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PULLBACKS OF BANACH BUNDLES

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ABSTRACT. Let S and T be compact Hausdorff spaces, $\alpha : S \to T$ a continuous map, and $\rho : F \to T$ a Banach bundle. The bundle $\pi : E \to S$ which is the pullback to S of ρ via α, has fibers given by $E_p = F_{\alpha(p)}$ for $p \in S$. The present paper investigates some properties, such as norm continuity and Hausdorfness, which the bundle π inherits from ρ via the map α. The main result presents a relation between the section spaces $\Gamma(\rho)$ and $\Gamma(\pi)$ via inductive tensor products.

This paper continues the study of Banach bundles which has been pursued by the authors in previous papers [**2, 3, 4, 5, 6**], to which the reader is referred for notations, definitions, and general information. This time the focus is upon pullback bundles. If $\pi : F \to T$ is a bundle of Banach spaces, and if $\alpha : S \to T$ is a continuous map, then there is a bundle of Banach spaces $\pi : E \to S$, called the *pullback* of the given bundle by α, with the following properties

1) if $p \in S$, then $E_p = \pi^{-1}(\{p\})$, the stalk in the pullback bundle over p, is an isomorphic copy of $F_{\alpha(p)} = \rho^{-1}(\{\alpha(p)\})$;

2) If σ is any section in $\Gamma(\rho)$, then its pullback by α, namely $\alpha^*(\sigma) = \sigma \circ \alpha : S \to E$ is a section in $\Gamma(\pi)$. (See Kitchen and Robbins [**3**] for a discussion of pullbacks.)

Our results are basically of three sorts. First, we investigate properties which the pullback bundle inherits from the given bundle. (If, for instance, the bundle $\rho : F \to T$ is norm continuous, is the pullback bundle $\pi : E \to S$ also norm continuous, is the pullback bundle $\pi : E \to S$ also norm continuous?) The main result (Theorem 5) presents a relation between the section spaces $\Gamma(\rho)$ and $\Gamma(\pi)$ via inductive tensor products. The substance of the theorem is summarized by the equation

$$C(S)\hat{\hat{\otimes}}_{C(T)}\Gamma(\rho) \cong \Gamma(\pi),$$

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whose exact meaning (together with necessary hypotheses) is explained later. This formula for $\Gamma(\pi)$, the pullback section space, is then exploited in the final theorem, which says that each element ω in $\mathrm{Hom}_T(\Gamma(\rho), C(T))$, the internal dual of the section space $\Gamma(\rho)$, can be pulled back by α in a natural way to yield an element $\alpha^*(\omega)$ in $\mathrm{Hom}_S(\Gamma(\pi), C(S))$, the internal dual for the section space of the pullback bundle.

For some of the results, compactness of the base spaces S and T is essential. As a matter of convenience, we shall assume throughout that S and T are compact Hausdorff spaces, even though these hypotheses are sometimes excessively strong. We begin, then, by making the following standing assumptions which will remain in force throughout the entire paper

1) S and T are compact Hausdorff spaces;

2) $\rho : F \to T$ is a bundle of Banach spaces;

3) $\alpha : S \to T$ is a continuous map. (Sometimes it will be further assumed that the map is surjective or open.)

4) $\pi : E \to S$ is the pullback of the bundle $\rho : F \to T$ by α.

We shall also continue to use the notation introduced two paragraphs earlier for pullbacks. If $\sigma \in \Gamma(\rho)$, then $\alpha^*(\sigma) = \sigma \circ \alpha$ is its pullback by α; $\alpha^*(\sigma)$ then belongs to the space $\Gamma(\pi)$. Similarly, if $g \in C(T)$, we define $\alpha^*(g)$ to be $g \circ \alpha$, the pullback of g by α; clearly, $\alpha^*(g)$ belongs to $C(S)$. The space $\Gamma(\rho)$ is a $C(T)$-module, while $\Gamma(\pi)$ is a $C(S)$-module. Additionally, the space $\Gamma(\pi)$ is a $C(T)$-module by pullback, the module multiplication being defined by the equation

$$\begin{aligned} g \cdot \sigma &= (g \circ \alpha)\sigma \quad \text{(pointwise product)} \\ &= \alpha^*(g)\sigma, \end{aligned}$$

for all $g \in C(T)$ and $\sigma \in \Gamma(\pi)$. When $\Gamma(\rho)$ and $\Gamma(\pi)$ are viewed as $C(T)$-modules, then the pullback map $\alpha^* : \Gamma(\rho) \to \Gamma(\pi)$ is a module homomorphism, that is, it is a contractive (of norm one or less) linear map which is $C(T)$-linear ($\alpha^*(g\sigma) = g \cdot \alpha^*(\sigma)$ for all $g \in C(T)$ and $\sigma \in \Gamma(\rho)$). If $\alpha : S \to T$ is surjective, the map $\alpha^* : \Gamma(\rho) \to \Gamma(\pi)$ is an isometry, and, while the map is rarely surjective, $\{\alpha^*(\sigma) : \sigma \in \Gamma(\rho)\}$ is a full set of sections of the pullback bundle. (*Proof.* Let x be any vector in the stalk $E_p = F_{\alpha(p)}$. Then there exists a section $\sigma \in \Gamma(\rho)$

such that $\sigma(\alpha(p)) = x$. Then $\{\alpha^*(\sigma)\}(p) = \sigma(\alpha(p)) = x$. Thus, every element of the fiber space E lies on a section of the form $\alpha^*(\sigma)$. So, by definition, $\{\alpha^*(\sigma) : \sigma \in \Gamma(\rho)\}$ is a full set of sections for the pullback bundle.) The space $C(S)$ can also be viewed as a $C(T)$-module, where

$$g \cdot f = \alpha^*(g)f = (g \circ \alpha)f$$

for all $g \in C(T)$ and $f \in C(S)$.

