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ON BEST COAPPROXIMATION IN NORMED LINEAR SPACES

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ABSTRACT. This article is a brief survey of the research concerning best coapproximation, a kind of approximation introduced by C. Franchetti and M. Furi. It is concerned with the existence and uniqueness of elements of best coapproximation by elements of linear subspaces, characterizations of elements of best coapproximation, characterizations of strict convexity in terms of best coapproximation, properties of the best coapproximation operator and best coapproximation on convex sets. Some unsolved and partially solved problems raised by persons working in this field have been mentioned.

1. Introduction. The main object of the theory of best approximation is solution to the following problem: Given a subset G of a normed linear space E and an element $x \in E$, find elements $g_0 \in G$ such that

$$||x - g_0|| \leq ||x - g|| \quad \text{for every } g \in G. \tag{1.1}$$

The set of all such elements $g_0 \in G$ (if any) satisfying (1.1) are called elements of best approximation of x by means of the elements of G and is denoted by $P_G(x)$ (see, e.g., [**21**] or [**22**]). Clearly, $P_G(x) = C_G(x) \cap G$, where $C_G(x) = \cap_{g \in G} \bar{b}_{||x-g||}(x), \bar{b}_r(x)$ denotes the closed ball with center x and radius r.

Recently, another kind of approximation from a subspace G, which naturally extends to any set, has been introduced by Franchetti and Furi [**10**], who have considered those elements $g_0 \in G$ (if any) for which

$$||g_0 - g|| \leq ||x - g|| \quad \text{for every } g \in G \tag{1.2}$$

and have denoted the set of all such elements $g_0 \in G$ by $R_G(x)$. Any $g_0 \in G$ satisfying (1.2), i.e., any $g_0 \in R_G(x)$, is called an element of "best coapproximation" of x by means of the elements of G. Clearly, $R_G(x) = B_G(x) \cap G$, where $B_G(x) = \cap_{g \in G} \bar{b}_{||x-g||}(g)$.

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This branch of approximation theory is comparatively very new. Only a few mathematicians have pursued this study initiated by Franchetti and Furi.

The material of this article is divided in the following way: In Section 2 we give certain notations and definitions. In Section 3 we discuss the existence and uniqueness of elements of best coapproximation. In Section 4 we shall give a relation between best coapproximation by elements of a linear subspace G and best approximation by elements of certain one-dimensional linear manifolds, and we shall use it to obtain, via theorems known for best approximation, some characterizations of elements of best coapproximation (i.e., some necessary and sufficient conditions in order that $g_0 \in R_G(x)$). Applying results of this section to characterizations and existence of elements of best coapproximation in spaces of continuous functions defined on compact spaces, best polynomial coapproximation in $C([0,1])$ can be completely described. In Section 5 we shall give some characterizations of strict convexity of a normed linear space E in terms of best coapproximation, and we shall compare them with some known characterizations of strict convexity in terms of best approximation. In Section 6 we shall give some properties of the best coapproximation operator $R_G : x \to R_G(x)$, also called metric co-projection [**3**]. It will be observed that if the space is a Hilbert space, given a subspace G of X, the two mappings P_G and R_G coincide and are everywhere defined and linear; but in the general setting of Banach spaces, they have (also when they are everywhere defined) a completely different behavior. This is because of the lack of symmetry of orthogonality in these spaces. In Section 7 we discuss various properties of the map R_C, where C is an arbitrary subset of a normed linear space. It will be observed that various properties of the map R_G, where G is a linear subspace, can only partially be extended to the map R_C, and the map R_C is also related to the orthogonal retraction defined in [**5**]. The problem of "strong approximation" for the map R_C, similar to that of the map P_C suggested in [**2**], is also considered in this section. Finally, at the end of this article we point out that best coapproximation seems to be useful both as a kind of approximation and because of its relations with some other notions, which can also be expressed in terms of $R_G(x)$. We also indicate some unsolved and partially solved problems.

In order to limit the size of the article, we have only mentioned the results. For proofs of these, one should refer to the respective cited references. We do not claim that the article is complete in itself. Some results have been omitted due to various reasons.

In this article we shall consider only real normed linear spaces. The extension of these results to complex scalars is easy.

2. Notations and definitions.

We begin with giving certain notations and definitions. All notations and terminology which is not given in this article can be found in the respective quoted papers.

In this article $\varnothing$ denotes the empty set, $\mathbf{R}$ stands for the set of real numbers, $E\backslash G$ is the complement of G in E, E^* stands for dual of the normed space E, $D(f)$ stands for domain of the mapping f, $x\perp G$ denotes that element x is orthogonal to every element of the set G, $B(x,r) = \{y \in E : ||x-y|| \leq r\}$, $\operatorname{dist}(x,A)$ denotes distance between x and A and iff stands for if and only if.

Definition 2.1. Let G be a subset of a normed linear space E and $x \in E$. Then the *elements of best approximation* of x, by means of the elements of G, are those $g_0 \in G$ (if any) for which

$$||x-g_0|| \leq ||x-g|| \quad \text{for all } g \in G. \tag{2.1}$$

The set of all such elements g_0 is denoted by $P_G(x)$. If $P_G(x) \neq \varnothing$, then $P_G^0(x)$ will denote an arbitrary point of $P_G(x)$.

