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ALMOST COMPLETELY DECOMPOSABLE TORSION-FREE GROUPS

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Recently, D. Arnold and C. Vinsonhaler were successful in developing a complete set of numerical invariants for certain pure subgroups and homomorphic images of completely decomposable abelian groups [**4**]. Their results extended work by F. Richman [**8**]. Let A_i be subgroups of the rationals, $\mathbf{Q}$, containing the integers, $\mathbf{Z}$. The groups that they consider are the strongly indecomposable groups of the form: $A_1 \oplus \cdots \oplus A_n/\langle(1,1,\dots,1)\rangle_*$; or $\operatorname{Ker}\sigma$, where $\sigma : A_1 \oplus \cdots \oplus A_n \to \mathbf{Q}$ is defined by $\sigma(a_1,\dots,a_n) = \sum a_i$.

In this paper we assume that A is almost completely decomposable (quasi-isomorphic to a finite direct sum of subgroups of $\mathbf{Q}$), and give conditions on the types of the quasi-summands of A which characterize when every pure subgroup and/or every torsion-free homomorphic image of A is again almost completely decomposable. Our results generalize a theorem of M.C.R. Butler [**6**].

Throughout, all groups considered are torsion-free abelian. If n is a positive integer ≥ 2, we write $\bar{n}$ for the set $\{1,2,\dots,n\}$. Suppose for each $i \in \bar{n}$, τ_i is a type. If $I \subseteq \bar{n}$ is nonempty, we write τ^I (respectively, τ_I) for $\sup\{\tau_i : i \in I\}$ (respectively, $\inf\{\tau_i : i \in I\}$). If $I = \{i,j\}$, we often write τ^{ij} or $\tau_i \vee \tau_j$ (respectively, τ_{ij} or $\tau_i \wedge \tau_j$) for τ^I (respectively, τ_I).

1. Almost completely decomposable homomorphic images.

We will assume that $A = A_1 \oplus \cdots \oplus A_n$ with $\mathbf{Z} \leq A_i \leq \mathbf{Q}$. Set $G\langle A\rangle = A/X$, where X is the pure subgroup generated by $(1,1,\dots,1)$.

Let $f : A_1 \oplus \cdots \oplus A_n \to G\langle A\rangle$ be the natural quotient map. If K is corank-1 in $G\langle A\rangle$ (i.e., if $G\langle A\rangle/K$ is rank-1 and torsion-free), set $\operatorname{cosupp}(K) = \{i \in \bar{n} : f(A_i) \leq K\}$. By [**2**, Theorem 1.4], there is a cobalanced embedding $\delta : G\langle A\rangle \to \oplus\{G\langle A\rangle/K : K$ is corank-1 and $\operatorname{cosupp}(K)$ is maximal with respect to inclusion$\}$, where δ is induced

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by the quotient maps $G\langle A\rangle \to G\langle A\rangle/K$.

Suppose $1 \le i < j \le n$ and $K_{ij} \le A$ is such that $A/K_{ij} \cong A_i + A_j$. Then $\overline{K}_{ij} = K_{ij}/X$ is the unique corank-1 subgroup of $G\langle A\rangle$ with maximal cosupp $(\overline{K}_{ij}) = \bar{n} - \{i,j\}$. Since $G\langle A\rangle/\overline{K}_{ij} \cong A_i + A_j$, the above discussion yields a cobalanced embedding $\delta : G\langle A\rangle \to \oplus\{A^{ij} : 1 \le i < j \le n\}$, where $A^{ij} = A_i + A_j$. Moreover, if $\pi_{ij} : \oplus\{A^{lm} : 1 \le l < m \le n\} \to A^{ij}$ is the projection map, δ can be chosen so that $\pi_{ij}\delta((a_1,\dots,a_n) + X) = a_i - a_j$. With the above notation we can show

Proposition 1.1. *Suppose $G\langle A\rangle$ has a rank-1 quasi-summand. Then, there exist distinct $i, j \in \bar{n}$ and nonempty disjoint subsets I and J of $\bar{n}$ such that $I \cup J = \bar{n}$ and $\tau^{ij} = \tau_I \vee \tau_J$.*

Proof. Let $\delta : G\langle A\rangle \to \oplus\{A^{ij} : 1 \le i < j \le n\} = D$ be the cobalanced embedding. If C is a rank-1 quasi-summand of $G\langle A\rangle$, then C is a quasi-summand of D, so $C \cong A^{ij}$ for some $i < j$.

