x}) \in \mathbf{S}_h(\phi)$.

Proof. The case $\boldsymbol{\theta} = \mathbf{0}$ was established in [**1**]. Applying this result to $\tilde{\phi}(\mathbf{x}) := e^{-\boldsymbol{\theta}\cdot\mathbf{x}}\phi(\mathbf{x})$ we find $\tilde{c} \in \mathbf{C}^{\mathbf{Z}_h^s}$, such that $\tilde{\phi} *_h \tilde{c} = p^{(\boldsymbol{\alpha})}(\mathbf{x})$. Thus, denoting $c_{\boldsymbol{\beta}} = e^{\boldsymbol{\theta}\cdot\boldsymbol{\beta}}\tilde{c}_{\boldsymbol{\beta}}$, $\boldsymbol{\beta} \in \mathbf{Z}_h^s$, one has

$$\phi *_h c = e^{\boldsymbol{\theta}\cdot\mathbf{x}}(\tilde{\phi} *_h \tilde{c}) = e^{\boldsymbol{\theta}\cdot\mathbf{x}} p^{(\boldsymbol{\alpha})}(\mathbf{x}). \qquad \square$$

Theorem 3.1. *Let* ϕ_1, ϕ_2 *be compactly supported distributions. Assume that there exists* $c \in \mathbf{C}^{\mathbf{Z}_h^s}$ *such that* $\phi_1 *_h c = e^{\boldsymbol{\theta}\cdot\mathbf{x}}\mathbf{x}^{\boldsymbol{\alpha}}$ *for some* $\boldsymbol{\theta} \in \mathbf{C}^s$, $\boldsymbol{\alpha} \in \mathbf{Z}_+^s$. *Then for* $\phi := \phi_1 * \phi_2$,

$$\phi *_h c = e^{\boldsymbol{\theta}\cdot\mathbf{x}}\left[\hat{\phi}_2(-i\boldsymbol{\theta})\mathbf{x}^{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\nu}<\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\nu}} (-i)^{\boldsymbol{\alpha}-\boldsymbol{\nu}} \hat{\phi}_2^{(\boldsymbol{\alpha}-\boldsymbol{\nu})}(-i\boldsymbol{\theta})\mathbf{x}^{\boldsymbol{\nu}}\right]. \tag{3.1}$$

Proof. The fact that ϕ_2 is compactly supported implies that $\hat{\phi}_2$ is a multiplier in $\hat{\mathcal{D}}'$. Therefore, since convolution commutes with the shift operator, we have, by Leibniz's rule,

$$\begin{aligned}[\phi *_h c]^\wedge &= [(\phi_2 * \phi_1) *_h c]^\wedge = [\phi_2 * (\phi_1 *_h c)]^\wedge = [\phi_2 * e^{\boldsymbol{\theta}\cdot\mathbf{x}}\mathbf{x}^\alpha]^\wedge \\ &= i^{|\boldsymbol{\alpha}|}\hat{\phi}_2(\cdot)\delta^{(\boldsymbol{\alpha})}(\cdot + i\boldsymbol{\theta}) \\ &= i^{|\boldsymbol{\alpha}|}\hat{\phi}_2(-i\boldsymbol{\theta})\delta^{(\boldsymbol{\alpha})}(\cdot + i\boldsymbol{\theta}) \\ &\quad + \sum_{\boldsymbol{\nu}<\boldsymbol{\alpha}} (-i)^{|\boldsymbol{\alpha}|}(-1)^{|\boldsymbol{\nu}|}\binom{\boldsymbol{\alpha}}{\boldsymbol{\nu}}\hat{\phi}_2^{(\boldsymbol{\alpha}-\boldsymbol{\nu})}(-i\boldsymbol{\theta})\delta^{(\boldsymbol{\nu})}(\cdot + i\boldsymbol{\theta}).\end{aligned}$$

Hence (3.1) is obtained by inversion. □

Remark 3.1. In order to apply Fourier transform to $e^{\boldsymbol{\theta}\cdot\mathbf{x}}\mathbf{x}^{\boldsymbol{\alpha}}$ we used its Gelfand–Shilov's extension [**14**]; nevertheless, the proof is based merely on elementary calculus with distributions and Fourier transform.

Remark 3.2. The conditions "ϕ_2 is compactly supported" and "$\hat{\phi}_2$ is a multiplier" are actually equivalent. To see it recall that $\hat{\phi}_2$ is a multiplier in $\hat{\mathcal{D}}'$ if and only if it is an entire function of exponential type which is slowly growing on $\mathbf{R}^s$. Therefore, the Paley–Wiener–Schwartz Theorem (see, e.g., [**14**, p. 162]) establishes this equivalence.

Substituting $\boldsymbol{\alpha} = \mathbf{0}$ in (3.1) one obtains

Corollary 3.1. *Let* ϕ, ϕ_1, ϕ_2 *be as in Theorem* 3.1. *Assume* $\phi_1 *_h c = e^{\boldsymbol{\theta}\cdot\mathbf{x}}$ *where* $\boldsymbol{\theta} \in \mathbf{C}^s$, $c \in \mathbf{C}^{\mathbf{Z}_h^s}$; *then*

$$\phi *_h c = \hat{\phi}_2(-i\boldsymbol{\theta})e^{\boldsymbol{\theta}\cdot\mathbf{x}}. \tag{3.2}$$

This last result generalizes the main assertion of [**18**, Theorem 5.1]. See also Proposition 2.5 here.