Definition 2.2. Let G be a subset of a normed linear space E and $x \in E$. Then the *elements of best coapproximation* of x by means of elements of G, are those $g_0 \in G$ (if any) for which

$$||g_0-g|| \leq ||x-g|| \quad \text{for all } g \in G. \tag{2.2}$$

The set of all such elements g_0 is denoted by $R_G(x)$. Clearly

$$R_G(x) = B_G(x) \cap G, \tag{2.3}$$

where $B_G(x) = \cap_{g\in G}\bar{b}_{||x-g||}(g)$. It follows from (2.3) that $R_G(x)$ is a bounded set (in fact (2.2) implies that $||g_0|| \leq ||x||$ for all $g_0 \in R_G(x)$). Also, $R_G(x)$ is closed if G is closed and it is convex if G is convex.

It is well known (see, e.g., [**20**]) that

$$g_0 \in P_G(x) \quad \text{iff } x - g_0 \perp G, \tag{2.4}$$

and it was observed in [**10**] that

$$g_0 \in R_G(x) \quad \text{iff } G \perp x - g_0. \tag{2.5}$$

It follows that, in Hilbert spaces, $R_G = P_G$ (see, e.g., [**10**]). We recall that here the notion of orthogonality is used in the sense of G. Birkhoff [**4**] (see also [**13**]), i.e., $x \perp y$ iff $||x + \alpha y|| \geq ||x||$ for all scalars α, and $G_1 \perp G_2$ means that $g_1 \perp g_2$ for all $g_1 \in G_1$ and $g_2 \in G_2$.

We consider R_G as a set-valued mapping from

$$D(R_G) = \{x \in E : R_G(x) \neq \phi\}$$

into G, and we denote by R_G^0 an arbitrary selection of R_G, i.e., a mapping of $D(R_G)$ into G such that $R_G^0(x) \in R_G(x)$ for all $x \in D(R_G)$. Clearly, $D(R_G) \supset G$ and $R_G(x) = \{x\}$ for all $x \in G$ and so $(R_G^0)^2(x) = R_G^0(x)$ for all $x \in D(R_G)$, i.e,. the set-valued mapping R_G is idempotent.

Definition 2.3. A subset A of a normed linear space E is said to be *admissible* [**7**] if it is the intersection of a family of closed balls. A *complete family of centers of* A is a set C such that

$$A = \cap\{B(x, r(x)) : x \in C\}$$

for a mapping $r : C \to [0, \infty[$. A closed subspace D of E is called a *diameteral space* if it contains a complete family of centers of A.

Definition 2.4. If E denotes a normed linear space over the real field $\mathbf{R}$, the tangent functional $\tau(x, y)$ from $E \times E$ into $\mathbf{R}$ is defined as

$$\tau(x, y) = \lim_{t \to 0^+} \frac{||x + ty|| - ||x||}{t}.$$

Definition 2.5. A set V in a normed linear space E is called a *linear manifold* if it is of the form $V = x_0 + G \equiv \{x_0 + g : g \in G\}$, where G

is a linear subspace of E. A closed linear manifold $H \subset E$ is called a *hyperplane* if there exists no closed linear manifold $H_1 \subset E$ such that $H \subset H_1$ and $H \neq H_1 \neq E$.

Definition 2.6. A normed linear space E is said to be *strictly convex* if $||x|| \leq r$, $||y|| \leq r$ imply $||(x+y)/2|| < r$ unless $x = y$ where $r > 0$ is a real number.

Definition 2.7. A *projection* is a mapping f which satisfies the following: whenever $p \in D(f)$, $t \geq 0$, and $p^* = f(p) + t(p - f(p)) \in D(f)$, then $f(p^*) = f(p)$.

Definition 2.8. If C is a convex subset of a Banach space E, a projection is a *retraction* r of C onto a subset F which, for each $x \in C$, maps each point of the ray $\{r(x) + t(x - r(y)) : t \geq 0\} \cap C$ onto the same point $r(x)$.

A retraction r of C onto F is *orthogonal* if, for each $p \in C$ and $y \in F$,

$$||(1-t)r(p) + tp - y|| \geq ||r(p) - y|| \quad \text{for all } t \geq 0.$$

Definition 2.9. We say that x_0 is *strongly unique* or belongs to $P_C(x)$, C a subset of E (or to $P_G(x)$, G a linear subspace of E), strongly if there exists an $r, 0 < r \leq 1$ such that

$$||x - y|| \geq ||x - x_0|| + r||x_0 - y||$$

for every $y \in C$ (for every $y \in G$), i.e.,

$$\tau(x - x_0, x_0 - y) \geq r||x - y||$$

for all $y \in C$ (for all $y \in G$). This for a linear subspace G is equivalent to

$$\tau(x - x_0, m) \geq r||m||$$

for every $m \in G$.

Definition 2.10. Let A be a bounded subset of a normed linear space X and V be any subset of X. For any $x \in X$, put

$$F_A(x) = \sup\{||x - y|| : y \in A\}.$$

The *Chebyshev radius* of A, $r(A)$, and the *Chebyshev radius of A w.r.t. V*, $r_V(A)$ (see, e.g., [**19**]), are defined as

$$r(A) = \inf_{x \in X} F_A(x), \qquad r_V(A) = \inf_{x \in V} F_A(x).$$

An element $x_0 \in X$ ($x_0 \in V$) is a *Chebyshev center* or a *center* of A (center of A w.r.t. V) if

$$F_A(x_0) = \inf_{x \in X} F_A(x), \qquad F_A(x_0) = \inf_{x \in V} F_A(x).$$

The set of centers (of centers w.r.t. V) of A is denoted by $E(A)$ ($E_V(A)$).

