ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 4, Winter 1992

K-THEORY OF ANALYTIC CROSSED PRODUCTS

CHAOXIN QIU

ABSTRACT. We prove the following theorem which is simultaneously a non-self-adjoint analogue of Conne's Thom Isomorphism and a generalization of a result of J. Peters. Suppose that G is a locally compact, compactly generated, abelian group and that Σ is a subsemigroup of G satisfying $(\Sigma^0)^- = \Sigma$ and $\Sigma \cap (-\Sigma) = \{0\}$. Then, for an arbitrary C^*-dynamical system (A, G, α),

$$K_i(\Sigma \times_\alpha A) \cong \begin{cases} K_i(A) & \text{if } G \text{ is discrete} \\ \{0\} & \text{otherwise} \end{cases} \qquad (i = 0, 1).$$

1. Introduction. K-Theory has revolutionized the study of operator algebras in the last few years [**2, 4, 1**]. Most work is, however, devoted to C^*-algebras and relatively little is known on the K-theory for non-self-adjoint Banach algebras. A few results in this direction can be found in [**12, 11**].

We will concentrate our attention on the computation of the K-groups of analytic crossed products. We will use terminology, notation and basic facts on K-theory and crossed product used in [**1, 10, 8**]. We recall here some details about analytic crossed products for the sake of the reader's convenience.

Let A be a C^*-algebra, let G be a locally compact group with left Haar measure μ and let α be a continuous homomorphism from G into $\mathrm{Aut}\,(A)$, the group of C^*-automorphisms of A with the topology of pointwise norm-convergence. Following the notation in [**10**], we denote the enveloping C^*-algebra of $L^1(G, A)$ by $G \times_\alpha A$ and call it the C^*-*crossed product determined by the* C^*-*dynamical system* (A, G, α).

Let Σ be a closed subsemigroup of G containing the identity e of G satisfying the following conditions:

Received by the editors on February 1, 1990 and in revised form on September 24, 1990.
This work is partially supported by the National Science Foundation.

1.1 Σ is the closure of its interior,

1.2 $\Sigma \cap \Sigma^{-1} = \{e\}$,

1.3 Σ generates G.

The set of functions in $L^1(G, A)$ that are supported on Σ, $L^1(\Sigma, A)$, is clearly a closed subalgebra of $L^1(G, A)$. The closure of $L^1(\Sigma, A)$ in $G \times_\alpha A$ will be denoted by $\Sigma \times_\alpha A$ and will be called the *analytic crossed product determined by* (A, G, α) and the *semigroup* Σ. In this note we always work with compactly generated, abelian locally compact groups without further explanation and thus we denote the operation in groups additively.

It is not difficult to show that the K_1-group of the algebra of compact operators in any nest algebra is trivial. Also, J. Peters [**11**] proved that

$$K_i(\mathbf{Z}_+ \times_\alpha C(X)) \cong K_i(C(X)) \qquad (i = 0, 1).$$

Consider the following special case, called the "standard nest algebra". Let H be a Hilbert space with orthonormal bias $\{e_n\}_{n \in \mathbf{Z}}$, let $\mathbf{N}$ be the nest with $M_n = \bigvee_{i \le n} \mathbf{C}\{e_i\}$, and let $K\mathrm{Alg}\,(\mathbf{N})$ be the set of compact operators in nest algebra $\mathrm{Alg}\,(\mathbf{N})$. It is easy to see that

$$K\mathrm{Alg}\,(\mathbf{N}) \cong \mathbf{Z}_+ \times_\alpha C_0(\mathbf{Z})$$

where $\mathbf{Z}$ acts by translation [**9**]. Both our argument and Peter's paper can be applied to prove the triviality of $K_1(K\mathrm{Alg}\,(\mathbf{N}))$. Looking through the arguments in the above works, we see that the idea behind them is to multiply each function f in $L^1(\mathbf{Z}_+, C_0(\mathbf{Z}))$ by some decay function with parameter $s \in [0, \infty]$, i.e., to define a homomorphism $H : \mathbf{R}_+ \times L^1(\mathbf{Z}_+, C_0(\mathbf{Z})) \to L^1(\mathbf{Z}_+, C_0(\mathbf{Z}))$ by

$$H(s, f) = f_s \quad \text{with } f_s(n) = e^{-sn} f(n).$$

We find that this same idea works for a large class of analytic crossed products. Our main theorem is