Formula (3.1) can be written as

$$\phi *_h c = e^{\boldsymbol{\theta}\cdot\mathbf{x}}\Big[\hat{\phi}_2(-i\boldsymbol{\theta})\mathbf{x}^{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\nu}>\mathbf{0}} \frac{(-i)^{|\boldsymbol{\nu}|}}{\boldsymbol{\nu}!}\hat{\phi}_2^{(\boldsymbol{\nu})}(-i\boldsymbol{\theta})D^{\boldsymbol{\nu}}(\mathbf{x}^{\boldsymbol{\alpha}})\Big],$$

thus we have

Corollary 3.2. *Let* ϕ, ϕ_1, ϕ_2 *be as in Theorem* 3.1. *Assume that* $\phi_1 *_h c = e^{\boldsymbol{\theta}\cdot\mathbf{x}} p(\mathbf{x})$ *where* $\boldsymbol{\theta} \in \mathbf{C}^s$, $p(\mathbf{x}) \in \pi$ *and* $c \in \mathbf{C}^{\mathbf{Z}_h^s}$; *then*

$$\phi *_h c = e^{\boldsymbol{\theta}\cdot\mathbf{x}} \left[\hat{\phi}_2(-i\boldsymbol{\theta}) p(\mathbf{x}) + \sum_{\boldsymbol{\nu}>\mathbf{0}} \frac{(-i)^{|\boldsymbol{\nu}|}}{\boldsymbol{\nu}!} \hat{\phi}_2^{(\boldsymbol{\nu})}(-i\boldsymbol{\theta}) p^{(\boldsymbol{\nu})}(\mathbf{x}) \right]. \tag{3.3}$$

From Proposition 3.1 and Corollary 3.2 we conclude

Corollary 3.3. *Let* ϕ, ϕ_1, ϕ_2 *be as in Theorem* 3.1. *Assume that* $e^{\boldsymbol{\theta}\cdot\mathbf{x}} p(\mathbf{x}) \in \mathbf{S}_h(\phi_1)$ *and* $\hat{\phi}_2(-i\boldsymbol{\theta}) \neq 0$. *Then* $e^{\boldsymbol{\theta}\cdot\mathbf{x}} p(\mathbf{x}) \in \mathbf{S}_h(\phi)$.

Proof. By induction on $\deg p(\mathbf{x})$. The case $p(\mathbf{x}) \equiv 0$ is trivial. Assume $\deg p(\mathbf{x}) \geq 0$. Since $e^{\boldsymbol{\theta}\cdot\mathbf{x}} p(\mathbf{x}) \in \mathbf{S}_h(\phi_1)$, there exists $c \in \mathbf{C}^{\mathbf{Z}_h^s}$ such that $\phi_1 *_h c = e^{\boldsymbol{\theta}\cdot\mathbf{x}} p(\mathbf{x})$. Therefore (3.3) implies

$$e^{\boldsymbol{\theta}\cdot\mathbf{x}} p(\mathbf{x}) = \hat{\phi}_2^{-1}(-i\boldsymbol{\theta}) \left[\phi *_h c - e^{\boldsymbol{\theta}\cdot\mathbf{x}} \sum_{\boldsymbol{\nu}>\mathbf{0}} \frac{(-i)^{|\boldsymbol{\nu}|}}{\boldsymbol{\nu}!} \hat{\phi}_2^{(\boldsymbol{\nu})}(-i\boldsymbol{\theta}) p^{(\boldsymbol{\nu})}(\mathbf{x}) \right]. \tag{3.4}$$

By Proposition 3.1 $e^{\boldsymbol{\theta}\cdot\mathbf{x}} p^{(\boldsymbol{\nu})}(\mathbf{x}) \in \mathbf{S}_h(\phi_1)$ for every $\boldsymbol{\nu} < \boldsymbol{\alpha}'$. Thus the induction hypothesis gives $e^{\boldsymbol{\theta}\cdot\mathbf{x}} p^{(\boldsymbol{\nu})}(\mathbf{x}) \in \mathbf{S}_h(\phi)$, $\boldsymbol{\nu} > \mathbf{0}$, and so from (3.4) we conclude $e^{\boldsymbol{\theta}\cdot\mathbf{x}} p(\mathbf{x}) \in \mathbf{S}_h(\phi)$. □

Denote now by $\pi_{<k}$ the space of all polynomials of degree $< k$. Let ϕ_1, ϕ_2 be compactly supported functions with $\hat{\phi}_1(\mathbf{0}) = \hat{\phi}_2(\mathbf{0}) = 1$ and assume $\pi_{<k} \subset \pi_h(\phi_1)$, $\pi_{<l} \subset \pi_h(\phi_2)$. Then, using Strang-Fix conditions it can be proved that $\pi_{<k+l} \subset \pi_h(\phi_1 * \phi_2)$ [**6, 1**]. For $l = 0$ it follows that

$$\pi_{<k} \subset \pi_h(\phi_1) \Longrightarrow \pi_{<k} \subset \pi_h(\phi_1 * \phi_2). \tag{3.5}$$

The extension of (3.5) to compactly supported *distributions* is straightforward and can also be used to derive Corollary 3.3. Note also that if ϕ_2 *is* a compactly supported integrable *function*, (3.5) can be obtained by a straightforward manipulation of $\int_{\mathbf{R}^s} (\mathbf{x} - \mathbf{t})^{\boldsymbol{\alpha}} \phi_2(\mathbf{t})\, d\mathbf{t}$.

4. Translates of exponential box splines. Here, choosing ϕ to be the EB-spline $B_h(\Gamma)$, we utilize the results of the previous section in order to characterize $\mathbf{H}_h(\Gamma)$. This allows us to settle completely the question of linear independence for the translates of $B_h(\Gamma)$.

