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PROJECTIVE SUMMANDS OF NOT SINGULAR MODULES

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1. Introduction. Let R be an associative ring with identity. In [**3**], Harada defined the term non-cosmall as follows: For a left R-module M over an associative ring R, M is non-cosmall if and only if there is an epimorphism of a free left R-module onto M such that the kernel is not essential. It turns out that this is equivalent to saying that the module is not singular, and we will adopt that term in this paper. Oshiro [**6**] and Harada [**3**] considered the rings R for which every left not singular module has a projective direct summand and called this condition $(*)^*$. A module M is called an extending module (or is said to have the extending property) if every nonzero submodule of M is essential in a direct summand of M. Oshiro [**6**] gives a complete description of the rings which satisfy $(*)^*$ in the case that the ring has acc on left annihilators. He calls these rings co-H-rings. His theorem states that a ring R is a left co-H-ring if and only if every projective left R-module is an extending module if and only if every left R-module is a direct sum of a projective module and a singular module, if and only if every essential extension of a projective R-module is projective. We are concerned with rings which satisfy the more general condition that every finitely generated not singular module has a nonzero projective direct summand, and we will denote this condition by $(F)^*$. If the ring is left nonsingular and has finite left uniform dimension, we will show that the ring R satisfies $(F)^*$ if and only if R is an FGSP ring in the sense of Goodearl [**1**], i.e., a ring R is an FGSP ring if the singular submodule is a direct summand of every f.g. module. We will obtain results similar to those of Oshira and Harada for rings with finite uniform dimension. We also show that the bounded rings which satisfy $(F)^*$, which are semi-perfect with nil radical, and have a unique minimal projective module which cogenerates all the f.g. projective modules are FQF-3 rings. We also show that if R is a semi-perfect ring with $\cap J^n = 0$ which is FQF-3, and each of its homomorphic images is an FQF-3 ring, then each of the finitely generated modules over R

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decomposes into a direct sum of cyclic modules C_i, $i = 1, \dots, n$, such that for each i, C_i is either a uniserial nonsingular projective module, or is injective over the ring $R/\text{ann}(C_i)$. These rings mimic the integers to a very high degree.

2. Notation. In what follows, R is an associative ring with identity. All modules will be left R-modules unless it is specifically mentioned to the contrary. For an R-module M, we let $E(M)$ be the injective hull of M. The singular submodule of a module M will be denoted by $Z(M)$. For a submodule A of a module M, we will say that a submodule B of M is a compliment of A in M if B is maximal w.r.t. $A \cap B = 0$.

We say the module M is a *fgee* if every cofinitely generated essential extension of M is isomorphic to M: that is, if E is an essential extension of M such that E/M is finitely generated, then E is isomorphic to M. Projective fgees are of interest since, whenever they appear as a submodule of a finitely generated module, they are isomorphic to a direct summand of the module. The notion of a fgee module also seems to be dual to that of a flat module. Consider the following well-known characterization of flat modules: A module M is flat if and only if for any f.g. free module F and map $f : F \to M$ with kernel K, the following holds: For any k in K there exists an f.g. free module G and maps $h : G \to M$ and $g : F \to G$ such that $k \in \ker g$ and $gh = f$.

A ring R is said to be left FQF-3 if there exists a faithful module P, such that for each finitely generated faithful module M, there is a direct summand of M which is isomorphic to P. We will say a ring satisfies $(F)^*$ if every not singular finitely generated module has a nonzero projective direct summand.

3. The results. We first observe the following:

Proposition 1. *Let R be a ring which satisfies $(F)^*$ and such that the identity is the finite sum of indecomposable idempotents. Then R is the finite direct sum of principal indecomposable uniform modules.*

Proof. We first note that if the ring R has finite uniform dimension, then R is the direct sum of finitely many principal idempotent generated left ideals. Let e be one of these principal idempotents. We want to show that Re is uniform. Suppose Re contains two nonzero left ideals A and B with $A \cap B = 0$. Then Re/A is not singular and contains a nonzero projective direct summand. This induces a map of Re onto this projective module which must split, and this contradicts the indecomposability of Re. □

Corollary 2. *Any ring satisfying $(F)^*$ for which the identity is the finite sum of primitive idempotents has finite uniform dimension. In particular, any semi-perfect ring satisfying $(F)^*$ has finite uniform dimension.*

Corollary 3. *If R is a left nonsingular $(F)^*$-ring with finite left uniform dimension, then every finitely generated nonsingular module is projective.*