If V is a closed subspace of X, $a \in X$, and s is such that

$$\operatorname{dist}(a, V) = d < s,$$

the set

$$C_s = B(a, s) \cap V = \{v \in V : ||v - a|| \leq s\}$$

is called a *hypercircle*.

Definition 2.11. Let $S = \{u \in E : ||u|| = 1\}$ be the unit sphere in a Banach space E. E is said to be *smooth* if, for each $(x, y) \in S \times S$, the limit

$$\lim_{t \to 0} \frac{||x + ty|| - ||x||}{t} \tag{2.6}$$

exists. E is said to be *uniformly smooth* if the limit is uniform for each $(x, y) \in S \times S$.

If E is smooth, the *semi inner product* $[\cdot, \cdot]$ on E is the Gateaux differential of $(1/2)||\cdot||^2$ (with arguments reversed):

$$[y, x] = \lim_{t \to 0} \left(\frac{1}{2} \frac{||x + ty||^2 - ||x||^2}{t} \right).$$

This limit exists for each $x, y \in E$ because of existence of (2.6) for each $x, y \in S$.

If E is a Banach space with a semi-inner product (s.i.p.) $[\cdot,\cdot]$ and $x_1, x_2, \dots, x_n$ is a system of n elements of E, we associate with this system a number $\Gamma(x_1, x_2, \dots, x_n)$, called *generalized gramian determinant* of $x_1, x_2, \dots, x_n$ w.r.t. the s.i.p. $[\cdot,\cdot]$ given by

$$\Gamma(x_1, x_2, \dots, x_n) = \begin{vmatrix} [x_1, x_1] & \cdots & [x_n, x_1] \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ [x_1, x_n] & \cdots & [x_n, x_n] \end{vmatrix}.$$

3. Existence and uniqueness of elements of best coapproximation. In this section we discuss the existence and uniqueness of elements of best coapproximation. The following result on the existence of best coapproximation was proved in [**10**]:

Theorem 3.1. *If G is a closed subspace of a Hilbert space X or G is a one-dimensional subspace of a Banach space X, then $D(R_G) = X$.*

The following result proved in [**10**] gives a structure of elements of best coapproximation.

Theorem 3.2. *Let X be a real normed space, A an admissible subset of X, D a diametral space of A. Then $R_D(A) = D \cap A$.*

Using Theorem 3.2 the following theorem was proved in [**10**].

Theorem 3.3. *Let X be a real Banach space of dimension ≥ 3. Then the following properties are equivalent*:

(i) *X is a Hilbert space.*

(ii) *$R_G(x) \neq \phi$ for every hyperplane G and every $x \in X$.*

(iii) *$A \cap D \neq \phi$ for every admissible set A and every diameteral space D of A.*

The following sufficient condition for the existence of best coapproximation was given in [**14**]:

Theorem 3.4. *If, for every subspace G of a normed linear space X, there exists at least one element $x \in X \backslash G$ such that x has a best coapproximation in G, then for any subspace G of X every element of X has a best coapproximation in G.*

The following result proved in [**10**] gives the uniqueness of elements of best coapproximation.

Theorem 3.5. *If E is a smooth normed linear space, then $D(R_G)$ is a linear subspace of E and $R_G(x)$ is a singleton for each $x \in D(R_G)$.*

Next we correlate elements of best coapproximation with radii of hypercircles.

Suppose V is a closed subspace of a Banach space X, $a \in X$ and s is such that $\text{dist}(a, V) = d < s$. Let r_s be the Chebyshev radius of the hypercircle

$$C_s = B(a, s) \cap V = \{v \in V : ||v - a|| \leq s\}$$

and $l_s = r_s - s$. Let $l = \lim_s l_s$.

The following theorem proved in [**9**] relates l and elements of best coapproximation:

Theorem 3.6. *In any normed linear space X, $a \in D(R_V) \Rightarrow l \leq 0$. If X is a reflexive Banach space, $a \in D(R_V) \Leftrightarrow l \leq 0$.*

As a consequence of this result, we get the following characterization of reflexive Banach spaces of dimension ≥ 3.

Corollary 3.1 [**9**]. *If X is a real reflexive Banach space with $\dim X \geq 3$, then X is a Hilbert space iff $0 \geq l \equiv l(a, V)$ for every pair (a, V) with V a closed hyperplane and $a \in X$.*

Remark 3.1 [**9**]. By the above mentioned characterization of real Hilbert spaces it follows that the set

$$R_V(a) = \{v_0 \in V : ||v_0 - x|| \leq ||a - x||, x \in V\}$$

is empty in every non-Hilbert space X with $\dim X \geq 3$ for a suitable pair (a, V) with $a \in V$ and V a closed hyperplane; but if X is a reflexive Banach space,

$$V_\infty = \{v_0 \in V : ||v_0 - x|| \leq ||a - x|| + l, x \in V\}$$

is never empty. We conclude under these assumptions that

$$R_V(a) = \phi \Rightarrow l > 0.$$

Now suppose that V is a one-dimensional subspace of X. Since we work essentially in the two-dimensional subspace generated by a and V, we cover the case excluded in Corollary 3.1.

Let $s > d = \operatorname{dist}(a, V)$, then $E_V(C_s) = \{c_s\}$, where c_s is the middle point of the segment C_s.