Theorem 1.4. *Suppose that G is a compactly generated, abelian locally compact group and Σ is a subsemigroup of G satisfying the conditions* 1.1, 1.2, *and* 1.3. *Then*

$$K_i(\Sigma \times_\alpha A) \cong \begin{cases} K_i(A) & \text{if } G \text{ is discrete} \\ \{0\} & \text{otherwise} \end{cases} \qquad (i = 0, 1)$$

for all C^*-algebras A.

This result reminds us of the Pimsner-Voiculescu Exact Sequence and Conne's Thom Isomorphism [**2**].

In the next section, we will prove that the conclusion in Theorem 1.4 is true provided that there exists a continuous homomorphism $\theta : G \to \mathbf{R}$ such that $\theta(\Sigma) \subseteq \mathbf{R}_+$ and $\operatorname{Ker}(\theta) \cap \Sigma$ is trivial. We will conclude the proof of Theorem 1.4 by showing that such a θ exists when Σ satisfies 1.1, 1.2 and 1.3.

2. K-theory of analytic crossed products. We will use the terminology and notation in Section 1 without further explanation. We will also make the following additional assumption:

> There exists a continuous homomorphism $\theta : G \to \mathbf{R}$ such that $\theta(\Sigma) \subseteq \mathbf{R}_+$ and $\operatorname{Ker}(\theta) \cap \Sigma = \{0\}$.

This will be proved to hold for all pairs G, Σ under the above assumption in the next section.

The following lemma gives us a method to define a homotopy to be used in the computation of the K-groups of the crossed product $\Sigma \times_\alpha A$.

Lemma 2.1. *Let $f \in L^1(\Sigma, A)$. Define*

$$f_y(t) = e^{-y\theta(t)} f(t) \qquad t \in \Sigma,\ y \in [0, \infty);$$

and define

$$f_\infty(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ f(0) & \text{if } t = 0. \end{cases}$$

Then we have

(i) $f_y \in L^1(\Sigma, A)$ *for all* $y \in [0, \infty]$.

(ii) $(f * g)_y = f_y * g_y$,

(iii) $\lim_{y \to 0} f_y = f$, *and* $\lim_{y \to \infty} f_y = f_\infty$,

and

(iv) *the mapping $F : y \to f_y$ is continuous as a map from $[0, \infty]$ into $L^1(\Sigma, A)$.*

Proof. (i) is obvious. (ii) can be verified by a straightforward computation. (iii) and (iv) can be proved easily by using Lebesgue Dominated Convergence Theorem. □

Remark. From Lemma 2.1, we get a map H from $[0,\infty] \times L^1(\Sigma, A)$ into $L^1(\Sigma, A)$ defined by setting

$$H(y, f) = f_y \quad \text{for } y \in [0,\infty],\ f \in L^1(\Sigma, A).$$

Lemma 2.1 asserts that

(i) $H(y,\cdot)$ is a continuous homomorphism on $L^1(\Sigma, A)$, and

(ii) $H(\cdot, f)$ is also continuous.

We thus have the following byproduct.

Proposition 2.2. *For $i = 00$, 0 and 1,*

$$K_i(L^1(\Sigma, A)) \cong \begin{cases} K_i(A) & \text{if } G \text{ is discrete} \\ \{0\} & \text{otherwise.} \end{cases}$$

Proof. (i) On the K_{00}-group.

First, assume that G is discrete. Note that $\mu(\{0\}) = 1$. We are about to show that $K_{00}(L^1(\Sigma, A)) \cong K_{00}(L^1(\{0\}, A))$. It will result that $K_{00}(L^1(\Sigma, A)) \cong K_{00}(A)$.