For every complex valued $f : \mathbf{R}^s$ and compactly supported $\phi \in \mathcal{D}'$, let

$$\phi *_h' f := \phi *_h f|_{\mathbf{Z}_h^s}. \tag{4.1}$$

Let Γ be a defining set. Unless stated otherwise we always assume $\langle \Gamma \rangle = \mathbf{R}^s$. Define

$$\overline{\mathbf{H}}_h(\Gamma) = \{f \in \mathbf{C}(\mathbf{R}^s) \mid B_h(\Gamma) *_h' f \text{ is entire}\}. \tag{4.2}$$

Proposition 4.1.

$$\mathbf{H}_h(\Gamma) \subset \tilde{\mathbf{H}}(\Gamma) \subset \overline{\mathbf{H}}_h(\Gamma). \tag{4.3}$$

Proof. From Proposition 2.4 we know that the restriction of $B_h(\Gamma \mid \cdot)$ to each $\Gamma - h$ cell is in $\tilde{\mathbf{H}}(\Gamma)$, so $\mathbf{H}_h(\Gamma) \subset \tilde{\mathbf{H}}(\Gamma)$. For the other inclusion let $\boldsymbol{\theta} \in \Theta(\Gamma)$ and $f(\mathbf{x}) \in \mathbf{H}_h(\Gamma_{\boldsymbol{\theta}})$. From (2.4) it is easily concluded that $f(\mathbf{x}) = e^{\boldsymbol{\theta}\cdot\mathbf{x}} p(\mathbf{x})$ with $p(\mathbf{x}) \in \pi_h(X_{\boldsymbol{\theta}})$, and by [3, Corollary 1], $B_h(X_{\boldsymbol{\theta}}) *_h p \in \pi_h(X_{\boldsymbol{\theta}})$; thus $B_h(\Gamma_{\boldsymbol{\theta}}) *_h' f \in \mathbf{H}_h(\Gamma_{\boldsymbol{\theta}})$. Now (1.3) implies $B_h(\Gamma) = B_h(\Gamma \backslash \Gamma_{\boldsymbol{\theta}}) * B_h(\Gamma_{\boldsymbol{\theta}})$ whereas $B_h(\Gamma \backslash \Gamma_{\boldsymbol{\theta}})$ is a compactly supported distribution, so by (3.1),

$$B_h(\Gamma) *_h' f \in \operatorname{span}\{e^{\boldsymbol{\theta}\cdot\mathbf{x}} q^{(\boldsymbol{\nu})}(\mathbf{x}) \mid q \in \pi_h(X_{\boldsymbol{\theta}}), \boldsymbol{\nu} \in \mathbf{Z}_h^s\},$$

and, therefore, $f \in \overline{\mathbf{H}}_h(\Gamma)$. Since, clearly, $\overline{\mathbf{H}}_h(\Gamma)$ is a linear space the proof is completed. □

Theorem 4.1. *If* $\hat{B}_h(\Gamma \mid -i\boldsymbol{\theta}) \neq 0$ *for all* $\boldsymbol{\theta} \in \Theta(\Gamma)$, *then*

$$\mathbf{H}_h(\Gamma) = \tilde{\mathbf{H}}(\Gamma).$$

Proof. Let $\boldsymbol{\theta} \in \Theta(\Gamma)$, $p(\mathbf{x}) \in \pi_h(X_{\boldsymbol{\theta}})$. Since $\mathbf{H}_h(\Gamma_\theta) = e^{\boldsymbol{\theta}\cdot\mathbf{x}}\pi_h(X_{\boldsymbol{\theta}})$ we have $e^{\boldsymbol{\theta}\cdot\mathbf{x}} p(\mathbf{x}) \in \mathbf{H}_h(\Gamma_{\boldsymbol{\theta}})$. Now $B_h(\Gamma) = B_h(\Gamma \backslash \Gamma_{\boldsymbol{\theta}}) * B_h(\Gamma_{\boldsymbol{\theta}})$, so

$\hat{B}_h(\Gamma \mid -i\boldsymbol{\theta}) \neq 0$ implies $\hat{B}_h(\Gamma\backslash\Gamma_{\boldsymbol{\theta}} \mid -i\boldsymbol{\theta}) \neq 0$. Therefore, by Corollary 3.3, $e^{\boldsymbol{\theta}\cdot\mathbf{x}}p(\mathbf{x}) \in \mathbf{H}_h(\Gamma)$. $\square$

Remark 4.1. The same result is proved in [**13**] with the aid of Poisson's summation formula. In [**5**] it is shown that a weaker condition on the location of the nodes is still sufficient for the equality $\mathbf{H}_h(\Gamma) = \tilde{\mathbf{H}}(\Gamma)$ (compare with Example 4.2 below). We note that the proofs in [**13**] and [**5**] do not make use of the theory of polynomial box splines. Moreover, for polynomial box splines as well as real exponential box splines, a relatively elementary proof is available (see [**5**]). Theorem 4.1 can also be obtained via commutator theory, as pointed out to me by K. Jetter in a private communication.

Note that the condition $\hat{B}_h(\Gamma|-i\boldsymbol{\theta}) \neq 0$, $\boldsymbol{\theta} \in \Theta(\Gamma)$ is to say that the Laplace transform of $B_h(\Gamma)$ vanishes nowhere on $\Theta(\Gamma)$. The following simple example shows that this restriction in Theorem 4.1 cannot be removed.

Example 4.1. Let $s = 1$, $n \geq 2$, $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ with $x^j := x_{\gamma_j} = 1$ and $\lambda_j := \lambda_{\gamma_j} = 2\pi k_j i$, where $\{k_1, \ldots, k_n\}$ are distinct integers. In this case $\tilde{\mathbf{H}}(\Gamma) = \langle\{e^{2\pi k_j ix} \mid j = 1, \ldots, n\}\rangle$; hence, every $f \in \tilde{\mathbf{H}}(\Gamma)$ is periodic with period 1. But, since $n \geq 2$, $B_1(\Gamma \mid \cdot)$ is continuous; therefore, the periodicity of the functions in $\tilde{\mathbf{H}}(\Gamma)$ implies $B_1(\Gamma|\alpha) = 0$ for all $\alpha \in \mathbf{Z}$. Hence $e^{\lambda_{\gamma_j}x} \notin \mathbf{H}_1(\Gamma)$, $j = 1, \ldots, n$. In particular, $\mathbf{H}_1(\Gamma) \neq \tilde{\mathbf{H}}(\Gamma)$. This is an agreement with Theorem 4.1 since from (1.3) it is easily seen that $\hat{B}_1(\Gamma|2\pi k_j) = 0$, $j = 1, \ldots, n$. With somewhat more effort it can be shown that $\dim \mathbf{H}_1(\Gamma) = 1$.