All of the assertions in the previous paragraph are easily verified and well-known.

In addition to the given bundle $\rho : F \to T$ and its pullback bundle $\pi : E \to S$, a third bundle will enter our discussions, namely, the canonical bundle for $\Gamma(\pi)$ as a $C(T)$-module. We shall denote this bundle by $\xi : G \to T$. We begin with a description of this canonical bundle and the associated Gelfand morphism $\wedge : \Gamma(\pi) \to \Gamma(\xi)$. As a preliminary, we observe that the map $\alpha : S \to T$ yields a partition of S into closed subsets. For each $q \in T$, we define S_q to be the set $\alpha^{-1}(\{q\})$. To assure that the sets S_q are all nonempty, we assume that the map $\alpha : S \to T$ is surjective. Then, for each $q \in T$, S_q is a closed (and compact) subset of S, and $\{S_q : q \in T\}$ is a partition of S.

Proposition 1. *Assume that the map $\alpha : S \to T$ is not only continuous but surjective. Then, for each $q \in T$, the stalk $G_q = \xi^{-1}(\{q\})$ is isometrically isomorphic to $C(S_q, F_q)$, the space of all continuous maps from S_q to F_q. If $\sigma \in \Gamma(\pi)$, then its Gelfand representation $\sigma^\wedge$ is described by*

$$\sigma^\wedge(q) = \sigma|S_q = \text{ the restriction of } \sigma \text{ to } S_q,$$

for all $q \in T$ (where we use the natural identification of G_q with $C(S_q, F_q)$). The Gelfand morphism $\wedge : \Gamma(\pi) \to \Gamma(\xi)$ is a $C(T)$-linear isometric isomorphism.

Proof. It is easily checked that $\Gamma(\pi)$ is a $C(T)$-locally convex $C(T)$-module, and thus $\wedge : \Gamma(\pi) \to \Gamma(\xi)$ is an isometric isomorphism.

We now verify that $G_q \cong C(S_q, F_q)$. Note that, in doing so, we will also show that the restriction of the pullback bundle $\pi : E \to S$ to sets of the form $\alpha^{-1}(\{q\})$ $(q \in T)$ is trivial, with fibers F_q.

Fix $q \in T$. Given $\tau \in \Gamma(\pi)$, set $\phi(\tau) = \tau|S_q$. If $p \in S_q$, then $\tau(p) \in E_p = F_{\alpha(p)} = F_q$. Thus, $\phi(\tau)$ is a continuous map from S_q to F_q, that is,

$$\phi(\tau) \in C(S_q, F_q).$$

Clearly, $\|\phi(\tau)\| \leq \|\tau\|$. Thus, $\phi : \Gamma(\pi) \to C(S_q, F_q)$ is a norm-decreasing map. If $f \in C(S_q, F_q)$, then, according to the Tietze extension theorem for sections (see Kitchen and Robbins [**2**]) the local section $f : S_q \to E$ can be extended without increase of norm to a section $\sigma : S \to E$ in $\Gamma(\pi)$. Then $\phi(\sigma) = \sigma|S_q = f$. Thus, $\phi : \Gamma(\pi) \to C(S_q, F_q)$ is a surjective quotient map. It follows that ϕ induces an isometric isomorphism $\tilde{\phi} : \Gamma(\pi)/\ker\phi \to C(S_q, F_q)$. Now $\ker\phi$ consists of those sections in $\Gamma(\pi)$ which vanish on S_q. More importantly, $\ker\phi = I_q \cdot \Gamma(\pi)$, where $I_q = \{f \in C(T) : f(q) = 0\}$ is the maximal ideal in $C(T)$ corresponding to q and $I_q \cdot \Gamma(\pi)$ is the closed linear span of the set $\{g \cdot \sigma : q \in I_q, \sigma \in \Gamma(\pi)\}$.

The inclusion $I_q \cdot \Gamma(\pi) \subset \ker\phi$ is trivial. (Since $\ker\phi$ is a closed subspace, it suffices to show that $\ker\phi$ contains $g \cdot \sigma$, if $g \in I_q$ and $\sigma \in \Gamma(\pi)$. For all $p \in S_q$, however, $\alpha(p) = q$, so

$$(g \cdot \sigma)(p) = \{(g \circ \alpha)\sigma\}(p) = g(\alpha(p))\sigma(p) = 0 \cdot \sigma(p) = 0.$$