Define $\phi : L^1(\Sigma, A) \to L^1(\{0\}, A)$ by $\phi(a) = a|_{\{0\}}$, and define $\psi : L^1(\{0\}, A) \to L^1(\Sigma, A)$ by $\psi(b)(t) = b(0)$ if $t = 0$, 0 if $t \neq 0$. Let Φ and Ψ denote the homomorphism induced by ϕ and ψ on their K_{00}-groups respectively. It is clear that $\Phi \times \Psi = \mathrm{id}_{K_{00}(L^1(\{0\},A))}$. To show $\Psi \circ \Phi = \mathrm{id}_{K_{00}(L^1(\Sigma,A))}$, it suffices to show that if $[f](f = (a_{ij}) \in M_n(L^1(\Sigma, A)))$ is an idempotent in $K_{00}(L^1(\Sigma, A))$, then $[f]$ is homotopic to $[f_0]$ where $f_0 = (\psi(\phi(a_{ij}))) \in M_n(L^1(\Sigma, A))$.

Define $f_r = (a^r_{ij})$, for $r \in [0, 1]$, by

$$a^r_{ij}(t) = H\left[\frac{1}{\tan\left[\frac{\pi}{2} r\right]}, \alpha_{ij}\right].$$

From Lemma 2.1, $[f_r]$ is a path of idempotents in $M_n(L^1(\Sigma, A))$ joining f and f_0. It results that $[f] = [f_0]$ and hence,

$$K_{00}(L^1(\Sigma, A)) \cong K_{00}(L^1(\{0\}, A)).$$

If G is not discrete, then $\mu(\{0\}) = 0$. The above argument then leads to the conclusion that

$$K_{00}(L^1(\Sigma, A)) \cong K_{00}(L^1(\{0\}, A) \cong \{0\}.$$

(ii) On the K_0-group and K_1-group.

Let $L^1(\Sigma, A)^+$ be the initial Banach algebra obtained by adding a copy of $\mathbf{C}$ to $L^1(\Sigma, A)$. Extend H in the *Remark* preceding Proposition 2.2 to $L^1(\Sigma, A)^+$ by defining

$$H(y, f + c \cdot 1) = f_y + c \cdot 1 \qquad \text{for } c \in \mathbf{C}.$$

H satisfies (i) and (ii) in the above remark. Thus, by an argument similar to the one in (i), we see that

$$\begin{aligned} K_1(L^1(\Sigma, A)) &\cong K_1(L^1(\Sigma, A)^+) \cong K_1(L^1\{0\}, A)^+) \\ &\cong \begin{cases} K_1(A^+) \cong K_1(A) & \text{if } G \text{ is discrete} \\ K_1(\mathbf{C}) \cong \{0\} & \text{otherwise.} \end{cases} \end{aligned}$$

Moreover, since the following diagram is commutative

$$\begin{array}{ccc} L^1(\Sigma, A)^+ & \xrightarrow{\quad H(y,\cdot) \quad} & L^1(\Sigma, A)^+ \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ L^1(\Sigma, A)^+/L^1(\Sigma, A)+ & \xrightarrow{\quad \mathrm{id} \quad} & L^1(\Sigma, A)^+/L^1(\Sigma, A) \end{array}$$

for any $y \in [0, \infty]$, we have the commutative diagram

$$\begin{array}{ccccc} K_{00}(L^1(\Sigma, A)^+) & \xrightarrow{\ \Phi\ } & K_{00}(L^1\{0\}, A)^+) & \xrightarrow{\ \Psi\ } & K_{00}(L^1(\Sigma, A)^+) \\ \downarrow{\scriptstyle \pi_*^{(1)}} & & \downarrow{\scriptstyle \pi_*^{(2)}} & & \downarrow{\scriptstyle \pi_*^{(1)}} \\ K_{00}(\mathbf{C}) & \xrightarrow{\quad \mathrm{id} \quad} & K_{00}(\mathbf{C}) & \xrightarrow{\quad \mathrm{id} \quad} & K_{00}(\mathbf{C}) \end{array}$$