Nevertheless, the converse of Theorem 4.1 is not valid as is shown by the following example.

Example 4.2. Let $s = 1$, $|\Gamma| = 2$, $x^1 = 2$, $x^2 = 1$, $\lambda_1 = 0$, $\lambda_2 = \pi i$.

Applying Proposition 2.1 to this case we easily obtain

$$B_1(\Gamma\,|\,x) = \begin{cases} \frac{e^{\pi i x}-1}{2\pi i}, & 0 \le x < 1, \\ -\frac{1}{\pi i}, & 1 \le x < 2, \\ \frac{-e^{\pi i x}-1}{2\pi i}, & 2 \le x < 3, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, choosing $f_1(x) = 1$, $f_2(x) = xe^{\pi i x}$, one obtains

$$B_1(\Gamma) *_1' f_1 = -\frac{2}{\pi i},$$
$$B_1(\Gamma) *_1' f_2 = \frac{e^{\pi i x}}{\pi i}.$$

Since $\tilde{\mathbf{H}}(\Gamma) = \langle 1, e^{\pi i x}\rangle$ we conclude $\mathbf{H}_1(\Gamma) = \tilde{\mathbf{H}}(\Gamma)$ although

$\hat{B}(\Gamma\,|\,\pi) = 0$. For a necessary and sufficient condition for the equality $\mathbf{H}_h(\Gamma) = \tilde{\mathbf{H}}(\Gamma)$ see [**5**].

A set which plays an important role in the analysis of $B_h(\Gamma)$ is the set of all its "bases" to be defined as follows:

$$\mathbf{J}(\Gamma) = \{J \subset \Gamma \mid |J| = s, \langle J\rangle = \mathbf{R}^s\}. \tag{4.5}$$

Note that if $B_h(\Gamma)$ is a *polynomial* box spline, then $\Theta(\Gamma) = \{\mathbf{0}\}$, thus $\tilde{\mathbf{H}}(\Gamma)$ contains only polynomials, and so Proposition 4.1 implies $\mathbf{H}_h(\Gamma) = \pi_h(\Gamma)$. In [**11**] Dahmen and Micchelli computed the dimension of $\pi_h(\Gamma)$ (hence $\mathbf{H}_h(\Gamma)$) for polynomial box splines:

Theorem 4.2 [**11**, Theorem 2.1]. *Let $B_h(\Gamma)$ be a polynomial box spline. Then*

$$\dim \tilde{\mathbf{H}}(\Gamma) = \dim \mathbf{H}_h(\Gamma) = |\mathbf{J}(\Gamma)|.$$

Example 4.1 shows that for general EB-splines such a result is no longer valid. Instead we have

Corollary 4.1. *For any defining set* Γ,

$$\dim \tilde{\mathbf{H}}(\Gamma) = |\mathbf{J}(\Gamma)|. \tag{4.6}$$

Proof. It is obvious that

$$\dim \tilde{\mathbf{H}}(\Gamma) = \sum_{\boldsymbol{\theta} \in \Theta(\Gamma)} \dim \tilde{\mathbf{H}}(\Gamma_{\boldsymbol{\theta}}).$$

By Proposition 2.4 and Theorem 4.2

$$\dim \tilde{\mathbf{H}}(\Gamma_{\boldsymbol{\theta}}) = \dim \tilde{\mathbf{H}}(X_{\boldsymbol{\theta}}) = |\mathbf{J}(\Gamma_{\boldsymbol{\theta}})|,$$

so

$$\dim \tilde{\mathbf{H}}(\Gamma) = \sum_{\boldsymbol{\theta} \in \Theta(\Gamma)} |\mathbf{J}(\Gamma_{\boldsymbol{\theta}})|.$$

Now (4.6) follows from the fact that each $J \in \mathbf{J}(\Gamma)$ is contained exactly in one set $\mathbf{J}(\Gamma_{\boldsymbol{\theta}})$. □

Remark 4.2. Theorem 4.2 is extended in [**13**] to a very general setting which includes Corollary 4.1 as well. In [**5**] Corollary 4.1 (hence Theorem 4.2) is obtained as a limit of the so-called “simple case” with the aid of some elements of function theory.

Given a $\Gamma - h$ cell A, the following set is very useful for analyzing local properties of $B_h(\Gamma)$ and its translates:

$$b_h^\Gamma(A) = \{\boldsymbol{\alpha} \in \mathbf{Z}_h^s \mid A \subset \operatorname{supp} B_h(\Gamma \mid \cdot - \boldsymbol{\alpha})\}. \tag{4.7}$$

For PB-splines, $|b_h^\Gamma(A)|$ was computed in [**11**]. The result is as follows:

Proposition 4.2 [**11**, Theorem 3.1]. *Let* $B_h(\Gamma)$ *be a* polynomial *box spline. Then for every* $\Gamma - h$ *cell* A,

$$|b_h^\Gamma(A)| = \sum_{J \in \mathbf{J}(\Gamma)} |\det X_J|.$$

Combining Propositions 4.2 and 2.2 we conclude

Proposition 4.3. *Let* $B_h(\Gamma)$ *be an* EB-*spline and* A *a* $\Gamma - h$ *cell. Then*

$$|b_h^\Gamma(A)| \le \sum_{J \in \mathbf{J}(\Gamma)} |\det X_J|. \tag{4.8}$$

Equality in (4.8) cannot always hold as is shown in the following example.

Example 4.3. Take $s = 1$, $|\Gamma| = 2$, $x^1 = 1$, $x^2 = 2$, $\lambda_1 = 0$ and $\lambda_2 = 2\pi i$. In this case direct computation (use, e.g., Proposition 2.1 or [**18**, Theorem 4.2]) shows that

$$B_1(\Gamma\,|\,x) = \begin{cases} \frac{e^{2\pi i x}-1}{2\pi i}, & 0 \le x \le 1, \\ \frac{1-e^{2\pi i x}}{2\pi i}, & 2 \le x \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

So $|b_h^\Gamma(A)| = 2$ while $\sum_{J \in \mathbf{J}(\Gamma)} |\det X_J| = 3$.