Theorem 3.7 [**9**]. *$\{c_s\}$ is a convergent sequence, and if $c_0 = \lim_s c_s$, then $V \perp a - c_0$, i.e., $a \in D(R_V)$ and $c_0 \in R_V(a)$, we have also $l \leq 0$.*

Problem 3.1 [**9**]. Is it possible that $l^- = \inf l(a, V)/d(a, V) > 0$? If yes, since $R_V(a) \neq \phi \Rightarrow l \leq 0$ by Theorem 3.6, it must be $R_V(a) = \phi$ for all admissible pairs (a, V).

Next we associate to every system of n vectors $x_1, x_2, \ldots, x_n$ of a Banach space X, a number having some utility when dealing with the linear independence of these vectors, and with R_M if M is the subspace of X they generate.

Theorem 3.8 [**17**]. *If $\Gamma(x_1, x_2, \ldots, x_n) \neq 0$ for every semi-inner product and $x \in D(R_M)$, then every $x^0 \in R_M(x)$ is given by an expression of the type*

$$y = -\frac{1}{\Gamma(x_1, \ldots, x_n)} \begin{vmatrix} 0 & x_1 & \cdots & x_n \\ [x, x_1] & [x_1, x_1] & \cdots & [x_n, x_1] \\ \cdots & \cdots & \cdots & \cdots \\ [x, x_n] & [x_1, x_n] & \cdots & [x_n, x_n] \end{vmatrix}$$

for an s.i.p. $[\cdot, \cdot]$.

Remark 3.2. In the hypothesis of this theorem, we have

$$\frac{\Gamma(x, x_1, \dots, x_n)}{\Gamma(x_1, x_2, \dots, x_n)} = [x - x^0, x] = ||x||^2 - ||x^0||\,||x|| \geq 0.$$

From here we see that a necessary condition in order that $x \in D(R_M)$ is that $\Gamma(x, x_1, \dots, x_n)$ and $\Gamma(x_1, \dots, x_n)$ have the same sign.

Problem 3.2 [**17**]. Is $\Gamma(x, x_1, \dots, x_n) \geq 0$ a sufficient condition in order that $x \in D(R_M)$, where $M = [x_1, \dots, x_n]$?

4. Characterizations of elements of best coapproximation. In this section we shall give some necessary and sufficient conditions in order that $g_0 \in R_G(x)$. To this end, we first make the following basic observation (see [**20**]) relating best approximation and best coapproximation. This will permit us to apply the known characterization theorems of the theory of best approximation.

Theorem 4.1. *Let E be a normed linear space, G a linear subspace of E and $x \in E \backslash G$. Then*

$$R_G(x) = \{g_0 \in G : g_0 \in \cap_{g \in G} P_{\langle g_0, x \rangle}(g)\}, \tag{4.1}$$

where $\langle g_0, x\rangle = \{\alpha x + (1-\alpha) g_0 : \alpha \in \mathbf{R}\}$ is the linear manifold spanned by g_0 and x.

Corollary. *Let E be a normed linear space, G a linear subspace of E and $x \in E \backslash G$. For an element $g_0 \in G$ we have $g_0 \in R_G(x)$ iff*

$$G \subset P^{-1}_{[x - g_0]}(0) = \{z \in E : 0 \in P_{[x-g_0]}(z)\},$$

where $[x - g_0] = \{\alpha(x - g_0) : \alpha \in \mathbf{R}\}$.

Using the above observations the following theorem on the characterization of elements of best coapproximation was proved in [**20**].

Theorem 4.2 [**20**]. *let E be a normed linear space, G a linear subspace of E and $x \in E \backslash G$. For an element $g_0 \in G$ the following statements are equivalent*:

(i) $g_0 \in R_G(x)$.

(ii) *For each* $g \in G$ *there exists a functional* $f^g \in E^*$ *such that*

$$||f^g|| = 1 \tag{4.2}$$

$$f^g(x) = f^g(g_0) \tag{4.3}$$

$$f^g(g) = ||g||. \tag{4.4}$$

(iii) *The same as in* (ii)*, with* (4.4) *replaced by*

$$|f^g(g)| = ||g||. \tag{4.5}$$

Remarks 4.1. (a) Condition (4.3) (for all $g \in G$) can also be written in the form

$$x - g_0 \in \bigcap_{g \in G} \ker f^g, \tag{4.6}$$

whence

$$\operatorname{dom} R_G \subset G + \bigcap_{g \in G} \ker f^g. \tag{4.7}$$

(b) If E is smooth, then for each $g \in G$, there exists one and only one $f^g \in E^*$ satisfying (4.2) and (4.4).

(c) Theorem 4.2 (equivalence (i) $\Leftrightarrow$ (ii)), admits the following geometric interpretation: An element $g_0 \in G$ satisfies $g_0 \in R_G(x)$ iff, for each $g \in G$, there exists a hyperplane H^g supporting the ball $B(0, ||g||)$ at g and such that $x - g_0 + g \in H^g$.

By considering, for each $g \in G$, the functional f^{g-g_0} instead of f^g in Theorem 4.2, one obtains the following nicer geometric interpretation: An element $g_0 \in G$ satisfies $g_0 \in R_G(x)$ iff, for each $g \in G$, there exists a hyperplane H_g supporting the ball $B(g, ||g - g_0||)$ at g_0 and such that $x \in H_g$.

