ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 2, Spring 1992

ON A MEASURE OF SYMMETRY FOR STATIONARY RANDOM SEQUENCES

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ABSTRACT. Coefficients measuring "departure from exchangeability" are defined and shown to be equivalent to "conditional uniform strong mixing." The result shows that the conditional independence conclusion of the de Finetti theorem has a stability property under "small perturbations" of the exchangeability assumption.

0. Introduction. Part of the celebrated de Finetti theorem says that if $\{X_k\}_{k=1,2,\dots}$ is a random sequence such that $\{X_k\}_{k=1,2,\dots}$ and $\{X_1,\dots,X_n,X_{n+m},X_{n+m+1},\dots\}$ have the same distribution for every $n,m\geq 1$, then the random variables $\{X_k\}$ are conditionally independent (see [**1**]). In this note we define "coefficients of symmetry," which measure "departure from exchangeability" of a stationary random sequence. The coefficients, defined by (1) below are nonnegative numbers and equal to zero for exchangeable sequences only. We show that if the coefficients tend to zero, then the distant past and the future of the random sequence become asymptotically "conditionally independent" in the appropriate sense. Namely, we show that a conditional variant of the so-called ϕ-mixing condition and "asymptotic exchangeability" as defined by (1) below are equivalent. Theorem 1 below can be interpreted as stability of the de Finetti theorem.

While there are several other measures of weak dependence used in the literature (for definitions see, e.g., [**5**]), our proof seems to permit only two of them (ϕ-mixing and the so-called ψ-mixing) to be obtained as "measures of departure" from conditional independence in the form given by Theorem 1 below.

Our result might be of some interest to limit theorems, see, e.g., [**6, 8**] for limit theorems under exchangeability; it also might help to

Received by the editors on July 21, 1989, and in revised form on December 7, 1989.
1980 AMS *Mathematics subject classification.* Primary 60G99.
Keywords and phrases. Measures of dependence, measures of exchangeability, de Finetti's representation.

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approximate stationary sequences by exchangeable random variables, compare [**2, 3, 4**].

1. Notation. Let $\{X_n\}_{n\in\mathbf{Z}}$ be a stationary sequence on a probability space $(\Omega, \mathcal{M}, P)$. For $-\infty \le a < b \le +\infty$ denote by $\mathcal{F}_a^b$ the σ-field generated by $\{X_n : a \le n \le b\}$. By $S : \Omega \to \Omega$ we denote the forward shift corresponding to $\{X_n\}$. Transformation S is assumed to be measurable and measure preserving (one can take, e.g., $\Omega = \mathbf{R}^{\mathbf{Z}}$ with $S(\{x_k\}) = (\{x_{k+1}\})$. For any $m \ge 1$ define the "coefficient of symmetry"

$$(1) \quad \pi(m) = \sup\left\{\frac{|P(A\cap S^{-k}(B)) - P(A\cap B)|}{P(A)} : A \in \mathcal{F}_{-\infty}^0, P(A) > 0, B \in \mathcal{F}_m^\infty, k \in \mathbf{N}\right\}.$$

We shall say that a sequence $\{X_n\}$ is "asymptotically exchangeable" if $\lim_{m\to\infty} \pi(m) = 0$.

Notice that $S^{-k}(B) \in \mathcal{F}_{m+k}^\infty$ for every $k \ge 1$, and hence $\pi(m)$ is a nonincreasing function of m; in particular, $\lim_{m\to\infty} \pi(m)$ always exists. Also, if $\pi(1) = 0$, then $\pi(m) = 0$ for all $m \ge 1$; under stationarity, this is known to be equivalent to exchangeability, see, e.g., [**1**].

2. The main result. The following theorem shows that π is essentially a "conditional variant" of the so-called uniform strong mixing, or ϕ-mixing condition; for the definition and properties, see, e.g., [**5**]. The result shows that "asymptotically exchangeable" random sequences are "conditionally asymptotically independent"; hence, the conditional independence conclusion of de Finetti's theorem has a "stability property" under "small perturbations" of the exchangeability assumption.

Theorem 1. *For every $m \ge 1$ and each $A \in \mathcal{F}_{-\infty}^0$, $B \in \mathcal{F}_m^\infty$,*

$$(2) \qquad |P(A\cap B|\mathcal{I}) - P(A|\mathcal{I})P(B|\mathcal{I})| \le \pi(m)P(A|\mathcal{I}) \ a.s.,$$

where $\mathcal{I}$ is the invariant σ-field

$$\mathcal{I} = \{A \in \mathcal{M} : S^{-1}(A) = A\}.$$

Remark 1. A partial converse to Theorem 1 is also true: Suppose $\{\phi(n)\}$ is a sequence of (deterministic) numbers such that for every $m \geq 1$, $A \in \mathcal{F}_{-\infty}^{0}$ and $B \in \mathcal{F}_{m}^{\infty}$ we have