Therefore,

$$\begin{aligned} K_0(L^1(\Sigma, A)) &= K_0(L^1(\Sigma, A)^+, L^1(\Sigma, A)) \cong \operatorname{Ker} \pi_*^{(1)} \\ &\cong \operatorname{Ker} \pi_*^{(2)} \cong K_0(L^1(\{0\}, A)^+, L^1(\{0\}, A)) \\ &\cong \begin{cases} K_0(A^+, A) \cong K_0(A) & \text{if } G \text{ is discrete} \\ \{0\} & \text{otherwise.} \end{cases} \quad \square \end{aligned}$$

Remark. To establish our main result (Theorem 1.4), we need to extend the map H from $L^1(\Sigma, A)$ onto $\Sigma \times_\alpha A$ so that

(i) for each y, $H(y, \cdot)$ is a continuous homomorphism on $\Sigma \times_\alpha A$,

(i) for each $x \in \Sigma \times_\alpha A$, the function $H(\cdot, x)$ is continuous on $[0, \infty]$.

Since the norms on $L^1(\Sigma, A)$ and $\Sigma \times_\alpha A$ are different, the extension is not at all trivial. The following lemma is the key to making that extension.

Lemma 2.3. *Suppose G is discrete and $y \in [0, \infty]$. Then there exists $\mu_y \in M(\hat{G})$ so that $\hat{\mu}_y(t) = e^{-y\theta(t)}$ for $t \in \Sigma$, where $\hat{\mu}_y$ denotes the Fourier transformation of μ_y (see* [**6**, Chapter 6]*). In the case when G is not discrete, the above assertion is true for $y \in [0, \infty]$.*

Proof. We discuss this in three cases.

Case 1. Suppose $y = 0$. We may define μ_0 as the point measure at identity of $\hat{G}$.

Case 2. Suppose $y \in (0, \infty)$. Since θ is a continuous group homomorphism from G to $\mathbf{R}$, we may define $\hat{\theta} : \mathbf{R} \to \hat{G}$ as the dual group homomorphism. Hence, for each given y, $\hat{\theta}$ induces a measure $\mu_y \in M(\hat{G})$ corresponding to the measure in $M(\mathbf{R})$ determined by the Poisson kernel P_{iy}. Then it is straightforward to check that such a μ_y does work.

Case 3. Let $y = \infty$ and let G be discrete. Then $\hat{G}$ is compact and abelian (cf. [**6**], Theorem 23.17) and, thus, we may define μ_∞ as the Haar measure on $\hat{G}$ (cf. [**6**], Lemma 23.19). $\square$

We now may make the expected extension.

Proposition 2.4. *There exists a mapping H from $[0,\infty]\times(\Sigma\times_\alpha A)$ into $\Sigma\times_\alpha A$ satisfying the following:*

(i) *for each y, $H(y,\cdot)$ is a continuous homomorphism of $\Sigma\times_\alpha A$, and*

(ii) *for each x, $H(\cdot,x)$ is a continuous function on $[0,\infty]$.*

Proof. If G is discrete, define $H(y,x)=x*_{\hat{\alpha}}\mu_y$ for $x\in\Sigma\times_\alpha A$ and $y\in[0,\infty]$ where μ_y is given in Lemma 2.3. And if G is not discrete, define $H(y,x)=x*_{\hat{\alpha}}\mu_y$ if $0\le y<\infty$ and 0 if $y=\infty$.

Note that, for $f\in L^1(\Sigma,A)$ and $y\in[0,\infty)$,

$$\begin{aligned}H(y,f)(t)&=\int_{\hat{G}}\hat{\alpha}_{\hat{t}}(f)(t)d\mu_y(\hat{t})\\&=\int_{\hat{G}}\langle t,\hat{t}\rangle\,f(t)d\mu_y(\hat{t})\\&=f(t)\cdot\int_{\hat{G}}\langle t,\hat{t}\rangle\,d\mu_y(\hat{t})\\&=f(t)\cdot\hat{\mu}_y(t)\\&=e^{-y\theta(t)}f(t).\end{aligned}$$

It is also clear that

$$H(y,f)(t)=e^{-y\theta(t)}f(t)$$

for $f\in L^1(\Sigma,A)$ and $y=\infty$.