Remark 4.2 [**20**]. Another characterization of elements of best coapproximation, in terms of the derivatives of the norm, which is a consequence of some more general result of [**18**], admits the following equivalent formulation: Under the assumptions of Theorem 4.2, we have

$g_0 \in R_G(x)$ iff, for each $g \in G$, there exists a functional $h^g \in E^*$ such that

$$||h^g|| = 1 \tag{4.8}$$

$$h^g(x - g_0) \geq 0 \tag{4.9}$$

$$h^g(g_0 - g) = ||g_0 - g||. \tag{4.10}$$

Geometrically, it means that, for each $g \in G$, there exists a hyperplane H_g supporting the ball $B(g, ||g - g_0||)$ at g_0 and such that x and $B(g, ||g - g_0||)$ are on different sides of H_g.

Using the fact that dim $\langle g_0, x \rangle = 1$, one obtains the following characterization theorem.

Theorem 4.3 [**20**]. *Let E be a normed linear space, G a linear subspace of E and $x \in E \backslash G$. An element $g_0 \in G$ satisfies $g_0 \in R_G(x)$ iff, for each $g \in G$, there exist two extremal points f_1^g, f_2^g of $B_{E^*} = \{f \in E^* : ||f|| \leq 1\}$ and two numbers $\lambda_1^g, \lambda_2^g \geq 0$ with $\lambda_1^g + \lambda_2^g = 1$, such that*

$$\lambda_1^g f_1^g(x) + \lambda_2^g f_2^g(x) = \lambda_1^g f_1^g(g_0) + \lambda_2^g f_2^g(g_0) \tag{4.11}$$

$$\lambda_1^g f_1^g(g) + \lambda_2^g f_2^g(g) = ||g||. \tag{4.12}$$

Remark 4.3. [**20**]. In the case of complex scalars, Theorem 4.3 holds with three f_i^g and three λ_i^g $(i = 1, 2, 3)$ instead of two.

Remark 4.4. Using Theorem 4.3, Papini and Singer [**20**] deduced a characterization of elements of best coapproximation in the space $E = C(Q)$ of all real-valued continuous functions on a compact space Q, endowed with the supremum norm $||x|| = \max_{q \in Q} |x(q)|$, and some consequences, in $E = C(Q)$, of this characterization. They also gave some applications of these results to characterization and existence of elements of best coapproximation in $E = C([a, b])$; in particular, they have completely described best polynomial coapproximation in $C([0, 1])$.

5. Characterizations of strict convexity in terms of best coapproximation. In this section we shall give some characterizations of strict convexity of a normed linear space E in terms of best coapproximation and we shall compare them with some known characterizations of strict convexity in terms of best coapproximation.

The following characterization of strict convexity was given in [**20**]:

Theorem 5.1. *Let E be a normed linear space. The following statements are equivalent*:

(i) *For every linear subspace G of E, every $x \in E\backslash G$ and every $g_0 \in R_G(x)$, we have*

$$||g_0 - g|| < ||x - g||, \qquad g \in G. \tag{5.1}$$

(ii) *Same as* (i), *for every linear subspace G of E with* $\dim G = 1$.

(iii) *For every linear subspace G of E, every $x \in E\backslash G$ and every $g_0 \in R_G(x)$, we have*

$$||g_0|| < ||x||. \tag{5.2}$$

(iv) *Same as* (iii), *for every linear subspace G of E with* $\dim G = 1$.

(v) *E is strictly convex.*

Remark 5.1 [**20**]. It is interesting to compare Theorem 5.1 with the following well known characterizations of strict convexity in terms of best approximation (see, e.g., [**21**]):

Let E be a normed linear space. The following statements are equivalent:

(i) For every linear subspace G of E, every $x \in E\backslash G$ and every $g_0 \in P_G(x)$ we have

$$||x - g_0|| < ||x - g|| \qquad (g \in G, g \neq g_0). \tag{5.3}$$

(ii) Same as (i), for every linear subspace G of E, with $\dim G = 1$.

(iii) E is strictly convex.

6. Best coapproximation operator. In the present section we shall give some properties of the set-valued mapping $R_G : x \to R_G(x)$, defined on the subset

$$\text{dom}\, R_G = \{x \in E : R_G(x) \neq \phi\}$$

of E into G. We denote by R_G^0 an arbitrary selection of R_G, i.e., a mapping of $D(R_G)$ into G such that $R_G^0(x) \in R_G(x)$ for all $x \in D(R_G)$.

The following theorem was proved in [**10**].

Theorem 6.1. *The set-valued mapping R_G has the following properties*:

(i) $x \in D(R_G) \Rightarrow \lambda x \in D(R_G)$ *and* $R_G(\lambda x) = \lambda R_G(x)$ *for every scalar* λ, *i.e., the set-valued mapping* R_G *is homogenous.*

(ii) $D(R_G) \supset G$ *and* $R_G(x) = \{x\}$ *for all* $x \in G$.

(iii) $||R_G^0(x)|| \leq ||x||$ *for all* $x \in D(R_G)$ (*the result is true for any subset* G *of* E *with* $0 \in G$).

(iv) *If* $x \in D(R_G)$ *and* $P_G(x) \neq \phi$, *then* $||x - R_G^0(x)|| \leq 2||x - P_G^0(x)||$.

(v) *If* X *is smooth, then* R_G *is single-valued on* $D(R_G)$ *and it is a norm one linear projection of* $D(R_G)$ *onto* G.