If B is a rank-1 pure subgroup of $G\langle A\rangle$, define supp $(B) = \{lm : 1 \le l < m \le n$ and $\pi_{lm}\delta(B) \ne 0\}$ and supp $(G\langle A\rangle) = \{$supp $(B) : B$ is a pure rank-1 subgroup of $G\langle A\rangle\}$. Set $S_\delta = \{S \in$ supp $(G\langle A\rangle) : S$ is minimal with respect to inclusion$\}$, say $S_\delta = \{S_1,\dots,S_t\}$. For each u, $1 \le u \le t$, let G_u be the unique rank-1 pure subgroup of $G\langle A\rangle$ with supp $(G_u) = S_u$. By [**2**, Theorem 1.2], there is a balanced surjection $\phi : G_1 \oplus \cdots \oplus G_t \to G\langle A\rangle$.

Since C is projective with respect to $G_1 \oplus \cdots \oplus G_t \to G\langle A\rangle$, $C \cong G_k$ for some k, $1 \le k \le t$. We will show type $(G_k) = \tau_I \vee \tau_J$ for some I and J. Suppose $0 \ne b = (b_1,\dots,b_r) + X \in G_k$. Set $I = \{i \in \bar{n} : b_i = b_1\}$ and $J = \bar{n} - I$. Let $a = (a_1,\dots,a_n) + X$, where $a_i = 1$ if $i \in I$ and 0 otherwise. Thus, supp $(\langle a\rangle_*) \subseteq$ supp G_k so by minimality, $\langle a\rangle_* = G_k$. Moreover, since the embedding $\delta : G\langle A\rangle \to \oplus\{A^{uv} : 1 \le u < v \le n\}$ is pure, type $G_k =$ type $\langle a\rangle_* = \inf\{\tau^{uv} : u \in I, v \in J\} = \tau_I \vee \tau_J$, completing the proof. □

For $I \subseteq \bar{n}$, set $A_I = \oplus\{A_i : i \in I\}$. Let $A \sim B$ denote the quasi-isomorphism of the finite rank groups A and B.

Proposition 1.2. *Suppose $A = A_1 \oplus \cdots \oplus A_n$ where, for each $i \in \bar{n}$, $\mathbf{Z} \le A_i \le \mathbf{Q}$ and type $(A_i) = \tau_i$. Assume there exist distinct $i_0, j_0 \in \bar{n}$ and a nontrivial partition $\bar{n} = I \cup J$ such that $i_0 \in I$ and $\tau^{i_0 j_0} \le \tau^{ij}$ for all $i \in I$ and $j \in J$.*

(a) *If $j_0 \in J$, $G\langle A\rangle \sim G\langle A_I\rangle \oplus G\langle A_J\rangle \oplus G\langle A_{\{i_0,j_0\}}\rangle$.*

(b) *If $j_0 \in I$, $G\langle A\rangle \sim G\langle A_I\rangle \oplus G\langle A_{J\cup\{j_0\}}\rangle$.*

Proof. First, part (a) is Lemma 1 in [**7**]. As such, we shall omit the proof. To see how (b) follows from (a), we first observe that if $\tau_i \ge \tau_j$ for $i \ne j$, then $G\langle A\rangle \sim A_i \oplus G\langle A_{\bar{n}-\{i\}}\rangle$ by applying (a) with $I = \{i\}$ and $J = \bar{n} - \{i\}$.

Now, set $\tau_{n+1} = \tau_{j_0}$ and let $A_{n+1} \le \mathbf{Q}$ be a group of type τ_{n+1} containing $\mathbf{Z}$. Then $I \cup (J \cup \{n+1\}) = \overline{n+1}$ is a nontrivial partition and $\tau_{j_0} \vee \tau_{n+1} = \tau_{j_0} \le \tau^{ij}$ for all $i \in I$ and $j \in J \cup \{n+1\}$. Thus, by part (a), $G\langle A_{\overline{n+1}}\rangle \sim G\langle A_I\rangle \oplus G\langle A_{J\cup\{n+1\}}\rangle \oplus G\langle A_{\{j_0,n+1\}}\rangle$. Note $G\langle A_{\{j_0,n+1\}}\rangle \cong A_{n+1}$ and $G\langle A_{\overline{n+1}}\rangle \sim A_{n+1} \oplus G\langle A\rangle$ by our observation in the previous paragraph. Part (b) now follows. □