This proposition follows from Lemma 2.1 for $L^1(\Sigma,A)$ is dense in $\Sigma\times_\alpha A$ and $||x*_{\hat{\alpha}}\mu_y||\le||x||\cdot||\mu_y||$.

We conclude that the argument from Proposition 2.2 proves Theorem 1.4, subject to assumption (S).

3. Locally compact, compactly generated abelian semigroup. Recall from harmonic analysis that a locally compact, compactly generated abelian group G is topologically isomorphic to a "canonical" group $\mathbf{R}^a\times\mathbf{Z}^b\times F$ where a and b are nonnegative integers and F is a compact abelian group [**6**]. We are going to find a continuous homomorphism θ from G into $\mathbf{R}$ such that $\theta(\Sigma)\subseteq\mathbf{R}_+$ and

$\mathrm{Ker}\,(\theta) \cap \Sigma = \{0\}$. This is the assumption (S) in Section 2, and so, this will complete the proof of Theorem 1.4. Without loss of generality, we may assume that $G = \mathbf{R}^a \times \mathbf{Z}^b \times F$. We first consider the special case when $G = \mathbf{R}^a \times \mathbf{Z}^b$. We will need the following notation:

$$\mathbf{R}^N = \mathbf{R}^a \times \mathbf{R}^b \qquad (N = a+b),$$

$$h(\Sigma) = \{\sum_{i=1}^{k} \alpha_i x_i : \alpha_i \geq 0,\ x_i \in \Sigma \text{ for } i = 1, 2, \dots, k,\ k = 1, 2, \dots\},$$

$$\ell(\Sigma) = \text{ the linear subspace spanned by } \Sigma \text{ in } \mathbf{R}^N.$$

Lemma 3.2, below, guarantees that $h(\Sigma)$ is not equal to $\ell(\Sigma)$. Its proof requires an elementary observation in linear algebra.

Lemma 3.1. *Let $w_1, \dots, w_m \in Q^n$, and $\alpha_1, \dots, \alpha_n \in \mathbf{R}$ be such that $\sum_{j=1}^m \alpha_j w_j = 0$. If $\varepsilon > 0$ is given, there exist $\tilde{\alpha}_1, \dots, \tilde{\alpha}_n \in Q$ with $|\alpha_j - \tilde{\alpha}_j| < \varepsilon$, $1 \leq j \leq m$, and $\sum_{j=1}^m \tilde{\alpha}_j w_j = 0$.*

Proof. Let $A = [w_1, \dots, w_m]$ be the $n \times m$ matrix whose columns are the w_j's, and let $[\alpha] = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$. Then A induces linear transformations T_A from Q^m to Q^n and from $\mathbf{R}^m$ to $\mathbf{R}^n$. The dimension of the kernel of T_A, as a vector space over Q is the same as the dimension of the kernel of S_A as a vector space over $\mathbf{R}$, since both are determined by the reduced row-echelon form of A, which involves only rational operations on A. Thus if $\{v_1, \dots, v_d\}$ spans $\mathrm{Ker}\,(T_A)$ over Q, it also spans $\mathrm{Ker}\,(S_A)$ over $\mathbf{R}$. As $[\alpha]$ lies in the kernel of $S_A, [\alpha] = \sum_{j=1}^d t_j v_j$. Choose $\delta > 0$ such that if $|\tilde{t}_j - t_j| < \delta, 1 \leq j \leq d$, then $[\tilde{\alpha}] = \sum_{j=1}^d \tilde{t}_j v_j$ satisfies $|\alpha_j - \tilde{\alpha}_j| < \varepsilon$, $1 \leq j \leq m$. If t_j is also rational, $1 \leq j \leq d$ so is $\tilde{\alpha}_j$, and the proof is complete. □

Lemma 3.2. $-\Sigma^0 \cap h(\Sigma) = \phi$, *where Σ^0 is the interior of Σ in the topology of $G = \mathbf{R}^a \times \mathbf{Z}^b$.*