(iv) *If* X *is an inner-product space then* $R_G = P_G$. *If* X *is a Hilbert space, then* $D(R_G) = X$.

The following theorems were proved in [**17**] (see also [**10**]).

Theorem 6.2. (i) *If* M *is a one-dimensional subspace of* X, *then* $D(R_M) = X$ *and* R_M *admits a linear selection.*

(ii) X *is smooth iff, for every* (*or also, for every one-dimensional*) *subspace* M *of* X, R_M^0 *is uniquely determined over its domain.*

(iii) *If* X *is smooth, for every subspace* M *of* X, R_M^0 *is a norm one projection from* $D(R_M)$ (*which is a subspace*) *onto* M.

Theorem 6.3. *Let X be a Banach space with* $\dim(X) \geq 3$. *Then X is a Hilbert space iff one of the following conditions hold*:

(i) *X is smooth and $D(R_G) = X$ for every subspace G (or also, $D(R_G) = X$ and R_G is linear for every subspace G).*

(ii) *$D(R_G) = X$ for every hyperplane G.*

(iii) *There is a proper subspace Y of X such that, for every subspace G of X, X-isomorphic to Y, and for every x we have $P_G(x) \subset R_G(x)$.*

(iv) *X is strictly convex and there is a proper subspace Y of X such that for every subspace G of X, X-isomorphic to Y, R_G^0 is defined on X and $I - R_G^0$ is contractive.*

The following result on the composition of set-valued operators of best coapproximation was given in [**20**].

Theorem 6.4. *Let E be a normed linear space and G, G' linear subspaces of E, with $G \subset G'$. Then*

$$R_G(R_{G'}(x)) \subset R_G(x) \quad \textit{for all } x \in E.$$

It was also shown in [**20**] that in general the inclusion in Theorem 6.4 is strict. However, when E is smooth, the situation is different, viz.

Theorem 6.5 [**20**]. *Let E be a smooth normed linear space and G, G' linear subspaces of E, with $G \subset G'$. Then*

$$R_G(R_{G'}(x)) = R_G(x) \quad \textit{for all } x \in \mathrm{Dom}\,(R_{G'}), \tag{6.1}$$

$$||R_G(x)|| \leq ||R_{G'}(x)|| \quad \textit{for all } x \in \mathrm{Dom}\,(R_{G'}) \cap \mathrm{Dom}\,(R_G). \tag{6.2}$$

Remarks 6.1 [**20**]. (a) When E is not smooth, the inequalities (6.2) need not hold for selections from $R_G(x)$ and $R_{G'}(x)$.

(b) It is interesting to compare (6.2) with the following well-known inequalities (see [**21**]) for best approximation, where G, G' are linear subspaces of E with $G' \subset G$:

$$||x - \pi_G(x)|| \leq ||x - \pi_{G'}(x)||$$
$$\pi_G(x) \in P_G(x) \neq \phi,\ \pi_{G'}(x) \in P_{G'}(x) \neq \phi.$$

The following theorem proved in [**20**] shows that the mapping R_G is idempotent and quasi-additive.

Theorem 6.6. *Let E be a normed linear space and G a linear subspace of E. Then*

(a) *$R_G(R_G(x)) = R_G(x)$ for all $x \in \mathrm{Dom}\,(R_G)$, i.e., the set-valued mapping R_G is idempotent*;

(b) *If $x \in \mathrm{Dom}\,(R_G)$ and $g \in G$, then $x + g \in \mathrm{Dom}\,(R_G)$ and*

$$R_G(x+g) = R_G(x) + g,$$

i.e., the set-valued mapping R_G is quasi-additive.

Remarks 6.2 [**20**]. (a) The quasi-additivity and homogeneity of the mapping R_G permits us to apply to R_G some results of [**23**] on general quasi-additive homogenous projections. For example, from [**23**, Proposition 2.3] (extended, with the same proof, to the set-valued case), it follows that

$$g_0 \in R_G(\alpha x + (1-\alpha)g_0), \qquad g_0 \in R_G(x),\ \alpha \in \mathbf{R}; \tag{6.3}$$

of course, one can see this directly also, since $G \perp x - g_0$ implies $G \perp \alpha(x - g_0) = \alpha x + (1-\alpha)g_0 - g_0$, $\alpha \in \mathbf{R}$.

(b) The well-known fact that, for every one-dimensional subspace G of a normed linear space E, there exists a linear projection of norm 1 of E onto G reformulated in terms of R_G shows:

For every G with $dim\, G = 1$, we have $dom\, R_G = E$ and the operator R_G admits a linear selection.

(c) One can also study semi-continuity properties of the set-valued mapping R_G, e.g., it was observed in [**20**] that if G is closed, then the mapping R_G is upper (K)-semi-continuous.

(d) Since R_G is a bounded mapping, and $R_G^{-1}(0) = \{x \in E : 0 \in R_G(x)\}$ is closed, some results on the continuity of R_G (when R_G is single-valued) can be deduced from those of [**23**, Section 3].

7. Best coapproximation on convex sets. In this section we discuss various properties of the map R_C, where C is an arbitrary subset of a normed linear space. It will be observed that various properties of the map R_G, when G is a linear subspace, can only partially be extended to the map R_C, and the map R_C is also related to the orthogonal retraction defined in [**5**]. The problem of "strong approximation" for the map R_C, similar to that of the map P_C suggested in [**2**], is also considered in this section.