Proof. Let P be a finitely generated nonsingular module, and let F be a finitely generated free module which maps onto P. Let K be the kernel of this map. Since P is nonsingular, let A be a compliment to K. Since P is nonsingular, K is a compliment to A. It follows that A embeds as an essential submodule of P, and therefore P has finite uniform dimension. Now P has a projective direct summand P_1 with $P_1 \oplus Q = P$. If Q is not zero, then Q has a projective direct summand P_2, with $P = P_1 \oplus P_2 \oplus Q_2$. We can continue in this manner to take a summand of Q_2 which is projective in case Q_2 is not zero. This process must stop since P has finite uniform dimension. □

Theorem 4. *Let R be a left nonsingular ring with finite uniform dimension. Then R satisfies $(F)^*$ if and only if R is an* FGSP.

Proof. Let M be a nonsingular and finitely generated module. Let $f : F \to M \to 0$ be an epimorphism with F a finitely generated free module. Let $K = \ker f$. Since M is not singular, K is not essential in F. So there exists a submodule A which is maximal with respect to $A \cap K = 0$. Next, it follows that A embeds as an essential submodule into F/L, where L is the essential closure of K, i.e., L is the maximal essential extension of K in F. Such an L exists since R is nonsingular. Now A is nonsingular and has finite uniform dimension, so the same is true of F/L. We have that F/L is finitely generated and nonsingular, and so F/L is projective by the argument in the proof of Corollary 3. But it is clear that F/L is isomorphic to $M/Z(M)$ and so $Z(M)$ is a direct summand of M.

In [**1**, Theorem 2.15], Goodearl shows that if every nonsingular R-module is projective, then R is Morita equivalent to a finite product of full upper triangular matrix rings over division rings. If we could show that a ring which satisfies $(*)^*$ is left nonsingular and left perfect, every nonsingular module is projective, we could then apply Goodearl's result to obtain one of the equivalences of Phan [**8**, Theorem 7]. We will show that a left nonsingular left perfect ring which satisfies $(F)^*$ is left Artinian. It then follows easily that for these rings the injective hull of every projective module is projective. We will show that every finite dimensional projective module does have the extending property if R satisfies $(F)^*$ and has finite uniform dimension. Then for left perfect rings, $(F)^*$ will imply $(*)^*$ by Theorem 4 of [**8**]. □

Proposition 5. *Let R be a ring with finite uniform dimension. Then every finitely generated projective module has the extending property if and only if R satisfies $(F)^*$.*

Proof. Assume that R satisfies $(F)^*$. We proceed by induction on the uniform dimension of the f.g. projective P. Let P be a uniform projective. Then it is trivial that P has the extending property. Suppose every projective module of dimension less than k has the extending property and P has dimension k. Let A be a submodule of uniform dimension less than k. We can suppose the dimension of A is $k-1$ for, if not, we can add a submodule B with $B \cap A = 0$ to A with the appropriate uniform dimension so that $A+B$ has uniform dimension $k-1$. If we can show that $A+B$ is essential in a direct summand, this summand will have uniform dimension $k-1$ and will be projective. The induction hypothesis then will give us the desired direct summand of P in which A is essential. To this end, let C be a submodule maximal with respect to $C \cap A = 0$, and let D be maximal with respect to $D \cap C = 0$ and contains A. Now C is uniform, and the uniform dimension of D is still $k-1$. Hence, A is essential in D. Consider P/D. We claim C is embedded in P/D as an essential submodule. If the image of C is not essential, then this would contradict the maximality of D. We have that P/D is a uniform not singular module since D is not essential. It follows that P/D is projective and hence that D is a summand of P.

For the converse, suppose M is a not singular f.g. module. Let F be a f.g. free module and f an epimorphism of F onto M with $K = \ker f$. We have that K is essential in a direct summand Q of F. Since M is not singular, Q is not all of F. Hence, the module M has a nonzero projective quotient which is isomorphic to a summand of M. □

Proposition 6. *Let R be a ring which satisfies $(F)^*$ and has finite uniform dimension. Let $R = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n$, where each of the e_i is primitive. Let P be any uniform projective module. Then P embeds in one of the Re_i's.*

Proof. Let P be a uniform projective module. Then for some Q we have that $P \oplus Q = F$ is a free f.g. module and is isomorphic to the direct sum of copies of the Re_i's. If the projection of P onto each summand is not a monomorphism, then the image of P in F is singular. Since P is not singular, one of the projections must be a monomorphism, and we have the desired embedding. □