Proof. Suppose that there exists a $u \in \Sigma^0$ such that

$$-u = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$$

for some $k > 0$, $\{x_i\}_{i=1}^k \subseteq \Sigma$ and $\{\alpha_i\}_{i=1}^k \subseteq \mathbf{R}_+$. Let

$$u = (u^1, \dots, u^a, u^{a+1}, \dots, u^{a+b}),$$

and

$$x_i = (x_i^1, \dots, x_i^a, x_i^{a+1}, \dots, x_i^{a+b}), \qquad i = 1, 2, \dots, k,$$

be the coordinate representations of u and the x_i's. We then get a system of linear equations written in two portions—real and integer portions.

$$\text{(I)} \qquad -\begin{bmatrix} u^1 \\ \vdots \\ u^a \end{bmatrix} = \alpha_1 \begin{bmatrix} x_1^1 \\ \vdots \\ x_1^a \end{bmatrix} + \alpha_2 \begin{bmatrix} x_2^1 \\ \vdots \\ x_2^a \end{bmatrix} + \cdots + \alpha_k \begin{bmatrix} x_k^1 \\ \vdots \\ x_k^a \end{bmatrix},$$

$$\text{(II)} \qquad -\begin{bmatrix} u^{a+1} \\ \vdots \\ u^{a+b} \end{bmatrix} = \alpha_1 \begin{bmatrix} x_1^{a+1} \\ \vdots \\ x_1^{a+b} \end{bmatrix} + \alpha_2 \begin{bmatrix} x_2^{a+1} \\ \vdots \\ x_2^{a+b} \end{bmatrix} + \cdots + \alpha_k \begin{bmatrix} x_k^{a+1} \\ \vdots \\ x_k^{a+b} \end{bmatrix}.$$

We call (I) the "real portion" and (II) the "integer portion".

Since the topology of G is the product of the Euclidean topology on $\mathbf{R}^a$ and the discrete topology on $\mathbf{Z}^b$, we can find a small perturbation of the x_i's such that all the coordinates of the x_i's are rational numbers, for $(\Sigma^0)^- = \Sigma$ and $u \in \Sigma^0$. Denote the new system of equations by (I) and (II) again.

By Lemma 3.1, given $\varepsilon > 0$, there are rational β_i's with $|\beta_i - \alpha_i| < \varepsilon$, $1 \le i \le k$, such that (II) is satisfied with β_i in place of α_i. For ε sufficiently small, the element $u' = \sum_{j=1}^k \beta_j x_j$ has the property that nu' is in both $-\Sigma$ and Σ, for some positive integer n. Since their intersection is $\{0\}$, this is not possible. □

The following proposition is an analogue of the existence of states which appears in [**5**, Chapter 4]. The proof is basically the same as that used to prove [**5**, Corollary 4.4]. We simply outline the proof for the sake of completeness.

Proposition 3.3. *Suppose that* $\{u_1, \dots, u_n\}$ *is a basis of* $\ell(\Sigma)$ *such that* $u_i \in \Sigma^0$, $i = 1, 2, \dots, n$. *Let* $u = u_1 + u_2 + \cdots + u_n$. *Then* *there exists a linear (continuous) functional* $\theta : \mathbf{R}^N \to \mathbf{R}$ *such that* $\theta(h(\Sigma)) \subseteq \mathbf{R}^+$ *and* $\theta(u) = 1 > 0$.

Proof. We may assume that $\ell(\Sigma) = \mathbf{R}^N$ without loss of generality. Note that since $\mathbf{R}u \cap h(\Sigma) = \mathbf{R}_+ u$, there exists an $f : \mathbf{R}u \to \mathbf{R}$ which is a linear continuous functional satisfying $f(u) = 1$ and $f(\mathbf{R}(u) \cap h(\Sigma)) \subseteq \mathbf{R}^+$.