We are ready now to state and prove the main theorem of this section:

Theorem 4.3. *For any defining set* Γ *the following conditions are equivalent*:

(a) *If* $A \subset \mathbf{R}^s$ *is open,* $(B_h(\Gamma) *_h c)|_A \equiv 0$ *and* $c_{\boldsymbol{\alpha}} \ne 0$ *for some* $\boldsymbol{\alpha} \in \mathbf{Z}_h^s$*, then* $A \cap \operatorname{supp} B_h(\Gamma \mid \cdot - \boldsymbol{\alpha}) = \varnothing$,

(b) $B_h(\Gamma) *_h c \equiv 0$ *implies* $c \equiv 0$,

(c) $|\det X_J| = 1$ *for all* $J \in \mathbf{J}(\Gamma)$ *and* $\hat{B}_h(\Gamma \mid -i\boldsymbol{\theta}) \ne 0$ *for every* $\boldsymbol{\theta} \in \Theta(\Gamma)$,

(d) $|\det X_J| = 1$ *for every* $J \in \mathbf{J}(\Gamma)$ *and* $\mathbf{H}_h(\Gamma) = \tilde{\mathbf{H}}(\Gamma)$.

Conditions (a) and (b) are usually referred to as local and global linear independence, respectively. Note that the implication (d) $\Rightarrow$ (c) establishes a partial converse to Theorem 4.1.

Proof of Theorem 4.3. The implication (a) $\Rightarrow$ (b) is trivial. The fact that (b) implies $|\det X_J| = 1$ for all $J \in \mathbf{J}(\Gamma)$ was proved in [**18**, Theorem 5.2]. If $\hat{B}_h(\Gamma \mid -i\boldsymbol{\theta}) = 0$ for some $\boldsymbol{\theta} \in \Theta(\Gamma)$, then for $f(\mathbf{x}) = e^{\boldsymbol{\theta}\cdot\mathbf{x}}$, Proposition 2.5 gives

$$B_h(\Gamma) *'_h f = h^{-s}\hat{B}_h(\Gamma \mid -i\boldsymbol{\theta})f(\cdot) = 0,$$

hence (b) $\Rightarrow$ (c). (c) $\Rightarrow$ (d) by Theorem 4.1. Finally assume (d) and let A be a $\Gamma - h$ cell. Since $\mathbf{H}_h(\Gamma)$ contains only holomorphic functions, Corollary 4.1 and the assumption $\mathbf{H}_h(\Gamma) = \tilde{\mathbf{H}}(\Gamma)$ yield

$$\dim \langle \{B(\Gamma \mid \cdot - \boldsymbol{\alpha})|_A \mid \boldsymbol{\alpha} \in \mathbf{Z}_h^s\}\rangle \geq |\mathbf{J}(\Gamma)|.$$

But, since $|\det X_J| = 1$ for every $J \in \mathbf{J}(\Gamma)$, Proposition 4.3 implies $|b_h^\Gamma(A)| \leq |\mathbf{J}(\Gamma)|$, and therefore we must have

$$|b_h^\Gamma(A)| = |\mathbf{J}(\Gamma)| = \dim \langle \{B(\Gamma \mid \cdot - \boldsymbol{\alpha})|_A \mid \boldsymbol{\alpha} \in b_h^\Gamma(A)\}\rangle, \tag{4.9}$$

which shows that (a) holds. If A is not a $\Gamma - h$ cell, choose A_1 to be any open subset of A contained in a $\Gamma - h$ cell A_2. Since $(B_h(\Gamma) *_h c)|_{A_2}$ is holomorphic, the assumption $(B_h(\Gamma) *_h c)|_{A_1} \equiv 0$ implies $(B_h(\Gamma) *_h c)|_{A_2} \equiv 0$; thus, by the above proof $c_{\boldsymbol{\alpha}} \equiv 0$ for all $\boldsymbol{\alpha} \in b_h^\Gamma(A_2)$. But this is valid for every $\Gamma - h$ cell intersected by A so (a) follows and the proof is completed. □

Remark 4.3. Another way to prove the implication (c) $\Rightarrow$ (a) is by applying induction on $|\Gamma| \geq 1$. This method was used by Jia [**15,16**], in the polynomial case, for establishing the global as well as the local independence of the integer translates.

In view of Theorem 4.3, it is important to describe situations when the Laplace transform of $B_h(\Gamma)$ does not vanish at the nodes. We give here two simple sufficient conditions.

Theorem 4.4.

(a) *If* $\boldsymbol{\lambda}_\Gamma \subset \mathbf{R}$, *then* $\hat{B}_h(\Gamma \mid -i\boldsymbol{\theta}) \neq 0$ *for all* $\boldsymbol{\theta} \in \Theta(\Gamma)$.

(b) *For a given set of directions* X *and* $t > 0$ *there exists* $h_t > 0$ *such that, if* $X_\Gamma = X$, $||\boldsymbol{\lambda}_\Gamma||_\infty < t$ *and* $h < h_t$, *then* $B_h(\Gamma \mid -i\boldsymbol{\theta}) \neq 0$ *for all* $\boldsymbol{\theta} \in \Theta(\Gamma)$.

Proof. If $\boldsymbol{\lambda}_\Gamma \subset \mathbf{R}$, then clearly $\Theta(\Gamma) \subset \mathbf{R}^s$ and (a) follows directly from (1.2), (1.3).

