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FINITE CODIMENSIONAL IDEALS IN BANACH ALGEBRAS WITH ONE GENERATOR

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Let A be a complex Banach algebra and let M be a closed n-codimensional subspace of A with $n < \infty$. A.M. Gleason [**1**] and, independently, J.P. Kahane and W. Zelazko [**4**] proved that for $n = 1$, M is an ideal if and only if M consists exclusively of noninvertible elements. Equivalently, a linear functional F is multiplicative if and only if $F(f)$ is contained in the spectrum of f for any f in A. In the following years this result has been extended in various directions. In [**3**] one can find the history of this subject and a list of open problems. The main question here, still to large extent open, is when and how we can extend the Gleason-Kahane-Zelazko theorem for $n > 1$. In [**2**] we get a positive result for $A = C(S)$, but the examples given by C.R. Warner and R. Whitley [**5,6**] show that it fails in general.

In this note we give a partial solution to this question.

Theorem. *Let A be a Banach algebra with one generator and let M be a closed n-codimensional subspace of A with $n < \infty$. Assume that each element of M is contained in at least n distinct maximal ideals of A. Then M is an ideal, namely an intersection of n maximal ideals of A.*

Proof. A is a Banach algebra with one generator so there is an f in A such that the polynomials of f are dense in A. Considering $f + \lambda e$ in place of f, we can assume without loss of generality that $K = \sigma(f)$ is contained in $\{x + iy : x > 0\}$. We can also assume that K is infinite since otherwise $A = C(K)$ and the result (trivial in this case) follows from [**2**]. For any complex number λ the functions $f^{\lambda} = \exp(\lambda \ln f)$, $f^{\lambda+1}, \dots, f^{\lambda+n}$ are well defined and linearly independent so there is a nontrivial linear combination of these $n + 1$ functions contained in M. Assume there are two nonproportional such combinations. Then there

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is a nontrivial linear combination of functions $f^\lambda, f^{\lambda+1}, \dots, f^{\lambda+n-1}$ in M. We have

$$f^\lambda \sum_{j=0}^{n-1} b_j(\lambda) f^j = \sum_{j=0}^{n-1} b_j(\lambda) f^{\lambda+j} \in M.$$

Our assumption that any element of M is contained in at least n maximal ideals means that the Gelfand transform

$$z^\lambda \sum_{j=0}^{n-1} b_j(\lambda) z^j$$

of

$$f^\lambda \sum_{j=0}^{n-1} b_j(\lambda) f^j$$

has at least n distinct zeros in K. This is not possible since $0 \notin K$ and $\sum_{j=0}^{n-1} b_j(\lambda) z^j$ is a polynomial of degree at most $n-1$. Hence, for any $\lambda \in \mathbf{C}$ there is exactly one *n-tuple* $a_1(\lambda), \dots, a_n(\lambda)$ of complex numbers, with $a_n(\lambda) \neq 0$ and such that

$$\sum_{j=0}^{n-1} a_j(\lambda) f^{\lambda+1} + f^{\lambda+n} \in M \tag{1}$$

(we assume here, without loss of generality, that $a_n = 1$). We show now that $a_j(\cdot)$ are entire functions. To this end, let $F_1, \dots, F_n$ be continuous functionals on A such that $M = \cap_{j=1}^n \ker F_j$. The numbers a_j from (1) are the solutions of a system of n linear equations in n unknowns:

$$\sum_{j=1}^{n-1} a_j(\lambda) F_s(f^{\lambda+j}) + F_s(f^{\lambda+n}) = 0, \qquad s = 1, \dots, n.$$

Since this system has exactly one solution, its determinant is never zero and, from Cramer's formula, the solution is a well-defined fraction of combinations of the analytic functions $\lambda \mapsto F_s(f^{\lambda+j})$.

From (1) and the definition of M, for any $\lambda \in \mathbf{C}$ there are distinct complex numbers $c_1(\lambda), \dots, c_n(\lambda)$, all contained in K such that

$$\sum_{j=1}^{n-1} a_j(\lambda) f^{\lambda+j} + f^{\lambda+n} = f^{\lambda} \prod_{j=1}^{n} (f - c_j(\lambda)) \in M.$$

Hence, all the functions $a_j(\cdot)$ are bounded and entire, and thus constant, so the functions $c_j(\cdot)$ are also constant.

We proved that there are distinct complex numbers $c_1, \dots, c_n$ in K such that for any nonnegative integer k the element $f^k \prod_{j=1}^{n}(f - c_j)$ is in M. Since M is closed and linear combinations of f^k are dense in A, this shows that $M_1 = A \prod_{j=1}^{n}(f - c_j) \subseteq M$. We also have $M_1 \subseteq J = \{f \in A : f(c_j) = 0, j = 1, \dots, n\}$. Polynomials of f are dense in A so M_1 is dense in J, hence $J \subseteq M$ but the codimensions of these subspaces are the same so they are actually equal. □

Remarks. We assumed in our theorem that there is an f in A such that the linear combinations of nonnegative powers of f are dense in A. This assumption can be weakened. It is enough to assume that there is an f in A such that the linear combinations of powers of f (negative or positive) are dense in A. This can be easily deduced from our theorem by considering $B =$ *the closure, in* A, *of the algebra of all polynomials* of f and $M_k = \{gf^{-k} \in B : g \in M\}$, where k is an arbitrary integer.

The following is the main open problem related to our result.

Conjecture. *Let A be a Banach algebra (a Banach algebra with one generator), let M be a closed subspace of A of codimension $2 \leq n < \infty$, consisting of noninvertible elements only. Then M is contained in a nontrivial ideal of A.*

REFERENCES

1. A.M. Gleason, *A characterization of maximal ideals*, J. Analyse Math. **19** (1967), 171–172.

2. K. Jarosz, *Finite codimensional ideals in function algebras*, Trans. Amer. Math. Soc. **267** (1985), 779–785.

3. ———, *Generalizations of the Gleason-Kahane-Zelazko Theorem*, Rocky Mountain J. Math., to appear.

4. J.P. Kahane and W. Zelazko, *A characterization of maximal ideals in commutative Banach algebras*, Studia Math. **29** (1968), 339–343.

5. C.R. Warner and R. Whitley, *A characterization of regular maximal ideals*, Pacific J. Math. **30** (1969), 277–281.

6. ———, *Ideals of finite codimension in* $C[0,1]$ *and* $L^1(R)$, Proc. Amer. Math. Soc. **76** (1979), 263–267.

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