Let R be a ring which satisfies $(F)^*$ and has finite uniform dimension. As above, write $R = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n$, where each of the e_i's is primitive. Choose the indices of the summands so that for $i = 1, \dots, k$ we have Hom (Re_i, Re_j) is contained in the singular ideal of R if i is not equal to j, and for any $h > k$, Re_h embeds in one of the Re_i for $j \leq k$. Clearly, this can be done. Call a set constructed in the above manner a minimal primitive cogenerating set. When the ring satisfies $(*)^*$ and has finite uniform dimension, by [**6**] the principal indecomposables in any minimal primitive cogenerating set are all injective. We will say a ring has a unique minimal primitive cogenerating set if any two minimal primitive cogenerating sets have each element of one set isomorphic to one element of the other. In the more general setting we are concerned with, we obtain the following:

Proposition 7. *Let R be a ring satisfying $(F)^*$ with finite dimension. Let Re_i be a member of a minimal primitive cogenerating set. Let M be any finitely generated submodule of the injective hull of Re_i. Then M can be embedded in Re_i.*

Proof. We can assume that M contains Re_i. But then M is a uniform not singular module and is therefore a finitely generated indecomposable projective. It follows that M is embeddable in one of the members of the minimal primitive cogenerating set to which Re_i belongs. Since Re_i is contained in M, it follows that this member must be Re_i. □

With the hypothesis as in Proposition 7, we have the following Corollaries.

Corollary 8. *Any finitely generated submodule of $E(R)$ embeds in an f.g. free R-module.*

Proof. Let $\{Re_i, i = 1, \dots, n\} = C$ be a minimal primitive cogenerating set. Since each principal indecomposable embeds in one of the members of C, it follows that $E(R) \approx \oplus \sum E(Re_{ij})$, where each Re_{ij} is in C. If M is any f.g. submodule of $E(R)$, then clearly, $M \subset N = \oplus \sum N_{ij}$, where each N_{ij} is a cofinitely generated essential extension of Re_{ij}. Proposition 7 allows us to embed each of the N_{ij} in a finitely generated free R-module. This then gives the result. □

An R-module M is called torsionless if M embeds in a product of copies of R and is called Lambeck torsion free if it embeds in a product of copies of $E(R)$.

Corollary 9. *Every f.g. Lambeck torsion free module is torsionless.*

Proof. By considering the projections onto the direct factors of a product of copies of $E(R)$, the proof follows easily from Corollary 8. □

A left ideal L of the ring R is called dense if $\mathrm{Hom}_R(R/L, E(R)) = 0$, and is called closed if R/L is Lambeck torsion free. For any subset X of the ring R, let $l(X)$ be the left annihilator of X in R and let $r(X)$ be the right annihilator of X in R.

Corollary 10. *Let L be a left ideal of the ring R. Then*

a) *L is closed if and only if $L = l(X)$ for some set $X \subset R$;*

b) *L is dense if and only if $r(L) = 0$.*

Proof. We have that a) follows directly from Corollary 8. To prove b), notice that since R/L is finitely generated, by Corollary 8, $\mathrm{Hom}\,(R/L, E(R)) = 0$ if and only if $\mathrm{Hom}\,(R/L, R) = 0$ if and only if $r(L) = 0$. □

Corollary 11. *$E(R)$ is flat.*

Proof. Corollary 8 implies that $E(R)$ is the union of f.g. projective modules, and it is well known that this implies $E(R)$ is flat. To see this we notice that any finitely generated submodule of $E(R)$ is contained in a direct sum $\oplus \sum U_i$, $i = 1, \dots, 1$, where each U_i is a finitely generated essential extension of a principal indecomposable summand of R. By Proposition 7, we can embed each U_i in a cyclic uniform projective. It follows that the injective hull of each U_i contains a projective cyclic module which contains U_i. From this, we see that the finitely generated projective submodules of $E(R)$ are cofinal in the set of projective submodules of $E(R)$. □

Now we consider the case where R is semi-perfect and has a finitely generated radical. In this case we can show that the structure of the members of a minimal primitive cogenerating set are of three types.

Proposition 1.2. *If R is a semi-perfect ring with f.g. radical J such that $\cap J^n = 0$ and satisfies $(F)^*$, then for Re_i in a minimal primitive cogenerating set, either*

a) *Re_i is injective and a nonsingular uniserial module, or $Z(Re_i) = J^t e_i$ for some $t \leq \{$the uniform dimension of $R\}$;*

b) *Re_i is nonsingular and uniserial.*

Moreover, each principal indecomposable Re_i is either nonsingular and uniserial or for some $t > 0$ $Z(Re_i) = J^t e_i$ and all submodules of Re_i that properly contain $Z(Re_i)$ are projective.