For (b) note that, when $\int_0^h e^{(\lambda_\gamma - i\mathbf{x}_\gamma \cdot \boldsymbol{\theta})t} = 0$, one must have $h(\lambda_\gamma - i\mathbf{x}_\gamma \cdot \boldsymbol{\theta}) = 2\pi i j$ where $j \in \mathbf{Z}\backslash\{0\}$. Now if $\boldsymbol{\theta} \in \Theta(\Gamma)$ then there exists $J \in \mathbf{J}(\Gamma)$ such that $\boldsymbol{\theta} = \boldsymbol{\lambda}_J X_J^{-1}$, so $||\boldsymbol{\theta}||_\infty \le c_1 ||\boldsymbol{\lambda}_\Gamma||_\infty$ where c_1 depends only on X_J. Therefore, there exists c_2 such that for every $\gamma \in \Gamma$ and $\boldsymbol{\theta} \in \Theta(\Gamma)$, $|\lambda_\gamma - i\mathbf{x}_\gamma \cdot \boldsymbol{\theta}| \le c_2 ||\boldsymbol{\lambda}_\Gamma||_\infty$. Now (b) easily follows. □

In the univariate case, Theorem 4.4 (b) is closely related to a basic property of $\mathcal{L}$-splines (see [**21**, Theorem 10.5]).

5. Extensions of Marsden's Identity. In order to derive multivariate analogs of Marsden's Identity [**17**, Theorem 7] for EB-splines, we follow the approach of [**6, 7, 1**] and establish two recurrence formulae complementary one to the other.

Throughout this section $g_f := g_f^\phi := g_{f,h}^\phi$ denotes a solution of the equation $\phi *_h' g = f$ where $g, f \in \mathbf{H}_h(\phi)$. When $f(\mathbf{x}) = e^{\boldsymbol{\theta}\cdot\mathbf{x}} p(\mathbf{x})$ or $f(\mathbf{x}) = \mathbf{x}^{\boldsymbol{\alpha}}$ we may write $g_{\boldsymbol{\theta},p}, g_{\boldsymbol{\alpha}}$, respectively. For EB-spline $B_h(\Gamma)$ or PB-spline $B_h(X)$ the notations g_f^Γ, g_f^X are used.

Given $\mathbf{x}^{\boldsymbol{\alpha}} \in \pi_h(\phi)$, where $\boldsymbol{\alpha} \in \mathbf{Z}_+^s$ and $\hat{\phi}(\mathbf{0}) = 1$, Chui, Jetter and Ward established the following recurrence formula for the evaluation of $g_{\boldsymbol{\alpha}}$:

$$g_{\boldsymbol{\alpha}}(\mathbf{x}) = h^s\Bigg[\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\mathbf{j}\in\mathbf{Z}_h^s} \phi(\mathbf{j}) \sum_{\mathbf{0}\le\boldsymbol{\nu}\le\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\nu}} (-\mathbf{j})^{\boldsymbol{\alpha}-\boldsymbol{\nu}} g_{\boldsymbol{\nu}}(\mathbf{x})\Bigg]. \tag{5.1}$$

With the aid of Poisson summation formula, (5.1) can be transformed to [**7**]

$$g_{\boldsymbol{\alpha}}(\mathbf{x}) = h^s \mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\nu}<\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\nu}} (-i)^{|\boldsymbol{\alpha}-\boldsymbol{\nu}|} \hat{\phi}^{(\boldsymbol{\nu})}(\mathbf{0}) g_{\boldsymbol{\nu}}(\mathbf{x}). \tag{5.2}$$

Since [**7**, Lemma 2] $g_{\boldsymbol{\alpha}}^{(\boldsymbol{\alpha}-\boldsymbol{\nu})}(\mathbf{x}) = (\boldsymbol{\alpha}!/\boldsymbol{\nu}!) g_{\boldsymbol{\nu}}(\mathbf{x})$, (5.2) implies that for $p(\mathbf{x}) \in \pi_h(\phi)$ we have

$$g_p(\mathbf{x}) = h^s p(\mathbf{x}) - \sum_{\boldsymbol{\nu}>\mathbf{0}} \frac{(-i)^{|\boldsymbol{\nu}|}}{\boldsymbol{\nu}!} \hat{\phi}^{(\boldsymbol{\nu})}(\mathbf{0}) g_{p^{(\boldsymbol{\nu})}}(\mathbf{x}); \tag{5.3}$$

hence, we can easily conclude

Corollary 5.1. *If* $f(\mathbf{x}) = e^{\boldsymbol{\theta}\cdot\mathbf{x}}p(\mathbf{x}) \in \mathbf{H}_h(\phi)$ *and* $\hat{\phi}(-i\boldsymbol{\theta}) \neq 0$, *then*
(5.4)

$$g_{\boldsymbol{\theta};p}(\mathbf{x}) = \hat{\phi}(-i\boldsymbol{\theta})^{-1}\left[h^s f(\mathbf{x}) - \sum_{\boldsymbol{\nu}>\mathbf{0}} \frac{(-i)^{|\boldsymbol{\nu}|}}{\boldsymbol{\nu}!}\hat{\phi}^{(\boldsymbol{\nu})}(-i\boldsymbol{\theta})g_{\boldsymbol{\theta};p^{(\boldsymbol{\nu})}}(\mathbf{x})\right].$$

For EB-splines this last result can be combined with Proposition 2.5 to yield:

Algorithm 5.1. Let $B_h(\Gamma)$ be an exponential box spline. Assume $\hat{B}_h(\Gamma\,|\,-i\boldsymbol{\theta}) \neq 0$ and $f(\mathbf{x}) = e^{\boldsymbol{\theta}\cdot\mathbf{x}}p(\mathbf{x}) \in \mathbf{H}_h(\Gamma)$. Then $g_{\boldsymbol{\theta},p}$ can be computed recursively as follows:

(a) $g_{\boldsymbol{\theta},1}(\mathbf{x}) = h^s\hat{B}_h(\Gamma\,|\,-i\boldsymbol{\theta})^{-1}e^{\boldsymbol{\theta}\cdot\mathbf{x}}$;

(b) $g_{\boldsymbol{\theta},p}(\mathbf{x}) = \hat{B}_h(\Gamma\,|\,-i\boldsymbol{\theta})^{-1}\big[h^s f(\mathbf{x})$
$- \sum_{\boldsymbol{\nu}>\mathbf{0}}((-i)^{|\boldsymbol{\nu}|}/\boldsymbol{\nu}!)\hat{B}_h^{(\boldsymbol{\nu})}(\Gamma\,|\,-i\boldsymbol{\theta})g_{\boldsymbol{\theta},p^{(\boldsymbol{\nu})}}(\mathbf{x})\big]$.