Thus, $g \cdot \sigma$ vanishes on S_q, which means that $g \cdot \sigma \in \ker\phi$). To establish the reverse inclusion, we use approximate identities for I_q. For each open neighborhood V of q, we select a continuous function g_V in I_q such that g_V maps T into the interval $[0, 1]$ and $g_V = 1$ off V. If $\sigma \in \ker\phi$, then the net

$$\{g_V \cdot \sigma : V \text{ is a neighborhood of } q\}$$

converges (in the norm topology of $\Gamma(\pi)$) to σ, thereby showing that $\sigma \in I_q \cdot \Gamma(\pi)$. (*Proof.* Let $\varepsilon > 0$ be given. The set

$$U = \{p \in S : \|\sigma(p)\| < \varepsilon\}$$

is open (since $\|\sigma(\cdot)\| : S \to \mathbf{R}$ is upper semicontinuous) and contains S_q. Thus, $S \setminus U$ is closed and compact. The set $\alpha(S \setminus U)$ is therefore compact, and $q \notin \alpha(S \setminus U)$. Thus, $T \setminus \alpha(S \setminus U)$ is an open neighborhood of q. If V is an open neighborhood of q which is contained in $T \setminus \alpha(S \setminus U)$, then $\|(g_V \cdot \sigma)(p) - \sigma(p)\| < \varepsilon$ for every $p \in S$. For, if $p \in U$, then

$$\|(g_V \cdot \sigma)(p) - \sigma(p)\| = (1 - g_V(p))\|\sigma(p)\| < 1 \cdot \varepsilon = \varepsilon,$$

while, if $p \notin U$, then $\alpha(p) \notin V$, so

$$(g_V \cdot \sigma)(p) - \sigma(p) = g_V(\alpha(p))\sigma(p) - \sigma(p) = \sigma(p) - \sigma(p) = 0.$$

Thus, $\|g_V \cdot \sigma - \sigma\| < \varepsilon$. This proves that the net $\{g_V \cdot \sigma\}$ converges to σ.)

Now, by definition, $G_q = \Gamma(\pi)/I_q \cdot \Gamma(\pi) = \Gamma(\pi)/\ker\phi$, and

$$\sigma^\wedge = \sigma + I_q \cdot \Gamma(\pi).$$

Hence, $\tilde{\phi} : G_q \to C(S_q, F_q)$ is an isometric isomorphism, and under this natural identification of G_q with $C(S_q, F_q)$, $\sigma^\wedge(q) = \sigma|S_q$, by which we really mean

$$\tilde{\phi}(\sigma^\wedge(q)) = \tilde{\phi}(\sigma + \ker\phi) = \phi(\sigma) = \sigma|S_q. \qquad \square$$

Corollary 2. *Suppose that $\alpha : S \to T$ is surjective. If we view $C(S)$ as a $C(T)$-module, then for each point $q \in T$ the stalk above q in the canonical bundle for $C(S)$ can be identified with the space S_q in such a way that*

$$f^\wedge(q) = f|S_q.$$

Proof. We can view $C(S)$ as the section space of the canonical bundle $p : S \times \mathbf{C} \to S$, where $S \times \mathbf{C}$ has the product topology and p is a projection onto the first coordinate. $\square$

We first consider the Hausdorff condition.

Proposition 3. *If the given bundle $\rho : F \to T$ is Hausdorff, then the pullback bundle $\pi : E \to S$ is also Hausdorff. If $\rho : F \to T$ is Hausdorff, and if the map $\alpha : S \to T$ is surjective and open, then the canonical bundle $\xi : G \to T$ is also Hausdorff.*

Proof. If $\rho : F \to T$ is Hausdorff, then, by definition, the fiber space F is Hausdorff. Hence, $F \times S$, with the product topology, is Hausdorff. Now, E can be regarded as a subset of $F \times S$, namely,

$$E = \{(x, p) \in F \times S : \pi(x) = \alpha(p)\},$$

and thus E is Hausdorff.

Let us now suppose that the bundles $\rho : F \to T$ and $\pi : E \to S$ are Hausdorff and that the map $\alpha : S \to T$ is both open and surjective. To show that $\xi : G \to T$ is Hausdorff, it suffices to show that the nonzero vectors in the fiber space G form an open set. (See Gierz [**1**, Proposition 16.4].) Consider a nonzero element of G, say

$$\sigma^\wedge(q) = \sigma|S_q \neq 0,$$

where $\sigma \in \Gamma(\pi)$ and $q \in T$. Thus, $\sigma(p) \neq 0$ for some $p \in S_q$. Because the bundle $\pi : E \to S$ is Hausdorff, there is a neighborhood of $\sigma(p)$, say

$$\mathcal{U} = \{x \in E : \pi(x) \in V, \|x - \sigma(\pi(x))\| < \varepsilon\}$$

where V is a neighborhood of p, such that $\mathcal{U}$ consists entirely of nonzero vectors in E. Since α is open, $\alpha(V)$ is a neighborhood of $q = \alpha(p)$. Consider, now, the following neighborhood of $\sigma^\wedge(q)$ in G:

$$\mathcal{V} = \{y \in G : \xi(y) \in \alpha(V), \|y - \sigma^\wedge(\xi(y))\| < \varepsilon\}.$$

It suffices to show that $\mathcal{V}$ does not contain a zero vector. Suppose, to the contrary, that $\mathcal{V}$ contains the zero vector in stalk $G_{q'}$, where $q' \in \alpha(V)$. Then, since the zero vector is assumed to belong to $\mathcal{V}$, $\|\sigma^\wedge(q')\| < \varepsilon$, which means that $\|\sigma(p)\| < \varepsilon$ for all $p \in S_q$. In particular, $\|\sigma(p)\| < \varepsilon$ holds for all $p \in S_{q'} \cap V$. But this means that $\mathcal{U}$ contains the zero vector in each stalk E_p where $p \in S_{q'} \cap V$, and this is a contradiction. □

We next consider norm continuity.

