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MELNIKOV'S METHOD, STOCHASTIC LAYERS AND NONINTEGRABILITY OF A PERTURBED DUFFING-OSCILLATOR

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ABSTRACT. Melnikov's method is used to show that the perturbed Duffing-oscillator with Hamiltonian

$$H_\varepsilon(q,p,x,y) = \frac{p^2-q^2}{2} + \frac{q^4}{4} + \frac{x^2+y^2}{2} + \varepsilon q(p-y)$$

has a subharmonic of order m provided $\varepsilon \neq 0$ is sufficiently small and $h_0 > h_m$ where $h_0 = H_0(q,p,x,y)$ is the total energy of the unperturbed Duffing-oscillator and h_m is the energy of the resonant periodic orbit of order m of Duffing's equation; furthermore, for $\varepsilon \neq 0$ sufficiently small and $h_0 > 0$, this system is nonintegrable and there is a primary stochastic layer of width

$$d = \frac{2\pi\varepsilon\sqrt{h_0}\,\mathrm{sech}\,(\pi/2)}{\sqrt{p_0^2 + q_0^2(1-q_0^2)^2}} + O(\varepsilon^2)$$

near a point $(q_0,p_0) \neq (0,0)$ on the homoclinic manifold of the unperturbed system as well as resonant stochastic layers whose bandwidths $d_m = O(\sqrt{\varepsilon/m})$.

1. Introduction. The occurrence of stochastic regions in two-degree-of-freedom Hamiltonian systems has been a topic of mathematical interest for many years, from both a theoretical and an applied point of view. In their 1964 paper on the nonintegrability of galactic motions [**6**], Henon and Heiles presented a numerical study of a Hamiltonian system with

$$H(q,p,x,y) = \frac{p^2+q^2}{2} - \frac{q^3}{3} + \frac{y^2+x^2}{2} + x^2 q.$$

Their work clearly showed the appearance and evolution of stochastic regions with increasing energy levels. It also raised the question as to whether such behavior could be described analytically.

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Melnikov's method has proved to be an excellent mathematical tool for studying the stochastic regions that occur in perturbed Hamiltonian systems of the form

$$H_\varepsilon(q,p,x,y) = H_0(q,p,x,y) + \varepsilon H_1(q,p,x,y)$$

where

$$H_0(q,p,x,y) = F(q,p) + G(x,y)$$

and/or where the H_0 system is integrable. For example, Holmes [**7**] recently showed that the perturbed Henon-Heiles type Hamiltonian system with

$$H_\varepsilon(q,p,x,y) = \frac{p^2+\omega^2 q^2}{2} - \frac{q^3}{3} + \frac{y^2+\omega^2 x^2}{2} - x^2 q + \varepsilon(\alpha q^2 - \beta q^3),$$

in which case the H_0 system is integrable, has a stochastic layer whose width is accurately predicted by the theoretical results derived using Melnikov's method. And, more recently, Veerman and Holmes [**12**] derived a formula for the bandwidths of the resonant stochastic layers and applied their results to a perturbed Hamiltonian system consisting of two weakly coupled pendula with

$$H_\varepsilon(q,p,x,y) = \frac{p^2}{2} + (1-\cos q) + \frac{y^2}{2} + (1-\cos\omega x) + \varepsilon\frac{(x-q)^2}{2}.$$

In order to obtain the resonant bandwidths for this example, it was necessary to obtain the Melkinov function as a Fourier series. This added complication in obtaining the bandwidth formula, as well as the error introduced by truncating the Fourier series (after only one term in [**12**]), is unnecessary in this paper since the Melnikov function can be found in closed form for the problem considered here.

In this paper, we apply the Melnikov theory to a perturbed Hamiltonian system consisting of a harmonic oscillator coupled to a Duffing oscillator. The Hamiltonian is given by

$$H_\varepsilon(q,p,x,y) = \frac{p^2-q^2}{2} + \frac{q^4}{4} + \frac{y^2+\omega^2 x^2}{2} + \varepsilon q(p-y).$$

The Duffing oscillator defined by this Hamiltonian has not been previously studied; so all of the results concerning the occurrence and bandwidths of stochastic layers are new. It provides a particularly nice application of the recent Melnikov theory in that the formulas for the primary and resonant bandwidths are found in closed form and since the theoretical results are in excellent agreement with the numerical results for this problem. The appearance and evolution of the primary and resonant stochastic layers with increasing energy, as well as the widths of the primary and resonant stochastic layers, are accurately predicted by the theory. The numerical study of the Duffing oscillator combined with the analytical results contained in this paper constitute a fundamental contribution to our understanding of the Duffing oscillator. This paper therefore provides an easily understood introduction to the Melnikov theory and its applications for the uninitiated reader as well as a particularly nice application of the theory and some interesting new results on the Duffing oscillator for the expert in the field (cf. Theorems 2.1 and 2.2).

The Melnikov theory for perturbed Hamiltonian systems with two degrees of freedom,

$$\begin{aligned} \dot{q} &= \frac{\partial H_\varepsilon}{\partial p} & \dot{x} &= \frac{\partial H_\varepsilon}{\partial y} \\ \dot{p} &= -\frac{\partial H_\varepsilon}{\partial q} & \dot{y} &= -\frac{\partial H_\varepsilon}{\partial x} \end{aligned} \tag{1.1}$$

is particularly well developed and is clearly presented in [**5**]. The specific results used in this paper are established in [**4, 7, 8**]. We cite these results for easy reference in this introduction and then apply them to the perturbed Duffing-oscillator, with Hamiltonian $H_\varepsilon(q,p,x,y)$ given above, in Section 2. Numerical results for this example are given in Section 3 of this paper.

First of all, the Hamiltonian $H_\varepsilon(q,p,x,y)$ is constant on solution curves of (1.1) and thus the solution curves lie on three-dimensional manifolds

$$H_\varepsilon(q,p,x,y) = h \tag{1.2}$$

where the total energy h is a constant.