Application of Corollary 3.2 yields another result which is analogous to (5.4).

Corollary 5.2. *Let* ϕ, ϕ_1, ϕ_2 *be as in Theorem* 3.1. *Assume* $f(\mathbf{x}) = e^{\boldsymbol{\theta}\cdot\mathbf{x}}p(\mathbf{x}) \in \mathbf{H}_h(\phi_1)$ *and* $\hat{\phi}_2(-i\boldsymbol{\theta}) \neq 0$; *then*
(5.5)

$$g^{\phi}_{\boldsymbol{\theta},p}(\mathbf{x}) = \hat{\phi}_2(-i\boldsymbol{\theta})^{-1}\left[g^{\phi_1}_{\boldsymbol{\theta},p}(\mathbf{x}) - \sum_{\boldsymbol{\nu}>\mathbf{0}} \frac{(-i)^{|\boldsymbol{\nu}|}}{\boldsymbol{\nu}!}\hat{\phi}_2^{(\boldsymbol{\nu})}(-i\boldsymbol{\theta})g^{\phi}_{\boldsymbol{\theta},p^{(\boldsymbol{\nu})}}(\mathbf{x})\right].$$

Example 5.1. Let

$$\phi_1(x) = \begin{cases} 1, & -1 \le x < 0, \\ -1, & 0 \le x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

A simple calculation gives $\hat{\phi}_1(x) = \int_0^1 (e^{ixt} - e^{-ixt})\,dt$, hence $\hat{\phi}_1(0) = 0$ and Corollary 5.1 cannot be applied. Yet, for $g(x) = x$, $\phi_1 *_1' g = 1$.

Therefore, for a compactly supported distribution ϕ_2 we have by Corollary 5.2,

$$\phi *_1' (\hat{\phi}_2(0)^{-1}g) = 1,$$

where $\phi = \phi_1 * \phi_2$.

For EB-splines the space $\tilde{\mathbf{H}}(\Gamma)$ reflects the local structure of $B_h(\Gamma)$ and its translates, so it makes sense to define here $\tilde{\mathbf{H}}(\phi_1) = \langle 1 \rangle$. Thus we have $\mathbf{H}_1(\phi_1) = \tilde{\mathbf{H}}(\phi_1)$ although $\hat{\phi}_1(0) = 0$. Compare this with the equivalence of (c) and (d) in Theorem 4.3.

Assume ϕ is an EB-spline $B_h(\Gamma)$, $f(\mathbf{x}) = e^{\boldsymbol{\theta}\cdot\mathbf{x}}p(\mathbf{x}) \in \mathbf{H}_h(\Gamma)$ and $\hat{B}(\Gamma | -i\boldsymbol{\theta}) \neq 0$. Then, since $B_h(\Gamma) = B_h(\Gamma_{\boldsymbol{\theta}}) * B_h(\Gamma \backslash \Gamma_{\boldsymbol{\theta}})$, $f \in \mathbf{H}_h(\Gamma_{\boldsymbol{\theta}})$ and $B_h(\Gamma_{\boldsymbol{\theta}}) = e^{\boldsymbol{\theta}\cdot\mathbf{x}} B_h(X_{\boldsymbol{\theta}})$, (5.3) and (5.5) can be combined to obtain the following two step algorithm:

Algorithm 5.2. Let $B_h(\Gamma)$ be an EB-spline, $f(\mathbf{x}) = e^{\boldsymbol{\theta}\cdot\mathbf{x}}p(\mathbf{x}) \in \mathbf{H}_h(\Gamma)$, $\hat{B}_h(\Gamma \mid -i\boldsymbol{\theta}) \neq 0$. Then $g^{\Gamma}_{\boldsymbol{\theta},p}$ can be computed recursively as follows:

Step 1. $g^{\Gamma_{\boldsymbol{\theta}}}_{\boldsymbol{\theta},p} = e^{\boldsymbol{\theta}\cdot\mathbf{x}} g_p^{X_{\boldsymbol{\theta}}}$ where $g_p^{X_{\boldsymbol{\theta}}}$ is computed by (5.3) with $\phi = B_h(X_{\boldsymbol{\theta}})$.

Step 2. $g^{\Gamma}_{\boldsymbol{\theta},p}(\mathbf{x}) = \hat{B}_h(\Gamma \backslash \Gamma_{\boldsymbol{\theta}} \mid -i\boldsymbol{\theta})^{-1} \big[g^{\Gamma_{\boldsymbol{\theta}}}_{\boldsymbol{\theta},p}(\mathbf{x}) - \sum_{\boldsymbol{\nu} > \mathbf{0}} ((-i)^{|\boldsymbol{\nu}|}/\boldsymbol{\nu}!)$
$\hat{B}_h^{(\boldsymbol{\nu})}(\Gamma \backslash \Gamma_{\boldsymbol{\theta}} \mid -i\boldsymbol{\theta}) g^{\Gamma}_{\boldsymbol{\theta},p(\boldsymbol{\nu})}(\mathbf{x}) \big]$.

It should be noted that Algorithm 5.2 is sometimes useful even for PB-splines. In particular, if $\mathbf{x}^\circ \in X, p(\mathbf{x}) \in \pi(X \backslash \mathbf{x}^\circ)$ and $g^{X \backslash \mathbf{x}^\circ}_{p(\boldsymbol{\nu})}$ are known, g_p^X can be computed using only the simple derivatives of $\hat{B}(\{\mathbf{x}^\circ\} \mid \cdot)$.

