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FUCHS' PROBLEM 43

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What is the relationship between the abelian groups A and C, if $\operatorname{Ext}(A,B) \cong \operatorname{Ext}(C,B)$ for every abelian group B? This is problem 43 in [**3**]. In this note we give a complete solution to this problem when A, B and C are torsion-free abelian groups of finite rank. Our approach is to show that numerical invariants considered in [**5**] actually characterize the reduced finite rank torsion-free groups up to quasi-isomorphism.

This paper is essentially self-contained; however, the reader may wish to refer to [**1, 3**, and **4**]. For $B \leq A$, we say that B is a quasi-summand of A if for some $n \neq 0$ and $A' \leq A$, $nA \leq B \oplus A' \leq A$, and A is called strongly indecomposable in case A has no nontrivial quasi-summands. If $C \cong B$ and $nA \leq B \leq A$, then A and C are called quasi-isomorphic. As usual, set $QA = Q \otimes_Z A$ and regard $A \leq QA$.

Let $S_A(C)$ be the subgroup of C generated by $f(A)$ for all $f \in \operatorname{Hom}(A,C)$. A subgroup B of C will be said to be full in C if $\langle B\rangle_* = C$ where $\langle B\rangle_*$ denotes the pure subgroup of C generated by B.

Below all groups are torsion-free. The quasi-endomorphism ring of A is $QE(A)$ where $E(A)$ is the endomorphism ring of A. By the well-known result of J. Reid, $QE(A)$ is left Artinian if and only if A is quasi-isomorphic to a finite direct sum $A_1 \oplus \cdots \oplus A_n$ with each A_i strongly indecomposable. Moreover, if $\alpha \in QE(A_i)$, then α is invertible or α is nilpotent [**7**]. The proof of the main theorem will rest upon the

Lemma. *Let A and C be torsion-free groups with left Artinian quasi-endomorphism rings. If $S_A(C)$ is full in C and $S_C(A)$ is full in A, then A and C have an isomorphic nonzero quasi-summand.*

Proof. Let $E = E(A)$ and let R denote the nilradical of E. For $J =$ Jacobson radical of QE, $R = J \cap E$, and since QE is left Artinian, J (hence R) is nilpotent. Call $N = \langle RA\rangle_*$ which is the pure subgroup

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of A generated by $g(A)$ for all $g \in R$. If $R^n = 0$ and $R^{n-1} \neq 0$, then $R^{n-1}N = 0$ which proves that $A/N \neq 0$. Since A/N is torsion-free and $(S_C(A) + N)/N$ is full in A/N by the hypothesis, $S_C(A) \not\subseteq N$. This implies that there is an $f : C \to A$ with $f(C) \not\subseteq N$.

We may write $mC \leq C_1 \oplus \cdots \oplus C_k \leq C$ with each C_i strongly indecomposable and $m \neq 0$. It is easy to see that $mS_A(C) \leq S_A(C_1) \oplus \cdots \oplus S_A(C_k) \leq S_A(C)$. It must be that $f(S_A(C_1)) \not\subseteq N$ for some i. Otherwise, $\sum_{i=1}^k f(S_A(C_i)) \subseteq N$ which implies that $mf(S_A(C)) \subseteq N$, and since N is pure in A, $f(S_A(C)) \subseteq N$. But, for any $v \in C$, there is an $l > 0$ with $lv \in S_A(C)$ by the hypothesis, so that $f(lv) = lf(v) \in N$ which implies that $f(v) \in N$ by the purity of N. This contradicts $f(C) \not\subseteq N$. We may assume there is a map $g : A \to C_1$ such that $fg(A) \not\subseteq N$.

From the definition of N, $R \leq \mathrm{Hom}\,(A, N)$, so that $fg \notin R$. Now E/R is a full subring of the semi-simple ring QE/J so there are $h, h' \in E$ such that for $e = (hf)(gh')$, e is not nilpotent $\bmod R$ (QE/J is a direct product of matrix rings). Relabel hf and gh' as f and g, respectively, and restrict $f : C_1 \to A$.

We now have $gf \in \mathrm{End}\,(C_1)$. By the previously mentioned results of J. Reid, and by virtue of the fact that C_1 is strongly indecomposable, either $\alpha = gf$ is invertible in $QE(C_1)$ or else α is nilpotent. If $(gf)^n = 0$, then $e^{n+1} = f(gf)^n g = 0$, a contradiction. So α must be invertible. Consequently, there is an integer $s \neq 0$ so that $s\alpha^{-1} \in E(C_1)$ and $s1_{C_1} = s\alpha^{-1}gf$. Call $g' = s\alpha^{-1}g$.