Proposition 4. *Suppose that the given bundle $\rho : F \to T$ is norm continuous. Then the pullback bundle $\pi : E \to S$ is norm continuous. If, in addition, the map $\alpha : S \to T$ is open and surjective, then the canonical bundle $\xi : G \to T$ is also norm continuous.*

Proof. Consider a point x_0 in the fiber space E. We must show that the norm function $\|\cdot\| : E \to \mathbf{R}$ is continuous at x_0. Let $\varepsilon > 0$ be given. Set $p_0 = \pi(x_0)$. Then $x_0 \in E_{p_0} = F_{\alpha(p_0)}$. Choose a section

$\sigma \in \Gamma(\rho)$ such that $\sigma(\alpha(p_0)) = x_0$. Since $\|\sigma(\cdot)\|$ is continuous at $\alpha(p_0)$, there is a neighborhood V of $\alpha(p_0)$ such that

$$\Big| \|\sigma(q)\| - \|\sigma(\alpha(p_0))\| \Big| < \frac{\varepsilon}{2}$$

whenever $q \in V$. Since α is continuous, there is a neighborhood U of p_0 such that

$$p \in U \quad \text{implies} \quad \alpha(p) \in V.$$

Consider the following neighborhood of x_0:

$$\mathcal{U} = \{x \in E : \pi(x) \in U, \|x - \{\alpha^*(\sigma)\}(\pi(x))\| < \varepsilon/2.$$

Suppose that $x \in \mathcal{U}$. Setting $p = \pi(x)$, it follows that $p \in U$, so $\alpha(p) \in V$. Also,

$$\|x - \sigma(\alpha(p))\| = \|x - \{\alpha^*(\sigma)\}(\pi(x))\| < \frac{\varepsilon}{2}$$

and

$$\Big| \|\sigma(\alpha(p))\| - \|\sigma(\alpha(p_0))\| \Big| < \frac{\varepsilon}{2}.$$

Thus,

$$\begin{aligned} |\|x\| - \|x_0\|| &\le |\|x\| - \|\sigma(\alpha(p))\|| + |\|\sigma(\alpha(p))\| - \|x_0\|| \\ &\le \|x - \sigma(\alpha(p))\| + |\|\sigma(\alpha(p))\| - \|\sigma(\alpha(p_0))\|| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This proves that the norm function $\|\cdot\| : E \to \mathbf{R}$ is continuous at x_0.

Let us assume now that the map $\alpha : S \to T$ is open and surjective. To show that the canonical bundle $\xi : G \to T$ is norm continuous, it suffices to show that $\|\sigma^\wedge(\cdot)\|$ is a continuous function on T for each σ in $\Gamma(\pi)$. Set $f(p) = \|\sigma(p)\|$ for each $p \in S$. Since the pullback bundle is norm continuous, f belongs to $C(S)$. Also, for all $q \in T$,

$$\begin{aligned} \|\sigma^\wedge(q)\| &= \|\sigma|S_q\| = \sup\{\|\sigma(p)\| : p \in S_q\} \\ &= \sup\{|f(p)| : p \in S_q\} = \|f|S_q\| = \|f^\wedge(q)\|, \end{aligned}$$

where we view $C(S)$ as a $C(T)$-module. Now the canonical bundle for $C(S)$ as a $C(T)$-module is norm continuous (see Seda [**8**] and Kitchen and Robbins [**3**]). Thus, $\|\sigma^\wedge(q)\| = \|f^\wedge(q)\|$ varies continuously with q, and this completes the proof. □

We turn now to the formula for $\Gamma(\pi)$ as an inductive tensor product of the spaces $\Gamma(p)$, $C(S)$, and $C(T)$.

Theorem 5.
$$\Gamma(\pi) \cong C(S)\hat{\hat{\otimes}}_{C(T)}\Gamma(\rho).$$
By this we mean that there is an isometric isomorphism $\phi : (C(S)\hat{\hat{\otimes}}\Gamma(\rho))/K \to \Gamma(\pi)$, *where* K *is the closed linear span in* $C(S)\hat{\hat{\otimes}}\Gamma(\rho)$ *of elements of the form*
$$(g \cdot f) \otimes \sigma - f \otimes (g\sigma),$$
where $g \in C(T)$, $f \in C(S)$, *and* $\sigma \in \Gamma(\rho)$. *Moreover,*
$$\phi(f \otimes \sigma + K) = f\alpha^*(\sigma)$$
for all $f \in C(S)$ *and* $\sigma \in \Gamma(\rho)$.

Proof. The notation $C(S)\hat{\hat{\otimes}}_{C(T)}\Gamma(\rho)$ comes from an analogous situation involving projective tensor products. If M and N are Banach modules over a Banach algebra A, then $M \otimes_A N$ is defined to be the quotient space of $M\hat{\otimes}N$ modulo the closed linear span of elements of the form
$$(ax) \otimes y - x \otimes (ay),$$
where $a \in A$, $x \in M$, and $y \in N$.