Suppose that x and y can be expressed in terms of action-angle variables θ and I; i.e., suppose that there is an invertible, canonical change

FIGURE 1.

of variables $x = x(\theta, I)$, $y = y(\theta, I)$ under which the Hamiltonian $H_\varepsilon(q, p, x, y)$ takes the form

$$H_\varepsilon(q, p, \theta, I) = F(q, p) + G(I) + \varepsilon H_1(q, p, \theta, I).$$

Assume also that for $\varepsilon = 0$ the unperturbed Hamiltonian system,

$$\dot{q} = \frac{\partial F}{\partial p} \qquad \dot{\theta} = \frac{\partial G}{\partial I}$$
$$\dot{p} = -\frac{\partial F}{\partial q} \qquad \dot{I} = 0,$$

is completely integrable and that the following hypotheses are satisfied:

H1. $G'(I) > 0$ for $I > 0$

H2. The (q, p) phase plane contains a hyperbolic saddle with a homoclinic orbit Γ_0 filled with periodic orbits, cf. Figure 1.

The following theorem can then be proved as in [**4**].

Theorem 1.1. *Let hypotheses* H1 *and* H2 *hold and let γ_0 be the energy of the homoclinic solution, Γ_0, of the F-system; i.e., $\gamma_0 = F(q_0, p_0)$ for $(q_0, p_0) \in \Gamma_0$. For $h > \gamma_0$, let $l_0 = G^{-1}(h - \gamma_0)$, $\omega_0 = G'(l_0)$, and let $\{F, H_1\}(t - t_0)$ denote the Poisson bracket of F and H_1 evaluated along an orbit $(q_0(t - t_0), p_0(t - t_0), \omega_0 t, l_0)$ in the homoclinic manifold of the unperturbed system* (1.1) *with $\varepsilon = 0$. Then, for $0 < |\varepsilon| \ll 1$, if the Melnikov function*

$$M(t_0) = \int_{-\infty}^{\infty} \{F, H_1\}(t - to)\, dt \tag{1.3}$$

FIGURE 2. A homoclinic tangle.

has simple zeros, the Hamiltonian system (1.1) *has transverse homoclinic orbits on the energy surface* $H_\varepsilon = h$ *and* (1.1) *possesses no analytic second integral.*

Compare with [**5**, Theorem 4.8.4; **4**, Theorem 10.1; **7**, Theorem 1; **8**, Theorem 3.2]. Also see [**10, 2, 1**] for related background material. Recall that the Poisson bracket of F and H_1

$$\{F, H_1\} = \frac{\partial F}{\partial q}\frac{\partial H_1}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H_1}{\partial q}.$$

As in [**5**], the system (1.1) is said to have transverse homoclinic orbits when the stable and unstable manifolds of the hyperbolic periodic orbit of (1.1) intersect transversely; i.e., we have a transverse homoclinic point of the Poincaré map, $(q_0, p_0) \in W^s(0,0) \cap W^u(0,0)$; cf. [**4**, 177]. The existence of a transverse homoclinic point implies the existence of infinitely many such points and that the unstable manifold of the hyperbolic saddle at the origin accumulates on itself; i.e., we have a "homoclinic tangle" as shown in Figure 2 [**4, 5**]. The complicated behavior of trajectories of (1.1) in a neighborhood of the homoclinic manifold of the unperturbed system determines a stochastic region referred to as the "primary stochastic layer"; cf. [**5**, 222-225; **7**, 338, 344].

The Melnikov theory of Greenspan and Holmes [**4**] also shows that the maximum splitting of the stable and unstable manifolds, and hence the maximum width of the primary stochastic layer, near a point

$(q_0, p_0) \in \Gamma_0$ is given by

$$d = \frac{\varepsilon \sup M(t_0)}{\sqrt{F_q^2(q_0,p_0) + F_p^2(q_0,p_0)}} + O(\varepsilon^2). \tag{1.4}$$

Compare with equation (10.28) in [**4**] or equation (2.11) in [**7**].

We now discuss the behavior under small perturbations of the orbits of the unperturbed system that lie on concentric tori inside the homoclinic manifold generated by Γ_0; cf. Figure 1. First of all, the Kolmogorov-Arnold-Moser Theorem, [**5**, 219], guarantees that a measurable set of smooth, sufficiently irrational tori of positive Lebesgue measure is preserved for $0 < |\varepsilon| \ll 1$. On the other hand, the rational tori, defined by the resonance condition

$$T_F = \frac{m}{n} T_G \tag{1.5}$$

(where T_F and T_G denote the periods of the periodic orbits of the F and G-systems, respectively), generate subharmonic motions and stochastic layers of (1.1) for $0 < |\varepsilon| \ll 1$. We next outline the perturbation theory for these resonant stochastic layers.

Let $(q_\alpha(t), p_\alpha(t))$ denote the one-parameter family of periodic solutions of the F-system contained inside the homoclinic orbit Γ_0; cf. Figure 1. Let T_α denote the periods of these periodic orbits and $h_\alpha = F(q_\alpha(t), p_\alpha(t))$ their energies. Assume that T_α is a differentiable function of h_α and that hypothesis

H3: $$\frac{dT_\alpha}{dh_\alpha} > 0$$

is satisfied. Suppose that $T_G = T_0$, a constant. Then, in order to have subharmonic motions of period mT_0, it is necessary that the resonance condition (1.5), which has the form

$$T_\alpha = \frac{m}{n} T_0, \tag{1.6}$$

be satisfied. The following theorem, establishing the existence of subharmonic motions of (1.1), is proved in [**4**].