$$|P(A \cap B|\mathcal{I}) - P(A|\mathcal{I})P(B|\mathcal{I})| \leq \phi(m)P(A|\mathcal{I}) \text{ a.s.},$$

where $\mathcal{I}$ is the invariant σ-field. Then $\pi(m) \leq 2\phi(m)$, $m \geq 1$. Indeed, for each $k \geq 1$ we have

$$\begin{aligned} |P(A \cap S^{-k}(B)) - P(A \cap B)| &= |E\{P(A \cap S^{-k}(B)|\mathcal{I}) - P(A \cap B|\mathcal{I})\}| \\ &\leq |E\{P(A \cap S^{-k}(B)|\mathcal{I}) - P(A|\mathcal{I})P(S^{-k}(B)|\mathcal{I})\}| \\ &\quad + |E\{P(A \cap B|\mathcal{I}) - P(A|\mathcal{I})P(B|\mathcal{I})\} \leq 2\phi(m)P(A); \end{aligned}$$

we used invariance of $\mathcal{I}$ to assure $P(S^{-k}(B)|\mathcal{I}) = P(B|\mathcal{I})$ a.s.

3. Proof of Theorem 1. We prove a result of technical character first.

Lemma A. *Let $f, g : \Omega \to \mathbf{R}$ be nonnegative functions such that f is integrable $\mathcal{F}_{-\infty}^{0}$-measurable and g is bounded $\mathcal{F}_{m}^{\infty}$-measurable. Then*

$$|E\{fg \circ S^k\} - E\{fg\}| \leq \pi(m)E\{f\}\|g\|_\infty \tag{3}$$

for each $k \in \mathbf{N}$. (Here $\|g\|_\infty$ denotes the essential supremum of $|g|$.)

Proof. For each fixed $s, t \geq 0$, consider events $A = \{f \geq t\}$, $B = \{g \geq s\}$. Since for every $k \geq 1$, definition (1) implies

$$|P(A \cap S^{-k}(B)) - P(A \cap B)| \leq \pi(m)P(A);$$

therefore, we get

$$\begin{aligned} P(f \geq t, g \geq s) - \pi(m)P(f \geq t) &\leq P(f \geq t, g \circ S^k \geq s) \\ &\leq P(f \geq t, g \geq s) + \pi(m)P(f \geq t). \end{aligned}$$

Integrating this double inequality, we obtain

$$\int_0^\infty \int_0^{\|g\|_\infty} P(f \geq t, g \geq s)\, dt\, ds - \pi(m) \int_0^\infty \int_0^{\|g\|_\infty} P(f \geq t)\, dt\, ds$$
$$\leq \int_0^\infty \int_0^{\|g\|_\infty} P(f \geq t, g \circ S^k \geq s)\, dt\, ds$$
$$\leq \int_0^\infty \int_0^{\|g\|_\infty} P(f \geq t, g \geq s)\, dt\, ds$$
$$+ \pi(m) \int_0^\infty \int_0^{\|g\|_\infty} P(f \geq t)\, dt\, ds. \qquad \square$$

This ends the proof of the lemma, since by a well-known consequence of Fubini's theorem, we have

$$\int_0^\infty \int_0^{\|g\|_\infty} P(f \geq t, g \circ S^k \geq s)\, dt\, ds = E\{fg \circ S^k\},$$
$$\int_0^\infty \int_0^{\|g\|_\infty} P(f \geq t, g \geq s)\, dt\, ds = E\{fg\},$$
$$\int_0^\infty \int_0^{\|g\|_\infty} P(f \geq t)\, dt\, ds = \|g\|_\infty E\{f\}.$$

The following lemma is well known.

Lemma B. *Let $\mathcal{G} = \cap_{k \leq 0} \mathcal{F}^k_{-\infty}$ be the left tail σ-field. If $A \in \mathcal{I}$, the invariant σ-field of S, then for each $\varepsilon > 0$, there is $B_\varepsilon \in \mathcal{G}$ such that $P(I_A \neq I_{B_\varepsilon}) < \varepsilon$.*

Lemma C. *Let X, Y be integrable random variables and $\mathcal{F} \subset \mathcal{M}$ a σ-field. If for each $\mathcal{F}$-measurable bounded nonnegative Z we have $E\{XZ\} \leq \varepsilon E\{YZ\}$, then $E\{X|\mathcal{F}\} \leq \varepsilon E\{Y|\mathcal{F}\}$ a.s.*

This lemma is a consequence of the well-known implication which says that if $E\{(X - \varepsilon Y)Z\} \leq 0$ for every $\mathcal{F}$-measurable bounded nonnegative Z, then $E\{X - \varepsilon Y|\mathcal{F}\} \leq 0$ a.s.