To show the existence of the isomorphism ϕ, we begin by observing that $C(S)$ can be regarded as the section space of the trivial bundle $\gamma : S \times \mathbf{C} \to S$, where $S \times \mathbf{C}$ has the product topology and γ is coordinate projection. Consequently, the space $C(S)\hat{\hat{\otimes}}\Gamma(\rho) = \Gamma(\gamma)\hat{\hat{\otimes}}\Gamma(\rho)$ is isometrically isomorphic to $\Gamma(\gamma\hat{\hat{\otimes}}\rho)$, and the natural isomorphism $\theta : C(S)\hat{\hat{\otimes}}\Gamma(\rho) \to \Gamma(\gamma\hat{\hat{\otimes}}\rho)$ assigns to any product $f \otimes \sigma$ the pointwise tensor product $f \odot \sigma$ given by
$$(f \odot \sigma)(p, q) = f(p) \otimes \sigma(q)$$

for all $p \in S$ and $q \in T$. (See Kitchen and Robbins [**2**].) Now $f(p) \otimes \sigma(q)$ belongs to the stalk $\gamma^{-1}(\{p\}) \hat{\hat{\otimes}} F_q = \mathbf{C} \hat{\hat{\otimes}} F_q$. The latter product is naturally isomorphic to F_q, with a tensor product $\lambda \otimes y$ corresponding to the scalar multiple λy. Consequently, in the product bundle $\gamma \hat{\hat{\otimes}} \rho$ the stalk above the point (p, q) in $S \times T$ can be regarded as being F_q, and, so regarded, our isometric isomorphism $\theta : C(S) \hat{\hat{\otimes}} \Gamma(\rho) - \Gamma(\gamma \hat{\hat{\otimes}} \rho)$ is characterized by the equation

$$\{\theta(f \otimes \sigma)\}(p, q) = f(p)\sigma(q)$$

for all $f \in C(S)$, $\sigma \in \Gamma(p)$, $p \in S$, and $q \in T$.

We consider next the map ω which assigns to each section τ in $\Gamma(\gamma \hat{\hat{\otimes}} \rho)$ its restriction to the set

$$G_\alpha = \{(p, \alpha(p)) : p \in S\},$$

the graph of α. The set G_α is a closed subset of $S \times T$ and the restriction map ω is then a quotient map, which means that it induces an isometry $\tilde{\omega}$ on the quotient space $\Gamma(\gamma \hat{\hat{\otimes}} \rho)/L$, where L is the kernel of ω. (The fact that ω is a quotient map can best be argued by the application of the Tietze extension theorem for sections.) Obviously, the kernel of ω is the set of all sections in $\Gamma(\gamma \hat{\hat{\otimes}} \rho)$ which vanish on G_α. We will show a bit later that L is also the image of K under the map θ.

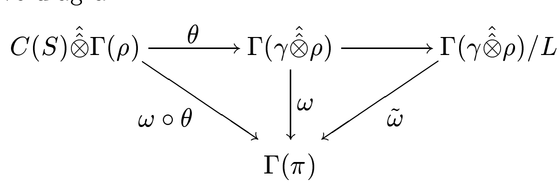
Note that the composed map $\omega \circ \theta$ carries a tensor product $f \otimes \sigma$ (with $f \in C(S)$ and $\sigma \in \Gamma(\rho)$) onto the section τ given by

$$\tau(p, \alpha(p)) = f(p)\sigma(\alpha(p)) = \{f\alpha^*(\sigma)\}(p).$$

Thus, τ is a section of the pullback bundle, only, instead of writing $\tau(p, \alpha(p))$, we now write $\tau(p)$, and we have

$$(\omega \circ \theta)(f \otimes \sigma) = \tau = f\alpha^*(\sigma).$$

The maps which we have considered can be exhibited in the following commutative diagram:



Now, sections of the form $(\omega \circ \theta)(f \otimes \sigma) = f\alpha^*(\sigma)$ are full in the pullback bundle $\pi : E \to S$, and their closed linear span, call it W, is clearly a $C(S)$-submodule of $\Gamma(\pi)$. By the Stone-Weierstrass theorem for bundles (see Gierz [**1**]), $W = \Gamma(\pi)$, which means that the range of $\omega \circ \theta$ is dense in $\Gamma(\pi)$. Clearly, the functions $\omega \circ \theta$, ω, and $\tilde{\omega}$ all have the same range. Since, however, $\tilde{\omega}$ is an isometry, it follows that the common range of $\omega \circ \theta$, ω, and $\tilde{\omega}$ must be all of $\Gamma(\pi)$. In other words, the maps $\omega \circ \theta$, ω, and $\tilde{\omega}$ are surjective.

Since $\omega \circ \theta : C(S)\hat{\hat{\otimes}}\Gamma(\rho) \to \Gamma(\pi)$ is a surjective quotient map, it induces an isometric isomorphism

$$\phi : (C(S)\hat{\hat{\otimes}}\Gamma(\rho))/\ker(\omega \circ \theta) \to \Gamma(\pi).$$

Clearly, $\ker(\omega \circ \theta) = \theta^{-1}(L)$, so to complete the proof, we must show that $\theta^{-1}(L) = K$, or equivalently that $L = \theta(K)$.

Since θ is a linear isometry, $\theta(K)$ is the closed linear span in $\Gamma(\gamma\hat{\hat{\otimes}}\rho)$ of sections of the form

$$\tau = (g \cdot f) \odot \sigma - f \odot (g\sigma)$$

where $g \in C(T)$, $f \in C(S)$, and $\sigma \in \Gamma(\rho)$. Note that the section τ vanishes on G_α, since

$$\begin{aligned}\tau(p, \alpha(p)) &= (g \cdot f)(p) \odot \sigma(\alpha(p)) - f(p) \odot (g\sigma)(\alpha(p))\\ &= g(\alpha(p))f(p)\sigma(\alpha(p)) - f(p)g(\alpha(p))\sigma(\alpha(p)) = 0\end{aligned}$$

for all $p \in S$. Hence, $\theta(K) \subset L$.