Call a finite list of (not necessarily distinct) types $\tau_1, \dots, \tau_n$ *well-formed* if and only if, for every subset $S \subseteq \bar{n}$ with $|S| \ge 2$, there exist distinct $i, j \in S$ and a nontrivial partition $I \cup J = S$ such that $\tau^{ij} = \tau_I \vee \tau_J$. For us, there are two important properties of well-formed lists which follow immediately from the definition. First, any subset of a well-formed list is well-formed. Second, we remark that if $\tau_1, \dots, \tau_n$ is well-formed, and $\tau_{n+1} = \tau_i$ for some $i \in \bar{n}$, then $\tau_1, \dots, \tau_n, \tau_{n+1}$ is well-formed. Consequently, the multiplicity of a given τ_i is irrelevant when considering whether a given list is well-formed. The importance of well-formed lists is revealed by

Theorem 1.3. *Suppose $A \sim A_1 \oplus A_2 \oplus \cdots \oplus A_n$, where each A_i is a rank-1 group of type τ_i. Then A/B is almost completely decomposable for all rank-1 pure subgroups B of A if and only if $\tau_1, \tau_2, \dots, \tau_n$ is well-formed.*

Proof. We may assume $\mathbf{Z} \le A_i \le \mathbf{Q}$ and that $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$. First suppose A/B is almost completely decomposable for all rank-1 pure B. If $S \subseteq \bar{n}$ with $|S| \geq 2$, define $b = (b_1, b_2, \ldots, b_n) \in A$ by $b_i = 1$ if $i \in S$ and $b_i = 0$ if $i \in \bar{n} - S$. Then $G\langle A_S \rangle$ is a direct summand of $A/\langle b \rangle_*$ and hence almost completely decomposable. From Proposition 1.1, we conclude that there is a partition $I \cup J$ of S and $i, j \in S$ so that $\tau^{ij} = \tau_I \vee \tau_J$. Therefore, $\tau_1, \ldots, \tau_n$ is well-formed.

Conversely, suppose $\tau_1, \tau_2, \ldots, \tau_n$ is well-formed. If $B = \langle (b_1, b_2, \ldots, b_n) \rangle_*$ is a rank-1 pure subgroup of A, set $T = \{i \in \bar{n} : b_i = 0\}$ and $U = \bar{n} - T$. Then $A/B \sim A_T \oplus G\langle A_U \rangle$. Since the list of τ_i's with $i \in U$ is well-formed, it suffices to show $G\langle A \rangle$ is almost completely decomposable. We do this by induction on $n = \text{rank } A$. If $n = 2$, there is nothing to prove. So, we may assume $n > 2$. Since $\tau_1, \tau_2, \ldots, \tau_n$ is well-formed, there exist distinct $i, j \in \bar{n}$ and a nontrivial partition $\bar{n} = I \cup J$ such that $\tau^{ij} = \tau_I \vee \tau_J$. Without loss, we may assume $i \in I$. We consider two cases.

Case 1. If $j \in J$, Proposition 1.2(a) implies $G\langle A \rangle \sim G\langle A_I \rangle \oplus G\langle A_J \rangle \oplus G\langle A_{\{i,j\}} \rangle$. Since both $|I| < n$ and $|J| < n$ and since sublists of well-formed lists are well-formed, both $G\langle A_I \rangle$ and $G\langle A_J \rangle$ are almost completely decomposable by induction. Thus, $G\langle A \rangle$ is almost completely decomposable.

Case 2. If $j \in I$, Proposition 1.2(b) implies $G\langle A \rangle \sim G\langle A_I \rangle \oplus G\langle A_{J \cup \{j\}} \rangle$. Then, $|I| < n$ and $|J \cup \{j\}| < n$ since $i \in J \cup \{j\}$. As in Case 1, $G\langle A \rangle$ is again almost completely decomposable by induction. □

Corollary 1.4. *Suppose $A \sim A_1 \oplus A_2 \oplus \cdots \oplus A_n$, where each A_i is a rank-1 group of type τ_i, and assume the list $\tau_1, \tau_2, \ldots, \tau_n$ is closed under supremums. Then, for every $m \geq 1$, any torsion-free homomorphic image of A^m is almost completely decomposable if and only if A/B is almost completely decomposable for all rank-1 pure subgroups B of A.*