Theorem 1.2. *Let hypotheses* H1–H3 *hold and for given positive integers m and n let $h_\alpha = F(q_\alpha(t), p_\alpha(t))$ be the energy of the periodic orbit $(q_\alpha(t), p_\alpha(t))$ with $\alpha = \alpha(m,n)$ satisfying the resonance condition* (1.6). *For $h > h_\alpha$, let $l_\alpha = G^{-1}(h - h_\alpha)$, $\omega_\alpha = G'(l_\alpha)$, and let $\{F, H_1\}\ (t - t_0)$ be the Poisson bracket of F and H_1 evaluated along the periodic orbit $(q_\alpha(t-t_0), p_\alpha(t-t_0), \omega_\alpha t, l_\alpha)$ lying on the rational tori of the unperturbed system* (1.1) *with $\varepsilon = 0$ defined by* (1.6). *Then for $0 < |\varepsilon| \ll 1$, if the subharmonic Melnikov function*

$$M^{m/n}(t_0) = \frac{T_0}{2\pi} \int_0^{mTo} \{F, H_1\}(t - to)\, dt \tag{1.7}$$

has j simple zeros $t_0 \in [0, T_0)$, the rational torus breaks into precisely j periodic orbits of period mT_0 on the energy surface $H_\varepsilon = h$; furthermore, j is an even integer, $j = 2k$, and k of these periodic orbits are elliptic and k are hyperbolic.

Compare with [**5**, Theorem 4.8.3; **4**, Theorem 10.2; and **7**, p. 344]. As Holmes [**7**] states, for $\varepsilon \neq 0$ sufficiently small, the stable and unstable manifolds of the hyperbolic periodic orbits generated by the rational tori, as in Theorem 1.2, are somehow packed between the invariant irrational tori which are preserved according to the KAM theory. Zehnder [**13**] established that, in the generic case, the stable and unstable manifolds of the hyperbolic periodic orbits intersect transversely. Thus, in the typical case, there are transverse homoclinic orbits, homoclinic tangles and nonintegrability occurring in a neighborhood of each rational torus, cf. Figure 3. (Also compare Figures 3 and 10 in this paper.) These regions around the rational tori are bounded by invariant irrational tori. They are referred to as resonant stochastic layers.

Note that the cross-section shown in Figure 3 contains two periodic orbits of period $3T_0$, one hyperbolic and one elliptic. These periodic orbits correspond to six fixed points of P^3 where P is the Poincaré map on this cross-section induced by solutions of (1.1), cf. [**5**, 214].

Asymptotic formulas for the widths of the resonant stochastic layers, i.e., for the resonant bandwidths, have recently been derived by Veerman and Holmes [**11, 12**]. Some resonant bandwidths for linearly coupled pendula are computed in [**12**].

According to [**12**, equation (2.13)], the *resonance bandwidth* on the

FIGURE 3. A resonant stochastic layer containing two subharmonic periodic orbits of order $m = 3$.

energy surface $H_\varepsilon = h$ is given in terms of the Melnikov function by

$$\Delta I_\alpha^{m,n} = 2\sqrt{\varepsilon}\left[\frac{2}{\bar{\omega}^{m,n}}(V_{\max}^{m,n} - V_{\min}^{m,n})\right]^{1/2} + O(\varepsilon) \tag{1.8}$$

where

$$V^{m,n}(\theta) = \frac{1}{2n\pi}\int M^{m/n}\left(\frac{T_0}{2\pi}\theta\right) d\theta \tag{1.9}$$

with $M^{m/n}(t_0)$ given by (1.7),

$$\bar{\omega}^{m,n} = -\frac{\partial h}{\partial I_\alpha}\frac{\partial}{\partial h}\left[\frac{T_0}{T_\alpha}\right] \quad \text{and} \quad \frac{\partial h}{\partial I_\alpha} = \frac{2\pi}{T_\alpha}. \tag{1.10}$$

(Note that there is a minor error [**12**, equation (2.15)] since $\omega_2/\omega_1 = m/n$ and not n/m in that equation.) Also, as in [**12**, equation (4.1)], in terms of the q-coordinate, the resonance bandwidth of order (m, n) is given by

$$\Delta q_{m,n} = \left.\frac{\partial q}{\partial h}\right|_{p=0}\frac{\partial h}{\partial I_\alpha}\Delta I_\alpha^{m,n}. \tag{1.11}$$

These formulas will be used in the next section to derive a formula for the resonance bandwidths of the Duffing-oscillator.

We end this summary with a result due to Chow, Hale and Mallet-Paret [**3**] which shows that the homoclinic bifurcation (which results in the primary stochastic layer) is the limit of a sequence of subharmonic saddle-node bifurcations (which result in resonant stochastic layers).

Theorem 1.3. *Let* $M^{m/1}(t_0) = M^m(t_0)$, *then*

$$\lim_{m\to\infty} M^m(t_0) = M(t_0).$$

Compare with Theorem 10.3 in [**4**].

2. A perturbed Duffing-oscillator. We now apply the results outlined in the introduction to an analysis of the stochastic layers of the perturbed Duffing-oscillator with Hamiltonian

$$H_\varepsilon(q,p,x,y) = \frac{p^2-q^2}{2} + \frac{q^4}{4} + \frac{y^2+\omega^2x^2}{2} + \varepsilon q(p-y). \tag{2.1}$$

Express x and y in terms of action-angle variables as

$$x = \sqrt{\frac{2I}{\omega}}\sin\theta \qquad y = \sqrt{2I\omega}\cos\theta.$$

This leads to the Hamiltonian function

$$\begin{aligned} H_\varepsilon(q,p,\theta,I) &= \frac{p^2-q^2}{2} + \frac{q^4}{4} + \omega I + \varepsilon q(p - \sqrt{2I\omega}\cos\theta) \\ &= F(q,p) + G(I) + \varepsilon H_1(q,p,\theta,I). \end{aligned} \tag{2.2}$$

The G-system is the familiar harmonic oscillator with periodic solutions of period $T_0 = 2\pi/\omega$. The (x,y) phase plane is filled with concentric ellipses,

$$\omega^2x^2 + y^2 = c^2,$$

centered at the critical point at the origin.

FIGURE 4. The phase plane for Duffing's equation.