It now suffices to show that $\theta(K)$ is dense in L, and we do so with the Stone-Weierstrass theorem for bundles.

Clearly, L is a closed $C(S \times T)$-locally convex submodule of $T(\gamma\hat{\hat{\otimes}}\rho)$. Hence, L is isometrically isomorphic to $\Gamma(\beta)$, where $\beta : H \to S \times T$ is the canonical bundle for L. It is easily argued that the stalks of the canonical bundle are described (to within isomorphism) by

$$H_{(p,q)} = \begin{cases} \theta, & \text{if } q = \alpha(p) \\ F_q, & \text{otherwise} \end{cases}$$

and for each $\sigma \in L$ the Gelfand representation $\sigma^\wedge$ is given by

$$\sigma^\wedge(p, q) = \sigma(p, q)$$

for all (p, q) in $S \times T$.

As for $\theta(K)$, we can first verify that it is a $C(S \times T)$-submodule. This requires a linearity and density argument. We know that finite sums of functions of the form $f \odot g$, where $f \in C(S)$ and $g \in C(T)$, are dense in $C(S \times T)$. Since $\theta(K)$ is a closed subspace, it suffices to show that $\theta(K)$ contains all section of the form

$$\tau = (f \odot g)\{(h \cdot k) \odot \sigma - k \odot (h\sigma)\}$$

where f and k belong to $C(S)$, g and h belong to $C(T)$, and σ belongs to $\Gamma(\rho)$. When τ is evaluated at a point (p, q) in $S \times T$, the following results

$$f(p)g(q)h(\alpha(p))k(p)\sigma(q) - f(p)g(q)k(p)h(q)\sigma(q),$$

which is equal to the section

$$(h \cdot fk) \odot (g\sigma) - (fk) \odot ((gh)\sigma)$$

evaluated at (p, q). Clearly, $\theta(K)$ contains τ.

To verify that $\theta(K) = L$, it suffices, according to the bundle version of the Stone-Weierstrass theorem, to show that $\theta(K)$ is stalkwise dense in the canonical bundle $\beta : H \to S \times T$ for L. Let $(p, q) \in S \times T$. If $q = \alpha(p)$, then the stalk $H_{(p,q)}$ is zero-dimensional, so there is nothing to prove. Suppose that $q \neq \alpha(p)$. Let $\sigma(q)$ be an arbitrary element in $H_{(p,q)} = F_q$, where $\sigma \in \Gamma(\rho)$. Let $f \in C(S)$ have the constant value 1, and let $g \in C(T)$ be such that $g(\alpha(p)) = 1$ and $g(q) = 0$. Then

$$\begin{aligned}\{(g \cdot f) \odot \sigma - f \odot (g\sigma)\}(p, q) \\ = g(\alpha(p))f(p)\sigma(q) - f(p)g(q)\sigma(q) = \sigma(q).\end{aligned}$$

This proves that $\theta(K)$ is stalkwise dense in the canonical bundle for L. Hence, by the bundle version of the Stone-Weierstrass theorem, $\theta(K) = L$.

Finally, we observe that our isometric isomorphism behaves in the manner advertised:

$$\phi(f \otimes \sigma + K) = (\omega \circ \theta)(f \otimes \sigma) = f\alpha^*(\sigma)$$

for all f in $C(S)$ and σ in $\Gamma(\rho)$. □

There is an addendum to the Theorem which further justifies the notation

$$C(S)\hat{\hat{\otimes}}_{C(T)}\Gamma(\rho) \cong \Gamma(\pi).$$

If M and N are Banach modules over a Banach algebra A, then, as we observed at the beginning of the proof, $M \otimes_A N$ is the space $M\hat{\otimes}N/W$, where W is the closed linear span of elements of the form

$$(ax) \otimes y - x \otimes (ay)$$

where $a \in A$, $x \in M$, and $y \in N$. An important feature of the Banach space $M \otimes_A N$ is that there is a natural way in which it is an A-module: if we denote by $x \otimes_A y$ the coset $x \otimes y + W$, as is customary, then the action of A on $M \otimes_A N$ is characterized by the identity

$$a(x \otimes_A y) = (ax) \otimes_A y = x \otimes_A (ay).$$

See, for example, [**7**].

Proposition 6. (Addendum to Theorem 5) *Given the data of Theorem* 5*, let us denote by* $f \otimes_T \sigma$ *the coset* $f \otimes \sigma + K$ *for* $f \in C(S)$ *and* $\sigma \in \Gamma(\rho)$*. Then there is a unique way in which* $C(S)\hat{\hat{\otimes}}_{C(T)}\Gamma(\rho)$ *can be made into a* $C(T)$*-module so that the identity*

$$(*) \qquad g \cdot (f \otimes_T \sigma) = (g \cdot f) \otimes_T \sigma = f \otimes_T (g\sigma)$$

holds for all $f \in C(S)$*,* $g \in C(T)$*, and* $\sigma \in \Gamma(\rho)$*. Moreover, with this module multiplication, the isometric isomorphism* $\phi : C(S)\hat{\hat{\otimes}}_{C(T)}\Gamma(\rho) \to \Gamma(\pi)$ *is* $C(T)$*-linear.* (*Also,* ϕ *is characterized by the equation* $\phi(f \otimes_T \sigma) = f\alpha^*(\sigma)$*.*)

Proof. Because of the definition of K it is clear that $(g \cdot f) \otimes_T \sigma = (g \cdot f) \otimes \sigma + K = f \otimes (g\sigma) + K = f \otimes_T (g\sigma)$. Moreover, *if* there is a module multiplication on $C(S)\hat{\hat{\otimes}}_{C(T)}\Gamma(\rho)$ for which $(*)$ holds, it must be unique since $C(S)\hat{\hat{\otimes}}_{C(T)}\Gamma(\rho)$ is the closed linear span of elements of the form $f \otimes_T \sigma$.