6. A remark on polynomial box splines. We show here that the observations made in Section 3 can be used to connect certain results from [**3**] and [**8**]. Throughout this section, we choose $h = 1$ and omit this subscript.

Let $X = \{\mathbf{x}^j\}_{j=1}^n$ be a fixed set of directions in $\mathbf{R}^s \backslash \{\mathbf{0}\}$. Given a subset $Y \subset X$ we denote as before,

$$D^Y = \prod_{\mathbf{x}^j \in Y} D_{\mathbf{x}^j},$$

where $D_{\mathbf{x}^j}$ is the directional derivative.

In [**3**] de Boor and Höllig showed that for a polynomial box spline $B(X)$, we have

$$\pi(X) = \pi \cap D(X), \tag{6.1}$$

where

$$D(X) = \cap\{\ker D^Y \mid Y \subset X, \langle X \backslash Y \rangle \neq \mathbf{R}^s\}. \tag{6.2}$$

In [**8**] Dahmen and Micchelli showed that $D(X)$ contains only polynomials, hence

$$\pi(X) = D(X). \tag{6.3}$$

Applying Corollary 3.3 to (6.2) we conclude

Corollary 6.1. *Let $B(X\,|\,\cdot)$ be a polynomial box spline. Assume $\phi(\cdot)$ is a compactly supported distribution and $\hat{\phi}(\mathbf{0}) \neq 0$. Then*

$$D(X) \subset \pi(B(X) * \phi). \tag{6.4}$$

In [**8**] Dahmen and Micchelli also examined compactly supported functions $\phi(\cdot)$ which satisfy

$$\hat{\phi}(\mathbf{x}) = \prod_{j=1}^{n} \hat{\rho}_j(\mathbf{x}^j \cdot \mathbf{x}), \tag{6.5}$$

where

$$\hat{\rho}_j(2\pi k) = 0 \qquad \forall k \in \mathbf{Z}\backslash\{0\}, \qquad j = 1, \dots, n, \tag{6.6}$$

and

$$\hat{\rho}_j(0) \neq 0, \qquad j = 1, \ldots, n. \tag{6.7}$$

They denoted by Λ the set of functions satisfying (6.5)–(6.7) and proved

Theorem 6.1 [**8**, Corollary 3.1]. *If $\phi \in \Lambda$, then*

$$D(X) \subset \pi(\phi).$$

Our aim is to reveal the nature of condition (6.6). For this purpose denote

$$\chi(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & \text{otherwise}, \end{cases}$$

and note that χ clearly satisfies (6.6)–(6.7). Moreover, given a compactly supported $\rho(\cdot)$, let

$$\tau(\cdot) = \sum_{k=0}^{\infty} \rho'(\cdot - k); \tag{6.8}$$

then obviously $\tau(\cdot)$ is well defined and

$$\tau * \chi = \left[\sum_{k=0}^{\infty} \rho'(\cdot - k)\right] * \chi = \rho * \sum_{j=0}^{\infty} \chi'(\cdot - j) = \rho * \delta = \rho, \tag{6.9}$$

thus $\hat{\rho} = \hat{\tau}\hat{\chi}$. Nevertheless, τ is not necessarily of compact support, hence $\hat{\tau}$ need not be defined in the pointwise sense. Yet, we have

Proposition 6.1. *Let ρ be a univariate compactly supported distribution. Define $\tau(\cdot)$ as in* (6.8)*. Then $\tau(\cdot)$ is compactly supported if and only if ρ satisfies* (6.6).

Proof. If τ is compactly supported, then $\hat{\tau}$ is analytic and the claim follows from (6.9). For the converse assume that ρ satisfies (6.6) and $\operatorname{supp} \rho \subset [a, b]$. Clearly, $\operatorname{supp} \tau \subset [a, \infty)$. Let $f(t)$ be a test function satisfying $\operatorname{supp} f(t) \subset (b - 1, \infty)$. Given $k \le -1$, one has

$$\rho'(\cdot - k)(f) = -\rho(f'(\cdot + k)) = 0;$$

thus, by Poisson's summation formula and (6.6),

$$\tau(f) = \Big[\sum_{k=0}^{\infty} \rho'(\cdot - k)\Big](f) = -\rho\Big(\sum_{k=0}^{\infty} f'(t+k)\Big) = -\rho\Big(\sum_{k=-\infty}^{\infty} f'(t+k)\Big)$$
$$= -\rho\Big(\sum_{k=-\infty}^{\infty} ike^{2\pi ikt}\hat{f}(2\pi k)\Big) = 0.$$

Hence, $\operatorname{supp}\tau \subset [a, b-1]$. □

Since $\hat{B}(X \mid \mathbf{x}) = \prod_{j=1}^{n} \hat{\chi}(\mathbf{x}^j \cdot \mathbf{x})$, Proposition 6.1 implies

Corollary 6.2. *For every $\phi \in \Lambda$ there exists a compactly supported distribution ψ such that $\hat{\psi}(\mathbf{0}) \neq 0$ and*

$$\phi = B(X) * \psi.$$

Consequently, Corollary 6.1 can be viewed as a generalization of Theorem 6.1.

Now we utilize Remark 3.2 to reveal the Fourier transform analog of Proposition 6.1.

Corollary 6.3. *Let T denote the space of all univariate entire functions of exponential type which grow slowly on $\mathbf{R}$. Assume $f(t)$ and $g(t) := itf(t)/(1-e^{-it})$ are entire. Then $f \in T$ if and only if $g \in T$.*

Proof. Since g is entire then $f(2\pi k) = 0$ for every $k \in \mathbf{Z}\backslash\{0\}$. Now, if $f \in T$, by Remark 3.2 and Proposition 6.1 there exist compactly supported distributions ρ, τ such that $\rho = \tau * \chi$ and $\hat{\rho} = f$. Since $\hat{\chi}(t) = (1-e^{-it})/(it)$ it follows that $g = \hat{\tau}$ and, since τ is compactly supported, $g \in T$. The converse implication is trivial. □

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