Proof. The necessity is clear, and to show the sufficiency, we may assume $\mathbf{Z} \leq A_i \leq \mathbf{Q}$ for each $i \in \bar{n}$ and $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$. By the remark preceding Theorem 1.3, the types of the summands of

$$A^m = (A_1 \oplus A_2 \oplus \cdots \oplus A_n) \oplus \cdots \oplus (A_1 \oplus A_2 \oplus \cdots \oplus A_n)$$

are a well-formed list. Theorem 1.3 implies all rank $mn-1$ torsion-free quotients of A^m are almost completely decomposable. By applying the pure embedding δ described at the beginning of this section, we conclude that the types of the rank-1 quasi-summands of a rank $mn-1$ torsion-free homomorphic image of A^m are of the form τ_i or τ^{ij}, $i, j \in \bar{n}$. Since $\tau_1, \tau_2, \ldots, \tau_n$ is closed under supremums and since sublists of well-formed lists are well-formed, the result follows by induction. □

2. The dual problem and a theorem of Butler. The results of section 1 all have duals but we intend only to present the duals to Theorem 1.3 and Corollary 1.4. A list of types $\tau_1, \ldots, \tau_n$ is called *co-well-formed* if for every $S \subseteq \bar{n}$ with $|S| \geq 2$ there are distinct $i, j \in S$ and a nontrivial partition $I \cup J = S$ satisfying $\tau_{ij} = \tau^I \wedge \tau^J$.

Theorem 2.1. *Assume $A \sim A_1 \oplus \cdots \oplus A_n$ with each A_i rank-1 of type τ_i. Then, every corank-1 subgroup of A is almost completely decomposable if and only if $\tau_1, \ldots, \tau_n$ is co-well-formed.*

Proof. Using the duality developed in [**5**], there is an almost completely decomposable group $B \sim B_1 \oplus \cdots \oplus B_n$ with type $(B_i) = \sigma_i$ such that every corank-1 subgroup of A is almost completely decomposable if and only if B/X is almost completely decomposable for each rank-1 pure subgroup X of B. Moreover, the correspondence $\sigma_i \to \tau_i$ extends to a lattice anti-isomorphism between the lattice of types generated by $\sigma_1, \ldots, \sigma_n$ and the lattice generated by $\tau_1, \ldots, \tau_n$. Therefore, the theorem follows from Theorem 1.3. □

Corollary 2.2. *Suppose $A \sim A_1 \oplus \cdots \oplus A_n$ with each A_i a rank-1 group of type τ_i. If $\tau_1, \ldots, \tau_n$ is closed under infimums, then for each $m \geq 1$, every pure subgroup of A^m is almost completely decomposable if and only if every corank-1 subgroup of A is almost completely decomposable.*

Suppose L is a finite sublattice of the lattice of all types. If $\alpha, \beta \in L$ with $\alpha < \beta$ and there is no τ in L with $\alpha < \tau < \beta$, then β is called a *cover* of α in L. If every element of L has at most two covers in L, we say L satisfies *Butler's condition*. In [**6**] it is proved that the finite lattice L satisfies Butler's condition if and only if every Butler group (i.e., a torsion-free homomorphic image or a pure subgroup of a completely decomposable group) with typeset contained in L is almost completely decomposable. The connection between Butler's theorem and our results is revealed by

Theorem 2.3. *Let L be a finite lattice of types and suppose $\tau_1, \dots, \tau_n$ is a list with each $\tau_i \in L$.*

(a) *If L satisfies Butler's condition, then $\tau_1, \dots, \tau_n$ is well-formed.*

(b) *If $T = \{\tau : \tau = \tau_i$ for some $i\}$ is a lattice, the following statements are equivalent*:

(i) *T satisfies Butler's condition.*

(ii) *$\tau_1, \dots, \tau_n$ is well-formed.*

(iii) *$\tau_1, \dots, \tau_n$ is co-well-formed.*

Proof. To prove (a) suppose $n \geq 3$ and set $\tau_i' = \tau_i \vee \inf\{\tau_j : j \neq i\}$ for each $i \in \bar{n}$. Suppose $\tau_1', \tau_2', \dots, \tau_n'$ are pairwise incomparable and let L' be the sublattice of L they generate. Let τ be a cover for $\tau_1' \wedge \tau_2' \wedge \cdots \wedge \tau_n'$ in L'. Set $S = \{\alpha \in L' : \alpha \geq \tau\}$ and $S^* = L' - S$. Since L' also satisfies Butler's condition, a routine lattice-theoretic argument shows that S^* is a linearly ordered sublattice of L'. Thus, there exists a unique i with $\tau_i' \in S^*$. Say $\tau_i' \in S^*$. Therefore, $\tau_1' \leq \tau_1' \vee \tau \leq \tau_1' \vee (\tau_2' \wedge \cdots \wedge \tau_n') = \tau_1'$. Consequently, $\tau_1' \notin S^*$, a contradiction. We conclude that $\tau_1', \dots, \tau_n'$ are not pairwise incomparable. Therefore, we may assume $\tau_1' \geq \tau_2'$.

For each i, select $\mathbf{Z} \leq A_i \leq \mathbf{Q}$ and set $A_i' = A_i + \cap\{A_j : j \neq i\}$. Then type $(A_i') = \tau_i'$. Since $\tau_1' \vee \tau_2' \leq \tau_1' \vee \tau_j'$ for $2 \leq j \leq n$, Proposition 1.2(a) implies $G' = G\langle A_1' \oplus \cdots \oplus A_n' \rangle$ has a rank-1 quasi-summand. Since $G' \cong G\langle A_1 \oplus \cdots \oplus A_n \rangle$, Proposition 1.1 completes the proof of (a).

To show that (ii) implies (i) in (b), suppose τ_1, τ_2, and τ_3 are all distinct covers of τ_4. Let $\mathbf{Z} \leq A_i \leq \mathbf{Q}$ have type τ_i. Then $G\langle A_1 \oplus A_2 \oplus A_3 \rangle$ is strongly indecomposable (see [**7**, Proposition 3]), a contradiction.

For (iii) implies (ii), let G be a torsion-free image of a direct sum of rank-1 groups with types belonging to T. Then G is a pure subgroup of an almost completely decomposable group with the types of the summands belonging to T [**1**, Proposition 4.2], so G is almost completely decomposable and (ii) holds. Finally, in case T is a lattice, it is well known that any pure subgroup of $\oplus A_i^k$ (with type $A_i = \tau_i$ and $k \geq 1$) is an epimorphic image of $\oplus A_i^m$ for some m. Thus (ii) implies (iii) is clear. $\square$

We conclude with two examples. They illustrate that the conditions well-formed and co-well-formed are independent if the list of types $\tau_1, \tau_2, \dots, \tau_n$ is not assumed to form a lattice.

Example 2.4. Let p, q and r be distinct prime numbers and set $A_1 = \mathbf{Z}$, $A_2 = \mathbf{Z}[1/p]$, $A_3 = \mathbf{Z}[1/q]$, $A_4 = \mathbf{Z}[1/r]$ and $\tau_i = \text{type}\,(A_i)$. It is easily verified that $\tau_1, \tau_2, \tau_3, \tau_4$ is co-well-formed but not well-formed. Therefore, if $A = A_1 \oplus \cdots \oplus A_4$, every corank-1 subgroup of A is almost completely decomposable. However, there exists a torsion-free quotient of rank 3 which is not almost completely decomposable. In fact, $A_1 \oplus \cdots \oplus A_4/ < \langle(0,1,1,1)\rangle_*$ has a strongly indecomposable quasi-summand of rank 2.

Example 2.5. Again let p, q and r be distinct primes and set $A_1 = \mathbf{Q}$, $A_2 = \mathbf{Z}_p$, $A_3 = \mathbf{Z}_q$ and $A_4 = \mathbf{Z}_r$, where the prime subscripts denote the respective localizations. If $\tau_i = \text{type}\ (A_i)$, note $\tau_1, \tau_2, \tau_3, \tau_4$ is well-formed but not co-well-formed. In this case, if $A = A_1 \oplus \cdots \oplus A_4$, $G\langle A\rangle$ is almost completely decomposable, yet $\operatorname{Ker}\sigma$, where $\sigma : A \to \mathbf{Q}$ is defined by $\sigma(a_1, \dots, a_n) = \sum a_i$, is strongly indecomposable.

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