To show the existence of the module multiplication on $C(S)\hat{\hat{\otimes}}_{C(T)}\Gamma(\rho)$, we simply transplant the $C(T)$-module multiplication from $\Gamma(\pi)$ by the isomorphism ϕ: if $g \in C(T)$ and if $\zeta \in C(S)\hat{\hat{\otimes}}_{C(T)}\Gamma(\rho)$ we define

$$g \cdot \zeta = \phi^{-1}(g \cdot \phi(\zeta)).$$

This immediately makes $C(S)\hat{\hat{\otimes}}_{C(T)}\Gamma(\rho)$ a $C(T)$-module, and it makes ϕ $C(T)$-linear. Moreover, $(*)$ is satisfied, since

$$\begin{aligned} g \cdot (f \otimes_T \sigma) &= \phi^{-1}(g \cdot \phi(f \otimes_T \sigma)) = \phi^{-1}(g \cdot (f\alpha^*(\sigma))) \\ &= \phi^{-1}(\alpha^*(g) f\alpha^*(\sigma)) = \phi^{-1}(f\alpha^*(g\sigma)) \\ &= \phi^{-1}(\phi(f \otimes_T (g\sigma))) = f \otimes_T (g\sigma), \end{aligned}$$

for all $g \in C(T)$, $f \in C(S)$, and $\sigma \in \Gamma(\rho)$. □

We shall now exploit Theorem 5 to show that there is a natural way of embedding the "internal dual" of the given bundle in the internal dual of the pullback bundle.

Theorem 7. *There is a natural norm-decreasing map*

$$\alpha^* : \mathrm{Hom}_T(\Gamma(\rho), C(T)) \to \mathrm{Hom}_S(\Gamma(\pi), C(S))$$

such that

$$\alpha^*(\omega)(f\alpha^*(\sigma)) = f\alpha^*(\omega(\sigma))$$

for all $\omega \in \mathrm{Hom}_T(\Gamma(\rho), C(T))$, $f \in C(S)$, *and* $\sigma \in \Gamma(\rho)$. *If* $\alpha : S \to T$ *is surjective, then the pullback map* $\alpha^* : \mathrm{Hom}_T(\Gamma(\rho), C(T)) \to \mathrm{Hom}_S(\Gamma(\pi), C(S))$ *above is an isometry.*

Proof. Let $\omega \in \mathrm{Hom}_T(\Gamma(\rho), C(T))$. We define a map $\omega' : C(S) \times \Gamma(\rho) \to C(S)$ by

$$\omega'(f, \sigma) = f\alpha^*(\omega(\sigma)).$$

It is easily verified that ω' is bilinear. Hence, we get an induced map $\omega' : C(S) \otimes \Gamma(\rho) \to C(S)$ such that

$$\omega'(f \otimes \sigma) = f\alpha^*(\omega(\sigma)).$$

Now, the algebraic tensor product $C(S) \otimes \Gamma(\rho)$ is a dense subspace of $C(S)\hat{\hat{\otimes}}\Gamma(\rho)$. To show that ω' can be extended to a bounded linear map on $C(S)\hat{\hat{\otimes}}\Gamma(\rho)$ it suffices for ω' to be bounded on $C(S) \otimes \Gamma(\rho)$ with respect to the inductive tensor product norm $\| \|_{\wedge\atop\wedge}$. Consider an arbitrary element of $C(S) \otimes \Gamma(\rho)$, say $\Sigma_{i=1}^n f_i \otimes \sigma_i$. Then

$$\begin{aligned}\|\omega'(\Sigma_{i=1}^n f_i \otimes \sigma_i)\| &= \|\Sigma_{i=1}^n f_i \alpha^*(\omega(\sigma_i))\| \\ &= \sup\{|\Sigma_{i=1}^n f_i(p)\{\omega(\sigma_i)\}(\alpha(p))| : p \in S\} \\ &\leq \sup\{|\Sigma_{i=1}^n f_i(p)\{\omega(\sigma_i)\}(q)| : (p,q) \in S \times T\} \\ &= \|\Sigma_{i=1}^n f_i \odot \omega(\sigma_i)\| = \|\Sigma_{i=1}^n f_i \otimes \omega(\sigma_i)\|_{\wedge\atop\wedge} \\ &\leq \|\omega\| \|\Sigma_{i=1}^n f_i \otimes \sigma_i\|_{\wedge\atop\wedge},\end{aligned}$$

since $\| \|_{\wedge\atop\wedge}$ is a uniform cross-norm. Hence, ω' can be uniquely extended to a bounded linear map $\omega' : C(S)\hat{\hat{\otimes}}\Gamma(\rho) \to C(S)$; moreover, $\|\omega'\| \leq \|\omega\|$.