The F-system is equivalent to Duffing's equation. The (q, p) phase plane consists of periodic solutions, three critical points, $(0, 0)$, $(\pm 1, 0)$, and two homoclinic orbits $\Gamma_0^{\pm}$. The integral curves of the F-system are given by

$$\frac{p^2 - q^2}{2} + \frac{q^4}{4} = h_\alpha$$

where the constants h_α are the energy levels of the solutions of Duffing's equation. The critical point at the origin and the two homoclinic orbits $\Gamma_0^{\pm}$ correspond to $h_\alpha = \gamma_0 = 0$; the critical points $(\pm 1, 0)$ correspond to $h_\alpha = -1/4$; and the periodic orbits inside $\Gamma_0^{\pm}$ correspond to $-1/4 < h_\alpha < 0$. Compare with Figure 4.

The periodic solutions inside the homoclinic orbit Γ_0^{+} are given by

$$\begin{aligned} q_\alpha(t) &= \sqrt{\frac{2}{2-\alpha^2}}\, dn\left(\frac{t}{\sqrt{2-\alpha^2}}, \alpha\right) \\ p_\alpha(t) &= -\frac{\sqrt{2}\alpha^2}{2-\alpha^2}\, sn\left(\frac{t}{\sqrt{2-\alpha^2}}, \alpha\right)\, cn\left(\frac{t}{\sqrt{2-\alpha^2}}, \alpha\right) \end{aligned} \tag{2.3}$$

for $0 < \alpha < 1$ where sn, cn, and dn are the Jacobi elliptic functions, cf. [**5**, equation (4.6.9), **4**, equation (10.5.4)]. (Note that there is a minor error in [**5**, equation (4.6.9)]. Equation (10.54) in [**4**] is correct.) The periodic orbits inside the homoclinic orbit Γ_0^- are given by reversing the signs in the above equations for $q_\alpha(t)$ and $p_\alpha(t)$. The periods of these periodic solutions for $0 < \alpha < 1$ are given by

$$T_\alpha = 2K(\alpha)\sqrt{2-\alpha^2} \tag{2.4}$$

where

$$K(\alpha) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\alpha^2 t^2)}} \tag{2.5}$$

is the complete elliptic integral of the first kind. Since

$$h_\alpha = \frac{p_\alpha^2(t) - q_\alpha^2(t)}{2} + \frac{q_\alpha^4(t)}{4}$$

is constant and since $q_\alpha(0) = \sqrt{2/(2-\alpha^2)}$ and $p_\alpha(0) = 0$, it follows that the energies of the periodic solutions of Duffing's equation inside $\Gamma_0^\pm$

$$h_\alpha = \frac{\alpha^2 - 1}{(2-\alpha^2)^2} \tag{2.6}$$

with $-1/4 < h_\alpha < 0$ for $0 < \alpha < 1$. Also, since $sn(\tau, \alpha) \to \tanh \tau$, $cn(\tau, \alpha) \to \operatorname{sech} \tau$ and $dn(\tau, \alpha) \to \operatorname{sech} \tau$ as $\alpha \to 1$, it follows that

$$q_0(t) = \lim_{\alpha \to 1} q_\alpha(t) = \sqrt{2} \operatorname{sech} t \tag{2.7}$$

and

$$p_0(t) = \lim_{\alpha \to 1} p_\alpha(t) = -\sqrt{2} \operatorname{sech} t \tanh t.$$

These are the usual parametric equations for the homoclinic orbit Γ_0^+, cf. [**5**, equation (4.5.20)].

From equation (2.4) for the periods T_α and the fact that $T_0 = 2\pi/\omega$, the resonance condition, equation (1.5), becomes

$$K(\alpha)\sqrt{2-\alpha^2} = \frac{\pi m}{\omega n}. \tag{2.8}$$

FIGURE 5.

The function $K(\alpha)\sqrt{2-\alpha^2}$ is shown in Figure 5.

For $\omega = 1$ and $n = 1$, there are two sequences of solutions of (2.8), α_m and α_m^+, $m = 1, 2, 3, \ldots$, which approach 1 from below and from above, respectively, as $m \to \infty$, cf. Figure 5. Numerical solution of (2.8) with $\omega = 1$ and $n = 1$ yields $\alpha_1^2 \cong .9651$ and $\alpha_2^2 \cong .999977$. From (2.6), the corresponding energies are $h_1 \cong -.0326$ and $h_2 \cong -.000023$.

These solutions α_m (and α_m^+) of the resonance condition (2.8) indicate that there is a sequence of resonant stochastic layers which accumulate on the primary stochastic layer in a neighborhood of the homoclinic manifold. In fact, for $\varepsilon \neq 0$ sufficiently small, we can use Theorems 1.1

and 1.2 to establish that there are resonant stochastic layers arbitrarily close to the primary stochastic layer. To do this, we first compute the subharmonic Melnikov function (1.7) with $F(q,p)$ and $H_1(q,p,\theta,I)$ given by (2.2) and $q_\alpha(t), p_\alpha(t)$ given by (2.3) with $\alpha = \alpha(m,n)$ a solution of (2.8):

$$\begin{aligned}\omega M^{m/n}(to) &= \int_0^{2\pi m/\omega} \{F, H_1\}(t-to)\,dt \\ &= -\int_0^{2\pi m/\omega} [p_\alpha^2(t) + q_\alpha^2(t) - q_\alpha^4(t) \\ &\quad - p_\alpha(t)\sqrt{2I\omega}\cos\omega(t+to)]\,dt \\ &= -\int_0^{2\pi m/\omega} [p_\alpha^2(t) + q_\alpha^2(t) - q_\alpha^4(t) \\ &\qquad\qquad + p_\alpha(t)\sqrt{2(h_0-h_\alpha)}\sin\omega t\sin\omega t_0]\,dt\end{aligned}$$