It was observed in [**18**] that the map R_C satisfies the following properties similar to the results in [**10**]:

(i) $C \subset \operatorname{Dom}(R_C)$ and $R_C(x) = \{x\}$ for every $x \in C$;

(ii) $R_C(x)$ is closed if C is closed;

(iii) $R_C(x)$ is convex if C is convex;

(iv) If $x \in \operatorname{Dom}(R_C)$, then $R_C(x)$ is bounded;

(v) If $x^0 \in R_C(x)$, then $x^0 \in R_C(tx + (1-t)x^0)$, for $t \geq 1$;

(vi) $C \subset R_C(x)$ whenever the diameter of C is smaller than $\operatorname{dist}(x, C)$.

Another kind of map R'_C—the so-called "orthogonal retraction" was defined in [**5**]. If $x' \in C$, we say that $x' \in R'_C(x)$ if

$$\tau(x' - y, x - x') \geq 0 \quad \text{for every } y \in C. \tag{7.1}$$

These maps obviously satisfy the properties (i), (ii), (iv), (v). Corollary 7.1 below will imply that (iii) is also satisfied.

The following relationship between R'_C and R_C was observed in [**18**].

Theorem 7.1. *$x' \in R'_C(x)$ implies $x' \in R_C(x)$, and also $x' \in R'_C(tx + (1-t)x')$ for $t \geq 0$.*

The following two properties—property (A) and property (B) below were considered in [**18**] and are sufficient for $R_C(x) = R'_C(x)$.

(A). Suppose that $R_C(x)$ satisfies:

$$\text{If } x^0 \in R_C(x), \text{ then } x^0 \in R_C(tx + (1-t)x^0) \text{ for } 0 \leq t \leq 1.$$

Then $R_C(x) = R'_C(x)$.

This property implies that $R_C(x)$ is contained in the boundary of C. Also, by Theorem 7.1 and property (A), one obtains the following:

Corollary 7.1 [**18**]. *$x^0 \in R'_C(x)$ iff $x^0 \in R_C(tx + (1-t)x^0)$ for $0 \le t \le 1$ (so, in view of (v), for every $t \ge 0$).*

(B). Suppose that $R_C(x)$ satisfies:

$$\text{If } x^0 \in R_C(x) \text{ and } y \in C, \text{ then } (1-t)x^0 + ty \in C \text{ for } t \ge 1.$$

Then $R_C(x) = R'_C(x)$.

From Property (B), we get

Theorem 7.2 [**18**]. *Let C' be a subset of normed space B such that if y_1 and y_2 belong to C', then also $ty_1 + (1-t)y_2 \in C'$ for $t \ge 1$. Then $R'_{C'} = R_{C'}$. In particular, $x^0 \in R_G(x)$ iff $\tau(g, x - x^0) \ge 0$ for every $g \in G$.*

Remark 7.1. Theorem 7.2 is similar to the well-known result: $x_0 \in P_G(x)$ iff $\tau(x - x_0, g) \ge 0$ for every $g \in G$. But it was observed in [**18**] that a result similar to $x_0 \in P_C(x)$ iff $\tau(x - x_0, x_0 - y) \ge 0$, for every $y \in C$, does not hold for the map R_C.

Remark 7.2. In Hilbert spaces, $R_C = R'_C = P_C$ for every C. If B is two-dimensional and C is closed, then R'_C (so also R_C) is always defined (see [**5**, Theorem 5]); in particular, R'_C exists whenever C is contained in a one-dimensional subspace of B. If B is smooth, then R'_C is single-valued and nonexpansive on its domain (see [**5**, Lemma 1 and Theorem 1]: In that terminology, R_C is a nonexpansive projection). If C is bounded and R_C is defined on B, the fulfillment of property (A) for every $x \in B$ is a very strong condition (see [**6, 11**]).

Next we speak of "strong approximation" for the map R_C, similar to that for the map P_C suggested in [**2**], considered in [**18**]. This concept of strongness has a different meaning from that of strong unicity for P_C.

Definition 7.1 [**18**]. We say that $x^0 \in R_C(x)$ (or $x^0 \in R_G(x)$) strongly, if $x \in C$ (or $x \in G$) and there exists an $r > 0$ ($r \le 1$) such that

$$||x^0 - y|| + r||x^0 - x|| \le ||x - y|| \quad \text{for every } y \in C \ (y \in G). \tag{7.1}$$

We say that $x' \in R'_C(x)$ strongly, if $x \notin C$ and there exists an $r > 0$ ($r \le 1$) such that

$$\tau(x' - y, x - x') \ge r||x - x'|| \quad \text{for every } y \in C, \ y \ne x'. \tag{7.2}$$

Clearly, (7.2) implies (7.1). If (7.1) is satisfied for x^0 and property (B) holds, then $\tau(x^0 - y, x - x^0) \ge r||x - x^0||$; in particular, for $R_G = R'_G$, (7.1) is equivalent to (7.2) and to

$$\tau(g, x - x^0) \ge r||x - x^0|| \quad \text{for every } g \in G, \ g \ne 0. \tag{7.3}$$

The definition given by (7.1) means that if a point if moved in C (or in G) from a strong approximation x^0, inside the ball of radius $r||x - x^0||$ and centered at x^0, all the points reached are still approximations. So the above concept of strongness has nothing to do with unicity, and the larger r is, the more x moves from x^0.

The following result given in [**18**] gives an upper bound for the Chebyshev radius of the set of strong approximation in the sense of (7.1) (so also for the set defined by (7.2)).