We observe that $K \subset \ker \omega'$. Since $\ker \omega'$ is a closed subspace, it suffices to show that $\ker \omega'$ contains all elements of the form

$$(g \cdot f) \otimes \sigma - f \otimes (g\sigma),$$

where $g \in C(T)$, $f \in C(S)$, and $\sigma \in \Gamma(\rho)$. But

$$\begin{aligned}\omega'((g \cdot f) \otimes \sigma) - f \otimes (g\sigma) &= (g \cdot f)\alpha^*(\omega(\sigma)) - f\alpha^*(\omega(g\sigma)) \\ &= \alpha^*(g) f \alpha^*(\omega(\sigma)) - f\alpha^*(g\omega(\sigma)) \\ &\quad \text{(since } \omega \text{ is } C(T)\text{-linear)} \\ &= \alpha^*(g) f \alpha^*(\omega(\sigma)) - f\alpha^*(g)\alpha^*(\omega(\sigma)) = 0.\end{aligned}$$

Since $K \subset \ker \omega'$, there is a unique linear map $\tilde{\omega} : C(S)\hat{\hat{\otimes}}\Gamma(\rho)/K \to C(S)$ such that

$$\tilde{\omega}(f \otimes_T \sigma) = f\alpha^*(\omega(\sigma)).$$

Moreover, $\|\tilde{\omega}\| = \|\omega'\| \leq \|\omega\|$. The above displayed equation closely resembles the equation

$$\phi(f \otimes_T \sigma) = f\alpha^*(\sigma)$$

which characterized the natural isometric isomorphism

$$\phi : C(S)\hat{\hat{\otimes}}_{C(T)}\Gamma(\rho) = C(S)\hat{\hat{\otimes}}\Gamma(\rho)/K \to \Gamma(\pi).$$

We now define $\alpha^*(\omega) : \Gamma(\pi) \to C(S)$ by $\alpha^*(\omega) = \tilde{\omega} \circ \phi^{-1}$. Then $\|\alpha^*(\omega)\| = \|\tilde{\omega}\| \leq \|\omega\|$; moreover, for all $f \in C(S)$ and $\sigma \in \Gamma(\rho)$,

$$\begin{aligned}\{\alpha^*(\omega)\}(f\alpha^*(\sigma)) &= (\tilde{\omega} \circ \phi^{-1})(\phi(f \otimes_T \sigma)) \\ &= \tilde{\omega}(f \otimes_T \sigma) = f\alpha^*(\omega(\sigma)).\end{aligned}$$

From the latter equation, it follows that $\alpha^*(\omega)$ is $C(S)$-linear. Thus, $\alpha^*(\omega)$ belongs to $\mathrm{Hom}_S(\Gamma(\pi), C(S))$.

Finally, if the map $\alpha : S \to T$ is surjective, we observe that

$$\begin{aligned}\|\alpha^*(\omega)\| &= \sup\{\|\alpha^*(\omega)(\sigma)\| : \sigma \in \Gamma(\pi), \|\sigma\| = 1\} \\ &\geq \sup\{\|\alpha^*(\omega)(\alpha^*(\tau))\| : \tau \in \Gamma(\rho), \|\tau\| = 1\} \\ &\quad \text{(recall that } \alpha^* : \Gamma(\rho) \to \Gamma(\pi) \text{ is an isometry} \\ &\quad \text{since } \alpha : S \to T \text{ is surjective)} \\ &= \sup\{\|\alpha^*(\omega(\tau))\| : \tau \in \Gamma(\rho), \|\tau\| = 1\} \\ &= \sup\{\|\omega(\tau)\| : \tau \in \Gamma(\rho), \|\tau\| = 1\} \\ &= \|\omega\|.\end{aligned}$$

Thus, $\|\alpha^*(\omega)\| = \|\omega\|$ and the proof is complete. □

For the sake of completeness, we give an example to show that the hypothesis in Propositions 3 and 4 that the map α be open is necessary, and that the map α^* of Theorem 7 need not be surjective.

Example 8. Let $S = [0, 1]$, let $T = \{z \in \mathbf{C} : |z| = 1\}$ be the unit circle in the complex plane, and define $\alpha : S \to T$ by $\alpha(p) = \exp(2\pi i p)$. Then α is evidently a continuous, but not open, surjection. If $\rho : F \to T$ is the trivial bundle, then $\Gamma(\rho) = C(T)$. The pullback $\pi : E \to S$ is also the trivial bundle, and $\Gamma(\pi) = C(S)$. The canonical bundle $\xi : G \to T$ of $\Gamma(\pi) = C(S)$ as a $C(T)$-module has fibers

$$G_q = \begin{cases} \mathbf{C}, & \text{if } q \neq 1 \\ \mathbf{C}^2, & \text{if } q = 1. \end{cases}$$

Then ξ is not Hausdorff (because the function $\dim : T \to \mathbf{R}$, given by $\dim(q)$ = dimension of G_q, is not lower semicontinuous; see Gierz [**1**, Theorem 18.3]) and hence not norm continuous, even though, of course, ρ satisfies both of these conditions.

We also have $\mathrm{Hom}_S(\Gamma(\pi), C(S)) \cong C(S)$ and $\mathrm{Hom}_T(\Gamma(\rho), C(T)) \cong C(T)$. It can be verified that the map $\alpha^* : \mathrm{Hom}_T(\Gamma(\rho), C(T)) \to \mathrm{Hom}_S(\Gamma\pi), C(S))$ of Theorem 7 is the ordinary isometric embedding of $C(T)$ into $C(S)$ via α. In particular, $\alpha^*(f)(0) = \alpha^*(f)(1)$ for each $f \in C(T)$, so that α^* is not surjective.

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