Proof of Theorem 1. Let $m \geq 1$ be fixed. We shall show that if $f \geq 0$ is integrable $\mathcal{F}^0_{-\infty}$-measurable, g is $\mathcal{F}^\infty_m$-measurable and $0 \leq g \leq 1$, then

$$|E\{fg|\mathcal{I}\} - E\{f|\mathcal{I}\}E\{g|\mathcal{I}\}| \le \pi(m)E\{f|\mathcal{I}\}. \tag{4}$$

This will end the proof, since putting into (4) characteristic functions of events A, B for f, g, respectively, will give (2).

By an approximation argument, it is enough to prove (4) for bounded functions of a finite number of arguments only, i.e., $f = f(X_k : -M \le k \le 0)$ and $g = g(X_k : m \le k \le m + M)$.

Let $h = h(X_k : k < -N - M)$ be a bounded nonnegative function. For each $k \ge 1$ by Lemma A we have

$$|E\{fg \circ S^k h\} - E\{fgh\}| \le \pi(m)E\{fh\}. \tag{5}$$

Since $h \ge 0$ is an arbitrary $\mathcal{F}_{-\infty}^{-N-M}$-measurable function, therefore (5) implies (by Lemma C applied twice to $X = fg \circ S^k - fg, Y = f$ and $X = fg - fg \circ S^k, Y = f$)

$$\begin{aligned} E\{fg|\mathcal{F}_{-\infty}^{-N-M}\} - \pi(m)E\{f|\mathcal{F}_{-\infty}^{-N-M}\} &\le E\{fg \circ S^k|\mathcal{F}_{-\infty}^{-N-M}\} \\ &\le E\{fg|\mathcal{F}_{-\infty}^{-N-M}\} + \pi(m)E\{f|\mathcal{F}_{-\infty}^{-N-M}\}. \end{aligned}$$

Hence, passing to a limit as $N \to \infty$, we get

$$\begin{aligned} &E\{fg|\mathcal{G}\} - \pi(m)E\{f|\mathcal{G}\} \\ &\qquad \le E\{fg \circ S^k|\mathcal{G}\} \le E\{fg|\mathcal{G}\} + \pi(m)E\{f|\mathcal{G}\}, \end{aligned} \tag{6}$$

where $\mathcal{G} = \cap_{n \le 0}\mathcal{F}_{-\infty}^{n}$ is the left tail σ-field.

Summation over k in (6) gives

$$\begin{aligned} E\{fg|\mathcal{G}\} - \pi(m)E\{f|\mathcal{G}\} &\le E\left\{fn^{-1}\sum_{k=1}^{n} g \circ S^k \middle| \mathcal{G}\right\} \\ &\le E\{fg|\mathcal{G}\} + \pi(m)E\{f|\mathcal{G}\} \end{aligned} \tag{7}$$

for each $n = 1, 2, \dots$.

From the ergodic theorem, see, e.g., [**7**, p. 178, Corollary 3.5.2], we have $n^{-1}\sum_{k=1}^{n} g \circ S^k \to E\{g|\mathcal{I}\}$ in L_1 as $n \to \infty$. Since f is bounded, we can pass in (7) to the limit as $n \to \infty$ and we obtain

$$E\{fg|\mathcal{G}\} - \pi(m)E\{f|\mathcal{G}\} \le E\{fE\{g|\mathcal{I}\}|\mathcal{G}\} \le E\{fg|\mathcal{G}\} + \pi(m)E\{f|\mathcal{G}\}. \tag{8}$$

Since for each invariant event A in $\mathcal{I}$ there is B in $\mathcal{G}$ such that $A = B$ a.s., see Lemma B, therefore $E\{.E\{.|\mathcal{G}\}|\mathcal{I}\} = E\{.|\mathcal{I}\}$ a.s. Hence, conditional expectation $E\{.|\mathcal{I}\}$ applied to each side of (8) gives (4). This concludes the proof. $\square$

REFERENCES

1. D. Aldous, *On exchangeability and conditional independence*, in *Exchangeability in probability and statistics* (G. Koch and F. Spizzichino, eds.), North Holland, Amsterdam, 1982.

2. I. Berkes, H.P. Rosenthal, *Almost exchangeable sequences of random variables*, Zeitschr. Wahrscheinlichkeitsh. **70** (1985), 473–507.

3. I Berkes, E. Peter, *Exchangeable random variables and subsequence principle*, Probability Theory Rel. Fields **73** (1986), 395–413.

4. R.C. Bradley, *Approximation theorems for strongly mixing random variables*, Michigan Math. J. **30** (1983), 69–81.

5. ———, *Basic properties of strong mixing conditions*, in *Dependence in probability and statistics* (E. Eberlein, M. Taqqu, eds.), Birkhäuser, Boston, 1986.

6. A.R. Dabrowski, *Strassen-type invariance principles for exchangeable sequences*, preprint, 1986.

7. W.F. Stout, *Almost sure convergence*, Academic Press, New York, 1974.

8. R.L. Taylor, P.Z. Daffer and R.F. Patterson, *Limit theorems for sums of exchangeable random variables*, Rowman & Allanheld, Totowa, New Jersey, 1985.

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