Let $\bar{K}$ be the collection of such pairs (K, g) where K is a linear subspace of $\mathbf{R}^N$ containing $\mathbf{R}u$, g is a linear functional satisfying $g|_{\mathbf{R}u} = f$ and $g(K \cap h(\Sigma)) \subseteq \mathbf{R}^+$. For (K, g) and (K', g') in $\bar{K}$, we define $(K, g) \leq (K', g')$ if $K \subseteq K'$ and $g'|_K = g$. By Zorn's Lemma, there exists a maximal element (K, g) in $\bar{K}$. It suffices now to show that $K = \mathbf{R}^N$. We are going to do this by contradiction. Suppose that $K \neq \mathbf{R}^N$. Then there exists a $x_0 (\neq 0) \in \mathbf{R}^N \backslash K$, where $\mathbf{R}^N \backslash K$ is the set theoretic complement of K in $\mathbf{R}^N$. Let $K' = K + \mathbf{R}x_0$. Define

$$p_{x_0} = \sup\{g(y)/m : y \in K,\, m \in \mathbf{R}^+,\, mx_0 - y \in h(\Sigma)\},$$

and

$$r_{x_0} = \inf\{g(y)/m : y \in K,\, m \in \mathbf{R}^+,\, y - mx_0 \in h(\Sigma)\},$$

It is easy to show that $-\infty < p_{x_0} \leq r_{x_0} < \infty$. Now, let q be a number between p_{x_0} and r_{x_0}, and define $g' : K' \to \mathbf{R}$ by

$$g'(k + rx_0) = g(k) + rq \quad \text{for all } k + rx_0 \in K'.$$

It is easy to show that $(K', g') \in \bar{K}$ and $(K', g') \underset{\neq}{\geq} (K, g)$, which is a contradiction. □

Furthermore, we have

Proposition 3.4. $\dim(\ell(\operatorname{Ker}\theta \cap \Sigma)) < N$.

Proof. If $\dim(\ell(\operatorname{Ker}\theta \cap \Sigma)) = N$, then $\dim(\operatorname{Ker}\theta) = N$. It follows $\operatorname{Ker}\theta \supset \Sigma$. But $u \in \Sigma$, and $u \notin \operatorname{Ker}\theta$. A contradiction. □

We now have

Proposition 3.5. *Suppose that $G = \mathbf{R}^a \times \mathbf{Z}^b$ and let $\mathbf{R}^N = \mathbf{R}^a \times \mathbf{R}^b$. Then there exists a linear functional $\theta : \mathbf{R}^N \to \mathbf{R}$ such that* $\operatorname{Ker}\theta \cap \Sigma = \{0\}$ *and $\theta(h(\Sigma)) \subseteq \mathbf{R}^+$.*

Proof. Let θ_1 be the linear functional from Proposition 3.3. Set $\Sigma_1 = \operatorname{Ker}\theta_1 \cap \Sigma$ and $N_1 = \dim \ell(\Sigma_1)$. By Proposition 3.4, $N_1 < N$. We may apply Proposition 3.3 to $\ell(\Sigma_1)$ again. Suppose that we have found $\theta_1, \dots, \theta_i$, $\Sigma_0(= \Sigma)$, $\Sigma_1, \dots, \Sigma_{i-1}$, and $N_0(= N)$, $N_1, \dots, N_i$ such that

(i) $\theta_j : \ell(\Sigma_{j-1}) \to \mathbf{R}$ is a continuous linear functional,

(ii) $\theta_j(h(\Sigma_{j-1})) \subseteq \mathbf{R}^+$,

(iii) $\dim \ell(\Sigma_j) = N_j < N_{j-1}$,

(iv) $\Sigma_j = \operatorname{Ker}\theta_j \cap \Sigma_{j-1}$.

Since N is finite, the inductive procedure will terminate after finitely many "iterations". Suppose i is the first index such that $N_i = 0$. We can use an argument similar to that used in Proposition 3.3 to extend all the θ_j's to $\mathbf{R}^N$ so that $\theta_j(h(\Sigma)) \subseteq \mathbf{R}^+$. We see that $\theta = \theta_1 + \cdots + \theta_i$ is the functional we want. □

Let us consider the general case, $G = \mathbf{R}^a \times \mathbf{Z}^b \times F$. We need the following lemma.