where $h_0 = H_0$ is the total energy of the unperturbed Duffing-oscillator with Hamiltonian (2.1) and $\varepsilon = 0$. The last equation follows from the fact that $p_\alpha(t)$ is an odd function and from the fact that $\omega I = h_0 - h_\alpha$ according to equation (2.2). It then follows by using $q_\alpha^4(t) = 4h_\alpha - 2p_\alpha^2(t) + 2q_\alpha^2(t)$ that

$$\begin{aligned}\omega M^{m/n}(t_0) &= -3\int_0^{2\pi m/\omega} p_\alpha^2(t)\,dt + \int_0^{2\pi m/\omega} q_\alpha^2(t)\,dt + 4h_\alpha\int_0^{2\pi m/\omega} dt \\ &\quad - \sqrt{2(h_0-h_\alpha)}\sin\omega t_0\int_0^{2\pi m/\omega} p_\alpha(t)\sin\omega t\,dt \\ &= -2[(2-\alpha^2)2E(\alpha) - 4(1-\alpha^2)K(\alpha)]/(2-\alpha^2)^{3/2} \\ &\quad + \left(\frac{2}{2-\alpha^2}\right)\int_0^{2\pi m/\omega} dn^2\left(\frac{t}{2-\alpha^2},\alpha\right)dt \\ &\quad + 4h_\alpha nT_\alpha - \sqrt{2(h_0-h_\alpha)}\sin\omega t_0 J_2(\alpha,\omega)\end{aligned}$$

where $E(\alpha)$ is the complete elliptic integral of the second kind and

$$J_2(\alpha,\omega) = \begin{cases} 0 & \text{for } n \neq 1 \\ \sqrt{2}\pi\omega\,\text{sech}\,\frac{\pi m K_1(\alpha)}{K(\alpha)} & \text{for } n = 1 \end{cases}$$

with

$$K_1(\alpha) = \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-(1-\alpha^2)t^2}}.$$

Compare with [**5**, equation (4.6.17)]. Since, by [**9**, equations on p. 97, 98],

$$\int_0^{2K(\alpha)} dn^2(u,\alpha)\,du = 2E(\alpha),$$

and since, from equation (2.4), $T_\alpha = 2K(\alpha)\sqrt{2-\alpha^2}$, it follows that for $n=1$

$$M^m(t_0) = -2\pi\sqrt{h_0 - h_\alpha}\sin\omega t_0 \operatorname{sech}\frac{\pi m K_1(\alpha)}{K(\alpha)} \tag{2.9}$$

wherein $\alpha = \alpha_m$, the solution of (2.8) with $n = 1$, and $h_\alpha = h_m$, the energy of the resonant periodic solution of order m of Duffing's equation, given by (2.6) with $\alpha = \alpha_m$. It then follows from (2.9) that for $n = 1$ and $h_0 > h_m$, the subharmonic Melnikov function $M^m(t_0)$ has two simple zeros $t_0 \in [0, 2\pi/\omega)$. It also follows from the above equations that for $n \neq 1$, $M^{m/n}(t_0) \neq 0$.

From equation (2.4) for T_α and equation (2.6) for h_α, it follows that for $0 < \alpha < 1$,

$$\frac{dT_\alpha}{dh_\alpha} = [(2-\alpha^2)K'(\alpha)/\alpha - K(\alpha)](2-\alpha^2)^{5/2}.$$

It can then be shown, using the definition of $K(\alpha)$ in equation (2.5), that this quantity is positive for $0 < \alpha < 1$. Theorem 1.2 then implies

Theorem 2.1. *For a given positive integer m and $0 < \omega < \sqrt{2}m$, let α_m be the solution of the resonance condition* (2.8) *with $n = 1$, and let h_m be the energy of the resonant periodic solution of order m of Duffing's equation, i.e., let h_m be given by* (2.6) *with $\alpha = \alpha_m$. Let h_0 be the total energy of the unperturbed Duffing oscillator* (1.1) *with Hamiltonian* (2.1) *and $\varepsilon = 0$, i.e., let $h_0 = H_0$. Then for a given $h_0 > h_m$, there exists an $\varepsilon_0 > 0$ such that for $0 < |\varepsilon| < \varepsilon_0$, the perturbed Duffing-oscillator* (1.1) *with Hamiltonian* (2.1) *has two subharmonic periodic orbits, one elliptic and one hyperbolic, of period $2\pi m/\omega$.*

It also follows from Melnikov's method and the above analysis that there are no subharmonic orbits of the above Duffing-oscillator if $n \neq 1$ or if $\sqrt{2}m \leq \omega$, cf. the remark at the bottom of p. 196 in [**5**].

Although it has not been proved for the perturbed Duffing-oscillator being studied here, we expect that, as in the generic case considered by Zehnder [**13**], the subharmonic orbits established in Theorem 2.1 generate resonant stochastic layers similar to the one depicted in Figure 3 for $m = 3$. The numerical results presented in Section 3 indicate that this is indeed the case, cf. Figures 7 and 8 in Section 3 of this paper.

The bandwidths of these resonant stochastic layers can be computed using the formulas in [**12**], i.e., equations (1.8)–(1.11) in this paper. For $n = 1$, it follows from equations (1.9) and (2.9) with $h_\alpha = h_m$ that

$$V^m(\theta) = \sqrt{h_0 - h_m}\cos\theta \operatorname{sech}\left[\frac{\pi m K_1(\alpha_m)}{K(\alpha_m)}\right].$$

Thus, from (1.8)

$$\Delta I_\alpha^m = 4\sqrt{\varepsilon}\left[\frac{\sqrt{h_0 - h_m}\operatorname{sech}[\pi m K_1(\alpha_m)/K(\alpha_m)]}{\bar{\omega}^m}\right]^{1/2} + O(\varepsilon)$$

where by (1.10)

$$\bar{\omega}_m = -\frac{2\pi}{T_\alpha}\frac{d}{d\alpha}\left[\frac{1}{T_\alpha}\right]\Big/\frac{dh}{d\alpha}.$$