For $K = \mathrm{Ker}\, g'$ and $A' = f(C_1)$ any $a \in A$ satisfies $sa = sa - f(g'(a)) + f(g'(a))$, and because $sa - f(g'(a)) \in K$, $sA \leq A' \oplus K \leq A$. Since f is a monomorphism, $A' \cong C_1$. □

For the remainder of the paper, assume A and C have finite rank. The p-rank of A, $r_p(A)$, is the Z/pZ-dimension of A/pA. Since $A/pA = Z_{(p)} \otimes_Z A$, if $0 \to A \to B \to C \to 0$ is pure exact, then $0 \to Z_{(p)} \otimes A \to Z_{(p)} \otimes B \to Z_{(p)} \otimes C \to 0$ is exact so $r_p(B) = r_p(A) + r_p(C)$. Also, $r_p(A) = 0$ for all p if and only if A is divisible. Let $\mu(A, C)$ denote the maximum rank of a strongly indecomposable quasi-summand of $A \oplus C$. By the Krull-Schmidt theorem, $\mu(A, C) = \max\{\mu(A, 0), \mu(C, 0)\}$. A reduced group is a group which contains no copies of Q.

Theorem. *Let A and C be reduced torsion-free groups of finite rank. The following are equivalent*:

(a) *A is quasi-isomorphic to C.*

(b) *$r_p(\mathrm{Hom}\,(A,B)) = r_p(\mathrm{Hom}\,(C,B))$ for all p and finite rank groups B.*

(c) *$r_p(\mathrm{Hom}\,(A,B)) = r_p(\mathrm{Hom}\,(C,B))$ for all p and all B of rank $\leq \mu(A,C)$.*

Proof. (a) $\to$ (b) Since, in this case, $\mathrm{Hom}\,(A,B)$ is quasi-isomorphic to $\mathrm{Hom}\,(C,B)$, and r_p is a quasi-isomorphism invariant (Theorem 0.2 in [**1**]), (b) follows.

(b) $\to$ (c) Clear.

(c) $\to$ (a) We will show that $S_A(C)$ is full in C; (a) will follow by the lemma and induction.

Let C_1 be a pure strongly indecomposable quasi-summand of C and $S_1 = \langle S_A(C_1)\rangle_*$. To simplify the argument, assume C_1 is a summand of C. Consider

$$0 \to \mathrm{Hom}\,(A,S_1) \to \mathrm{Hom}\,(A,C_1) \xrightarrow{\alpha} \mathrm{Hom}\,(A,C_1/S_1)$$

and

$$0 \to \mathrm{Hom}\,(C,S_1) \to \mathrm{Hom}\,(C,C_1) \xrightarrow{\beta} \mathrm{Hom}\,(C,C_1/S_1).$$

By the definition of $S_A(C_1)$, $\mathrm{Im}\,\alpha = 0$. By the hypothesis, $r_p(\mathrm{Hom}\,(C,C_1)) - r_p(\mathrm{Hom}\,(C,S_1)) = r_p(\mathrm{Im}\,\beta) = r_p(\mathrm{Im}\,\alpha) = 0$ for all p, so $\mathrm{Im}\,\beta$ is divisible.

Write $C = C_1 \oplus K$. Then $\mathrm{Ker}\,\beta = \mathrm{Hom}\,(C_1,S_1) \oplus \mathrm{Hom}\,(K,S_1)$ is a pure subgroup of $\mathrm{Hom}\,(C,C_1) = \mathrm{Hom}\,(C_1,C_1) \oplus \mathrm{Hom}\,(K,C_1)$ with a divisible cokernel. Hence, $\mathrm{Hom}\,(C_1,C_1)/\mathrm{Hom}\,(C_1,S_1)$ is a summand of a divisible group and is therefore divisible.

Let R_1 denote the nilradical of $\mathrm{End}\,(C_1)$ and $N_1 = \langle R_1C_1\rangle_* \leq C_1$. As in the lemma, $N_1 \neq C_1$. By the theorem of J. Reid, any endomorphism of C_1 is either in R_1 or else is a monomorphism. Hence, $R_1 = \mathrm{Hom}\,(C_1,N_1)$. By Reid's theorem, if $\mathrm{Hom}\,(C_1,S_1) \not\subseteq \mathrm{Hom}\,(C_1,N_1) = R_1$, then there is a monomorphism $f : C_1 \to C_1$ with $\mathrm{Im}\,f \leq S_1$. In this case, $\mathrm{rank}\,C_1 = \mathrm{rank}\,S_1$ implies $S_1 = C_1$ since S_1 is pure in C_1. We will show that $\mathrm{Hom}\,(C_1,S_1) \subseteq R_1$ is not possible.