Lemma 3.6. *Let Σ_1 be the projection of Σ onto $\mathbf{R}^a \times \mathbf{Z}^b$. Then $\Sigma_\cap(-\Sigma_1) = \{0\}$, and $(\Sigma_1^0)^- = \Sigma_1$.*

Proof. Since F is compact, the projection map from G onto $\mathbf{R}^a \times \mathbf{Z}^b$ is open and closed. The equation $(\Sigma_1^0)^- = \Sigma_1$ follows from $(\Sigma^0)^- = \Sigma$.

Assume that there is a $x(\neq 0) \in \Sigma_1 \cap (-\Sigma_1)$. Then there are $\lambda_1, \lambda_2 \in F$ such that (x, λ_1) and $(-x, \lambda_2) \in F$. Then

$$(0, \lambda_1 + \lambda_2) = (x, \lambda_1) + (-x, \lambda_2) \in \Sigma \cap (\{0\} \times F).$$

Since F is compact, $\Sigma \cap (\{0\} \times F)$ is a subgroup of $\{0\} \times F$. It results that $(0, -\lambda_1 - \lambda_2) \in \Sigma \cap (\{0\} \times F)$ and hence,

$$(-x, -\lambda_1) = (0, -\lambda_1 - \lambda_2) + (-x, \lambda_2) \in \Sigma.$$

This contradicts the fact that $\Sigma \cap (-\Sigma) = \{0\}$. □

We may now complete the proof of our main result, Theorem 1.4, by proving the following theorem.

Theorem 3.7. *Suppose that* $G = \mathbf{R}^a \times \mathbf{Z}^b \times F$ *and let* $\mathbf{R}^N = \mathbf{R}^a \times \mathbf{R}^b$. *There exists an* N*-tuple* $(\lambda_1, \dots, \lambda_N)$ *and a continuous homomorphism* $\theta : \mathbf{R}^N \times F \to \mathbf{R}$ *defined by*

$$(*) \qquad \theta(t) = \sum_{i=1}^{N} \lambda_i x_i \quad \text{for } t = (x_1, \dots, x_N, g) \in \mathbf{R}^N \times F$$

such that

$$\theta(\Sigma) \subseteq \mathbf{R}^+ \quad \text{and} \quad \operatorname{Ker} \theta \cap \Sigma = \{0\}.$$

Proof. Applying Proposition 3.5 to Σ_1 in $\mathbf{R}^N$, we get a liner functional $\theta_1 : \mathbf{R}^N \to \mathbf{R}$ defined by (*), for some N-tuple, which satisfies $\theta_1(\Sigma_1) \subseteq \mathbf{R}^+$ and $\operatorname{Ker} \theta_1 \cap \Sigma_1 = \{0\}$. Define θ by composing θ_1 with the projection map π from $\mathbf{R}^N \times F$ onto $\mathbf{R}^N$, i.e.,

$$\theta(t) = \theta_1(\pi(t)) \quad \text{for } t \in G.$$

Then θ has the form (*) and satisfies $\theta(\Sigma) \subseteq \theta_1(\Sigma) \subseteq \mathbf{R}^+$. Assume that $t = (x, g) \in \operatorname{Ker} \theta \cap \Sigma$. Then $x \in \operatorname{Ker} \theta_1 \cap \Sigma_1$ and hence $x = 0$. It follows that $t = (0, g) \in \Sigma \cap (\{0\} \times F)$. The subsemigroup $\Sigma \cap (\{0\} \times F)$ must be a group and hence, trivial for $\Sigma \cap (-\Sigma) = \{0\}$. It follows that $g = 0$ and therefore, $t = 0$. This concludes the proof. □

Acknowledgments. I would like to thank Professor Keren Yan for his encouragement and conversation during the pursuit of this research. I am grateful to the referee for calling our attention to Lemma 3.1 and its proof as a way to shorten an earlier version of this note. I would especially like to thank Professor Paul Muhly for introducing me to this area, for suggesting fruitful lines of inquiry in the early stages of this work, for his encouragement throughout this research and for his careful reading of the first version of this paper.

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Department of Mathematics, University of Iowa, Iowa City, IA 52242

Current address: Department of Mathematics, University of Nevada - Reno, Reno, NV 89557