It then follows from equation (2.4) for T_α and equation (2.6) for h_α that

$$\bar{\omega}_m = \frac{(2-\alpha^2)^{3/2}}{4m\alpha^3}[K'(\alpha)(2-\alpha^2) - \alpha K(\alpha)]$$

with $\alpha = \alpha_m$. Substituting the above formulas, together with

$$\left.\frac{\partial q}{\partial h}\right|_{p=0} = \frac{1}{\sqrt{1+4h_m}\sqrt{1-\sqrt{1+4h_m}}},$$

which follows from the energy integral for Duffing's equation, we find the following formula for *the resonance bandwidth of order m of the Duffing-oscillator*:

(2.10)

$$\Delta q_m = \frac{8\omega\sqrt{\varepsilon}}{\sqrt{m}} \cdot \left[\frac{\alpha_m^3\sqrt{h_0 - h_m}\operatorname{sech}[\pi m K_1(\alpha_m)/K(\alpha_m)]}{(1+4h_m)(1-\sqrt{1+4h_m})(2-\alpha_m^2)^{3/2}[K'(\alpha_m)(2-\alpha_m^2)-\alpha_m K(\alpha_m)]}\right]^{1/2} + O(\varepsilon).$$

We next consider the primary stochastic layer generated by the homoclinic orbits $\Gamma_0^{\pm}$ of Duffing's equation. If we compute the Melnikov function $M(t_0)$ either using (2.7) or using Theorem 1.3 and (2.9), we find that

$$M(t_0) = -2\pi\sqrt{h_0}\sin\omega t_0 \operatorname{sech}(\pi\omega/2).$$

Hence, from Theorem 1.1 and equation (1.4), we have

Theorem 2.2. *Let h_0 be the total energy of the unperturbed Duffing-oscillator with Hamiltonian* (2.1) *and $\varepsilon = 0$, i.e., let $h_0 = H_0$. Then, for a given $h_0 > 0$, there exists an $\varepsilon_0 > 0$ such that for $0 < |\varepsilon| < \varepsilon_0$, the perturbed Duffing-oscillator* (1.1) *with Hamiltonian* (2.1) *has a transverse homoclinic orbit and possesses no analytic second integral. The maximum width of the primary stochastic layer near a point $(q_0, p_0) \in \Gamma_0^{\pm}$ is given by*

$$d = \frac{2\pi\varepsilon\sqrt{h_0}\operatorname{sech}(\pi\omega/2)}{\sqrt{p_0^2 + q_0^2(1-q_0^2)^2}} + O(\varepsilon^2). \tag{2.11}$$

The result established in Theorem 2.2 is similar to the nonintegrability result established in [**7**] for a Henon-Heiles type of Hamiltonian system.

We note that if $h_0 < h_1 \cong -.0326$, the energy of the first subharmonic of Duffing's equation given by (2.6), and if $\varepsilon \neq 0$ is sufficiently small, then to first order the perturbed Duffing-oscillator with Hamiltonian (2.1) is completely integrable. As the total energy h_0 increases beyond h_1, subharmonics of arbitrarily high order appear provided $\varepsilon \neq 0$ is sufficiently small. However, for a given $h_0 > 0$ and $0 < |\varepsilon| \ll 1$ only a finite number of the resonant stochastic layers of order m appear since the higher order resonant stochastic layers are swallowed up by the primary stochastic layer. These phenomena are illustrated by the numerical results in the next section.

3. Numerical results. The numerical results presented in this section illustrate the theoretical results for the Duffing-oscillator obtained in the previous section. All of the numerical results were obtained using an HP 9845A desktop computer. A fourth-order Runge-Kutta numerical integration scheme was used to integrate the equations of motion

(1.1) for the perturbed Duffing-oscillator with Hamiltonian (2.1). Trajectories of (1.1), $(q(t), p(t), x(t), y(t))$, are analyzed by studying the Poincaré map P obtained from successive intersections of the trajectory with the plane $x = 0$ in the upward direction, i.e., points $(q(t), p(t))$ are plotted whenever $x(t) = 0$ and $y(t) > 0$. This is the same as the approach used by Henon and Heiles [**6**] and it yields a Poincaré map which is equivalent to the Poincaré map for Hamiltonian systems with two degrees of freedom described in [**5**], cf. the comment at the bottom of p. 214 in [**5**]. For given values of ε and h, the initial conditions (q_0, p_0, x_0, y_0) were determined by choosing a point (q_0, p_0) in the (q, p) plane (with $h - F(q_0, p_0) > 0$), setting $x_0 = 0$ and taking y_0 as the positive solution of the equation $H_\varepsilon(q_0, p_0, 0, y_0) = h$ (which is quadratic in y_0) with H_ε given by (2.1). It was observed that during the numerical integration, a step size of order .1 maintained the energy constant to four figure accuracy.

As noted in [**5, 6**], fixed points of P correspond to periodic solutions of (1.1), fixed points of P^m correspond to subharmonic orbits of order m and invariant closed curves of P correspond to invariant tori for (1.1). The elliptic periodic points of P are surrounded by islands of invariant closed curves of P. And regions densely filled with scattered points correspond to stochastic regions generated by an "ergodic trajectory," cf. [**6**]. The existence of stochastic regions indicates that (1.1) is nonintegrable. Throughout this section, the integer $n = 1$ and h denotes the total energy of the perturbed Duffing-oscillator, $h = H_\varepsilon = h_0 + O(\varepsilon)$.