Theorem 7.3. *The radius of the set of elements which belong strongly to $R_C(x)$, for a given r, is not larger than $(1 - r)d$.*

In general, we see that the radius of $R_C(x)$ is not larger than d. Moreover, if the space is smooth we recall that $R'_C(x)$ can contain at most one point, so in that case no element can belong to $R'_C(x)$ strongly, the same for $R_G(x)$ (a similar result holds for $P_G(x)$; see [**1**, Theorem 5]).

Concerning R'_C, the following result was proved in [**18**].

Theorem 7.4. *$x' \in R'_C(x)$ strongly iff the set*

$$A' = \{y \in C : \tau(x' - y, x - x') < ||y - x'||\}$$

contains no point of a certain sphere of positive radius, centered at x'.

If $\langle x_0, G\rangle$ denotes the linear span of x_0 and G, the following result is analogous to that of Proposition 1 of [**2**].

Theorem 7.5 [**18**]. *If x has a strong approximate x^0 from G, then so does any element in $\langle x, G\rangle$. More precisely, $x^0 \in R_G(x)$ implies $kx^0 + y \in R_G(kx + y)$ strongly with the same r for every $y \in G$ and $k \in \mathbf{R}$.*

Comments. The best coapproximation seems to be useful both as a kind of "approximation" and because of its relations with some other notions, which can be expressed in terms of $R_G(x)$. For example, $g_0 \in R_G(x)$ implies that the norm of the linear projection

$$P : G \oplus [x] \longrightarrow G \qquad (\text{where } [x] = \{\alpha x : \alpha \in R\}),$$

along $[x - g_0]$, i.e., of the operator

$$P(g + \alpha x) = g + \alpha g_0, \qquad g \in G,\ \alpha \in \mathbf{R}$$

is 1 and $P(x) = g_0$; conversely, if P is any linear norm 1 projection of $G \oplus [x]$ onto G, then $P(x) \in R_G(x)$. Thus, for $x \in E\backslash G$, we have $R_G(x) \neq \phi$ iff there exists a linear norm 1 projection from $G \oplus [x]$ onto G. Consequently, when G is closed, we have $R_G(x) \neq \phi$ for every $x \in E\backslash G$ iff there exists a linear norm 1 projection onto G from every subspace of E containing G as a hyperplane (in general, this does not imply the existence of a linear norm 1 projection from the whole space E onto G). Other concepts, which are defined with the aid of linear norm 1 projection, can also be expressed in terms of $R_G(x)$. For example, a sequence $\langle x_n\rangle$ in a Banach space E is a monotone basic sequence iff $0 \in R_{G_n}(x_{n+1})$, $n = 1, 2, \dots,$ where $G_n = [x_1, \dots, x_n]$ is the closed linear subspace of E spanned by $x_1, \dots, x_n$. Some further connections between $R_G(x)$ and monotone bases, as well as some other types of bases, have been given in [**17**]. Also, the notion of orthogonal retraction studied by Bruck [**5**] can be expressed in terms of $R_G(x)$ as follows: If G and C are subsets of E, with $G \subset C$, a mapping $r : C \to G$ is an orthogonal retraction of C onto G iff

$$r(x) \in R_G(\lambda x + (1-\lambda)r(x)), \qquad x \in C, \lambda \geq 0;$$

when $C = E$, this is an analogue of the "sun property" of G for the metric projection P_G (see, e.g., [**21**], also (6.3)).

Remark 5.1 shows that, in a certain sense, $r_G(x) \in R_G(x)$ behaves like $x - \pi_G(x)$ $(\in P_G^{-1}(0))$ where $\pi_G(x) \in P_G(x)$. In this connection we also recall that in a reflexive strictly convex space E, for every closed linear subspace G we have a natural decomposition $E = G \oplus P_G^{-1}(0)$, induced by π_G (see, e.g., [**22**, Proposition 3.1]). Correspondingly (in the light of Remarks 4.1(a), (b)), in a smooth space E, for every closed linear subspace G we have a decomposition

$$\operatorname{dom}(R_G) = G \oplus R_G^{-1}(0),$$

with R_G a linear norm 1 projection of $\operatorname{dom} R_G$ onto G along $R_G^{-1}(0)$ (hence $G \perp R_G^{-1}(0)$).

In 1961, V. Klee conjectured that there are nonconvex existence and uniqueness sets for the best approximation in infinite-dimensional Hilbert spaces. Although several fine results on this subject have been proved within the last two decades (see, e.g., [**16**]), we are still far from being able to give a definite answer. In contrast to this, for the best coapproximation, the existence and uniqueness sets in a Hilbert space are easily characterized. It was shown in [**3**] that for the best coapproximation in a Hilbert space the existence and uniqueness sets are the closed flats. But if the space is a strictly convex normed linear space, then every existence set for the best coapproximation is closed and convex.

To sum up the contents of this article, it is clear that very little has been done so far regarding best coapproximation. This theory can be developed to a large extent parallel to the theory of best approximation. Recently, this study has been taken by Geetha S. Rao and her pupil (see, e.g., [**8**]) when the underlying spaces are locally convex spaces with a family of semi norms and some results have been proved. It will be interesting to generalize the other known results on best coapproximation in such spaces and develop the theory further for such spaces. Also, perhaps it is possible to develop a parallel theory similar to the theory of farthest points. For known results in the theory of farthest points, one may refer to [**15**].

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