Suppose $I = \mathrm{Hom}\,(C_1, S_1) \subseteq \mathrm{Hom}\,(C_1, N_1) = R_1$. From above, $\mathrm{End}\,(C_1)/I$ is divisible, so $\mathrm{End}\,(C_1)/R_1$ is divisible. The Beaumont-Pierce principal theorem asserts that $\mathrm{End}\,(C_1)/R_1$ is a (group) quasi-summand of $\mathrm{End}\,(C_1)$ [**2**, Theorem 1.4]. But $\mathrm{End}\,(C_1)$ is reduced so $I \not\subseteq R_1$.

If $mC \leq C_1 \oplus \cdots \oplus C_k \leq C$ for strongly indecomposable groups C_i and some $m \neq 0$, then $\langle S_A(C_i)\rangle_* = C_i$ implies $\langle S_A(C)\rangle_* = C$. Therefore, $S_A(C)$ is full in C and by the symmetry $S_C(A)$ is full in A.

From the lemma, A and C have an isomorphic quasi-summand. If A is quasi-isomorphic to $G \oplus A'$ and C is quasi-isomorphic to $G \oplus C'$ with $G \neq 0$, then $r_p(\mathrm{Hom}\,(A, B)) = r_p(\mathrm{Hom}\,(G, B)) + r_p(\mathrm{Hom}\,(A', B)) = r_p(\mathrm{Hom}\,(G, B)) + r_p(\mathrm{Hom}\,(C', B)) = r_p(\mathrm{Hom}\,(C, B))$ for all p and B of rank $\leq \mu(A, C)$. Clearly, $\mu(A', 0) \leq \mu(A, 0)$ and $\mu(C', 0) \leq \mu(C, 0)$, so that $r_p(\mathrm{Hom}\,(A', B)) = r_p(\mathrm{Hom}\,(C', B))$ for all p and all B of rank $\leq \max\{\mu(A', 0), \mu(C', 0)\} = \mu(A', C')$. The result follows by induction on rank A. □

Recall that the outer type of A, $OT(A)$, is the supremum of the types of the rank-1 quotients of A. If $OT(A) = \mathrm{type}\,Q$, then $\mathrm{Ext}\,(A \oplus Q, B) \cong \mathrm{Ext}\,(A, B)$ for all B [**9**, Theorem 2.3].

Corollary. *Let A and C be torsion-free of finite rank. The following are equivalent*:

(a) $\mathrm{Ext}\,(A, B) \cong \mathrm{Ext}\,(C, B)$ *for all torsion-free groups B of finite rank.*

(b) $A = F \oplus A' \oplus D$ *and* $C = F' \oplus C' \oplus D'$ *with A' quasi-isomorphic to C', F and F' free, D and D' divisible, and the restriction that* $OT(A) = OT(C)$.

Proof. (a) $\rightarrow$ (b) Write $A = F \oplus A' \oplus D$ and $C = F' \oplus C' \oplus D'$ with F and F' free, D and D' divisible, and A' and C' reduced with $\mathrm{Hom}\,(A', Z) = \mathrm{Hom}\,(C', Z) = 0$. By Theorem 1,3 in [**5**], $OT(A) = OT(C)$ and $r_p(\mathrm{Hom}\,(A', B)) = r_p(\mathrm{Hom}\,(C', B))$ for all p and B. By our theorem, A' is quasi-isomorphic to C'.

(b) $\rightarrow$ (a) Clearly, $\mathrm{Ext}\,(A', B) \cong \mathrm{Ext}\,(C', B)$ for every B. If $OT(A) = OT(C) = \mathrm{type}\,Q$, then $\mathrm{Ext}\,(A' \oplus D, B) \cong \mathrm{Ext}\,(C' \oplus D', B)$ for all B

by Theorems 2 and 3 in [**9**]. Otherwise, $D = D' = 0$. In either case, $\text{Ext}(A, B) \cong \text{Ext}(C, B)$ for all B. □

As indicated by the referee, a similar problem was considered in [**6**]. Although their paper has a more general setting, in our context they show that two extension functors on the category of abelian groups, $\text{Ext}(A, \cdot)$ and $\text{Ext}(C, \cdot)$ are naturally equivalent if and only if $A \oplus F \cong C \oplus F'$ for some free groups F and F'. They impose no restrictions on A and C although they do require the isomorphishm $\text{Ext}(A, B) \cong \text{Ext}(C, B)$ to be natural.

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