The first major event that occurs for the perturbed Duffing-oscillator as h_0 increases from its minimum value of $-1/4$ (corresponding to the two critical points $(\pm 1, 0)$ of Duffing's equation) is the appearance of subharmonics. According to Theorem 2.1, the perturbed Duffing-oscillator (1.1) with Hamiltonian (2.1) and $\omega = 1$ has a first order subharmonic for h_0 greater than $h_1 \cong -.0326$, i.e,. for $h > h_1 + O(\varepsilon)$, and $\varepsilon \neq 0$ sufficiently small. Numerical experimentation confirms that, for $\varepsilon = .001$, a first order subharmonic appears at $h = -.0326$, cf. Figure 6. Furthermore, according to equation (2.3) $q \in [.26, 1.39]$ when $\alpha = \alpha_1$. Figure 6 shows a small island surrounding the first order subharmonic at $q \cong .26$ on the q-axis. The island surrounding the first order subharmonic grows with increasing ε and h. Figure 7 shows this growth at values of $\varepsilon = .02$ and $h = -.001$. Figure 7 also shows an

FIGURE 6. The appearance of a first order subharmonic for $\omega = 1$ and $\varepsilon = .001$ at $h = -.0326$.

asymmetry exhibited by the first order subharmonic, i.e., the elliptic fixed point of P corresponding to the first order subharmonic is closer to the origin than the hyperbolic fixed point of P for $q > 0$ while the reverse is true for $q < 0$. Figure 8 shows a stochastic region generated by a single ergodic trajectory for the case $\varepsilon = .02$, $h = -.001$. The existence of this region filled with scattered points inside the resonant stochastic layer of order one indicates that (1.1) is nonintegrable for $h > h_1$ and $0 < |\varepsilon| \ll 1$ as one would expect from the results for the generic case studied by Zehnder [**13**].

Even though higher order subharmonics exist for $h_0 > h_m$ and $\varepsilon \neq 0$ sufficiently small, they are extremely difficult to compute for $\omega = 1$ since they are tightly packed in a small neighborhood of the homoclinic orbits $\Gamma_0^{\pm}$, cf. Figure 5. For example, for $\alpha = \alpha_2$, $q \in [.0068, 1.4142]$ according to equation (2.3). However, for $\omega = 2$ and $m = 2$, the resonance condition (2.8) yields $\alpha \cong .9651$ and consequently $h_2 \cong -.0326$, and we observe second order subharmonics in the same

FIGURE 7. First order subharmonics for $\omega = 1$, $\varepsilon = .02$ and $h = -.001$.

region as the first order subharmonics that appeared in Figures 6 and 7. For $\omega = 2$, Figure 9 shows the second order subharmonic islands surrounding the two elliptic fixed points of P^2 which correspond to a single elliptic periodic orbit of (1.1) of period $2T_0$. There are two hyperbolic fixed points of P^2 in this case located at $q \cong .26$ and $q \cong 1.39$ on the q-axis. There is also an asymmetry exhibited in this case: the two elliptic fixed points of P^2 are located on the negative q-axis, and the two hyperbolic fixed points of P^2 are located off of the q-axis for $q < 0$. This is just the opposite of what happens for $q > 0$ as in Figure 9.

Figure 10 shows three third order subharmonic islands computed for $\omega = 3$, $\varepsilon = .01$ and $h = 0$. These three islands surround three elliptic fixed points of P^3 which correspond to a single elliptic periodic orbit of (1.1) of period $3T_0$. There are also three hyperbolic fixed points of P^3 which correspond to a single hyperbolic periodic orbit of (1.1) of period $3T_0$, cf. Figure 3.

Subharmonics of higher order can also be computed in this way. Note

FIGURE 8. A stochastic region generated by a single trajectory of (1.1) for $\omega = 1$, $\varepsilon = .02$ and $h = -.001$.

that for $\omega = 2$ the first order subharmonic is lost, as in Figure 9, since (2.8) has no solution for $m = 1$ and $\omega = 2$, cf. Figure 5. However, by a judicious choice of ω, both the first and second order subharmonics can be seen on the same plot. For example, with $\omega^2 = 1.97$, $\varepsilon = .001$ and $h = 0$, subharmonic islands corresponding to both first and second order subharmonics appear as in Figure 11.

The second major event that occurs for the perturbed Duffing-oscillator is the appearance of the primary stochastic layer in a neighborhood of the homoclinic manifold as h_0 increases beyond zero. According to Theorem 2.2, the perturbed Duffing-oscillator (1.1) with Hamiltonian (2.1) has a transverse homoclinic orbit and is nonintegrable for $h_0 > 0$ and $\varepsilon \neq 0$ sufficiently small. Figure 12 shows the primary stochastic layer generated by a single ergodic orbit of (1.1) for $\omega = 1$, $\varepsilon = .02$ and $h = .01$. Also evident in Figure 12 are two first order subharmonic islands surrounding the two elliptic periodic points at $q \cong .26$ and $q \cong -1.39$, respectively. The second order subharmon-

FIGURE 9. Second order subharmonics for $\omega = 2$, $\varepsilon = .02$ and $h = -.02$.

ics corresponding to $\alpha = \alpha_2$ have been swallowed up by the primary stochastic layer in this case. The cross-sections of several of the invariant tori surrounding the periodic points $(\pm 1, 0)$ are also shown in Figure 12.

The maximum width of the primary stochastic layer for the perturbed Duffing-oscillator is given by equation (2.11). For example, if $\omega = 1$ and $h = .07$, equation (2.11) has the form

$$d = 1.93\varepsilon + O(\varepsilon^2) \tag{3.1}$$

for $(q_0, p_0) = (\pm.25, \pm.25)$. The width of the primary stochastic layer was estimated for $\omega = 1$ and $h = .07$ near the point $(.25, .25)$ for several values of ε. For example, the primary stochastic layer for $\omega = 1$, $h = .07$ and $\varepsilon = .01$ is shown in a neighborhood of the origin in Figure 13. The maximum width of the primary stochastic layer near the point $(.25, .25)$, $d \cong .02$ is shown in Figure 13. This is in reasonably good agreement with equation (3.1) for $\varepsilon = .01$. The agreement is not as good near the points $(-.25, \pm.25)$ where the maximum width of the

FIGURE 10. Third order subharmonics for $\omega = 3$, $\varepsilon = .01$ and $h = 0$.

primary stochastic is about twice that predicted by (3.1). However, this asymmetry disappears as ε approaches zero. Figure 14 shows the straight line $d = 1.93\varepsilon$ and several points corresponding to values of the maximum width of the primary stochastic layer near the point $(.25, .25)$ estimated as in Figure 13. The agreement between the numerical results and equation (3.1) is seen to be quite good.

The width of the resonant stochastic layer of order m is given by equation (2.10). For example, if $m = 1$, $\omega = 1$ and $h = 0$, equation (2.10) has the form

$$\Delta q_1 = 1.62\sqrt{\varepsilon} + O(\varepsilon). \tag{3.2}$$

Measuring the resonance bandwidths of order $m = 1$, on the q-axis, in several figures similar to Figure 7 for various values of ε yields the results shown in Figure 15(a). The agreement between the numerical results and equation (3.2) is seen to be very good for ε sufficiently small. Similarly, for $m = 3$, $\omega = 3$ and $h = 0$, equation (2.10) has the form

$$\Delta q_3 = .253\sqrt{\varepsilon} + O(\varepsilon). \tag{3.3}$$

FIGURE 11. First and second order subharmonics for $\omega^2 = 1.97$, $\varepsilon = .001$ and $h = 0$.

FIGURE 12. The primary stochastic layer for $\omega = 1$, $\varepsilon = .02$ and $h = .01$.

FIGURE 13. The maximum width of the primary stochastic layer for $\omega = 1$, $\varepsilon = .01$ and $h = .07$.

FIGURE 14. Theoretical and numerical values for d, the maximum width of the primary stochastic layer for $\omega = 1$ and $h = .07$.

FIGURE 15. Resonance bandwidths.

Measuring the resonance bandwidths of order $m = 3$ in several figures similar to Figure 10 (where $\varepsilon = .01$ and $\Delta q_3 \cong .32$) yields the results shown in Figure 15(b). And, for $m = 1$, $\omega \cong 1.35$, corresponding to $\alpha_1 = .8$ in equation (2.8), and $h = 0$, equation (2.10) has the form

$$\Delta q_1 = 2.49\varepsilon + O(\varepsilon). \tag{3.4}$$

Measuring the resonance bandwidths of order $m = 1$ in several figures similar to Figure 11 for various values of ε gives the results shown in Figure 15(c). Once again, the agreement between the numerical results and the theoretical results is seen to be very good for sufficiently small

FIGURE 16. The primary stochastic layer for $\omega = 1$, $\varepsilon = .05$ and $h = .1$.

ε. In fact, the difference between the theoretical and numerical results for $\Delta q_m/\sqrt{\varepsilon}$ is $O(\sqrt{\varepsilon})$ as predicted by (3.2)–(3.4).

Finally, it was noted that as h_0 increased beyond zero, the width of the primary stochastic layer increased in size and subharmonics, corresponding to the solutions α_m^+ of the resonance condition (2.8), appeared outside the homoclinic manifold. Figure 16 shows the primary stochastic layer, generated by a single ergodic trajectory of (1.1) for $\omega = 1$, $\varepsilon = .05$ and $h = .1$. It is apparent that the resonant stochastic layer of order one has been engulfed by the primary stochastic layer; however, a first order subharmonic island still remains. Also, a pair of second order subharmonic islands (corresponding to α_2^+) can be seen near the points $p = \pm.25$ on the p-axis in Figure 16. Figure 17 shows an enlargement of one of these second order subharmonic islands in the primary stochastic layer, [cf. **6**, Figure 6].

FIGURE 17. A second order subharmonic for $\omega = 1$, $\varepsilon = .05$ and $h = .1$.

4. Concluding remarks. Melnikov's method offers an excellent mathematical tool for studying the stochastic layers and nonintegrability of perturbed Hamiltonian systems with two degrees of freedom as is evidenced by the results obtained for the perturbed Duffing-oscillator studied in this paper. The numerical results are in excellent agreement with the theoretical results for this problem which provides a particularly nice application of the theory.

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REFERENCES

1. A.A. Andronov, E.A. Leontovich, I.I. Gordon and A.G. Maier, *Theory of bifurcations of dynamic systems on a plane*, Israel Program for Scientific Translations, Jerusalem (1971).

2. V.I. Arnold, *Instability of dynamical systems with several degrees of freedom*, Sov. Math. Dokl. **5** (1964), 581–585.

3. S.N. Chow, J.K. Hale and J. Mallet-Paret, *An example of bifurcation to homoclinic orbits*, J. Differential Equations **37** (1980), 351–373.

4. B.D. Greenspan and P.J. Holmes, *Homoclinic orbits, subharmonics and global bifurcations in forced oscillations*, in *Nonlinear dynamics and turbulence*, Barenblatt, Iooss and Joseph (eds.), Pitman, London (1983), 172–214.

5. J. Guckenheimer and P.J. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, Second Printing, Springer-Verlag, New York, 1983.

6. M. Henon and C. Heiles, *The applicability of the third integral of motion: Some numerical experiments*, Astron. J. **69** (1964), 73–84.

7. P.J. Holmes, *Proof of nonintegrability for the Henon-Heiles Hamiltonian near an exceptional integrable case*, Physica **5**D (1982), 335–347.

8. P.J. Holmes and J.E. Marsden, *Horseshoes in perturbations of Hamiltonians with two degrees of freedom*, Comm. Math. Phys. **82** (1982), 523–544.

9. E. Jahnke and F. Emde, *Tables of functions with formulas and curves*, Fourth edition, Dover Publications, New York, 1945.

10. V.K. Melnikov, *On the stability of the center for periodic perturbations*, Trans. Moscow Math. Soc. **12** (1963), 1–57.

11. P. Veerman and P. Holmes, *The existence of arbitrarily many distinct periodic orbits in a two degree of freedom Hamiltonian system*, Physica **14**D (1985), 177–192.

12. ———, *Resonance bands in a two degree of freedom Hamiltonian system*, Physica **20**D (1986), 413–422.

13. E. Zehnder, *Homoclinic points near elliptic fixed points*, Comm. Pure Appl. Math. **26** (1973), 131–182.

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