Proof. Assume that Re_i is not injective. Then there is an element $q \in E(Re_i)$ and $q \notin Re_i$. Form $Re_i + Rq = M$. Now M is imbeddable in Re_i. If M is isomorphic to Re_i for all such q not in M, then Re_i is an fgee module and is uniserial by the proof in [**7**] of Proposition 3. If M embeds in Re_i as a proper submodule, we see that Je_i is a projective cyclic module since it contains M. We also have that under this embedding Re_i maps to a cyclic submodule T of itself. Now every submodule between T and Re_i is cyclic, projective and has a unique maximal submodule. It follows that Re_i/T is uniserial. Moreover, T must have the form $J^t e_i$ for some t. It follows that Re_i is uniserial and is nonsingular. Now assume Re_i is an injective. Then since $J^t e_i$ is uniform for all t, we have that $J^t e_i$ is either projective or singular for each $t > 0$. In case $J^t e_i$ is projective, $J^t e_i$ is cyclic and contains a unique maximal submodule. It follows that if $J^t e_i$ is not singular for all t, that Re_i is uniserial. If for some $t > 0$, $J^t e_i$ is singular, by taking the least such t we see that $J^t e_i = Z(Re_i)$. Moreover, all the submodules of Re_i which strictly contain $J^t e_i$ are projective, linearly ordered, and nonisomorphic. It follows that $t \leq \{\text{the uniform dimension of } R\}$. Finally, since each principal indecomposable embeds in one of the members of the minimal primitive cogenerating set, we see that every principal indecomposable has one of the forms cited in the theorem. □

Corollary 13. *If R is a left perfect left nonsingular ring which satisfies $(F)^*$, then R is left Artinian.*

Proof. When the ring is left perfect, then the ring has acc on left principal ideals by [**5**]. As we have seen, all f.g. submodules of a principal indecomposable are cyclic and, therefore, each of the principal indecomposable left R-modules is Noetherian and uniserial. But any left perfect left Noetherian ring is left Artinian. □

A ring R is called a left FQF-3 ring if there exists a faithful left module P which is a direct summand of every f.g. faithful module. Using the results of [**7**], we obtain the following:

Proposition 1.4. *Let R be a semi-perfect ring with f.g. radical J such that $\cap J^n = 0$. If R satisfies $(F)^*$, is essentially bounded and has a unique minimal primitive cogenerating set, then R is left FQF-3.*

Proof. For a ring which is semi-perfect, satisfies $(F)^*$, and has a unique minimal primitive cogenerating set, the elements of any minimal primitive cogenerating set are all fgees. To see this, as we have seen, if Re is an element of a minimal primitive cogenerating set with q in $E(Re)$, then $Re + Rq$ embeds in Re and is isomorphic to a principal indecomposable. We could replace Re with $Re + Rq$ and form a new minimal primitive cogenerating set. The uniqueness then gives us that Re and $Re + Rq$ are isomorphic. Now let M be an f.g. faithful module. Let A be the annihilator of $Z(M)$. We claim $Z(M)$ is f.g. Since M is faithful and f.g. and R is bounded, we see that $Z(M)$ is not M. We can write $M = P \oplus Y$, where $Z(Y) = Y$ and P is an f.g. projective. We have that Y is f.g. We claim that $Z(P)$ is f.g. Since P is isomorphic to a direct sum of principal indecomposables, the singular submodule of P is isomorphic to a finite direct sum of modules of the form $J^t e_i$ for primitive idempotents e_i, and each of these is f.g. since J is f.g. Since R is essentially bounded, we have that A is essential as a two-sided ideal. Now let Re be a member of a minimal primitive cogenerating set. If the kernel of every map of Re into M is essential, it follows that Re embeds into the product of copies of $Z(M)$. But this implies that Re is annihilated by A, which in turn implies that Re is singular, a contradiction. So each member of a minimal primitive cogenerating set is imbeddable in M; and since each member of a minimal primitive cogenerating set is a projective fgee, it follows that M contains a direct summand isomorphic to the direct sum of the members of a minimal primitive cogenerating set. This direct sum is then a faithful module which is isomorphic to a direct summand of each f.g. faithful module, and our ring is left FQF-3. □

A similar result holds for nonsingular rings which satisfy $(F)^*$.

Proposition 15. *Let R be a left nonsingular ring which satisfies $(F)^*$ and has finite uniform dimension. If R is essentially bounded and has a unique minimal primitive cogenerating set, then R is left* FQF-3.

Proof. Let M be any f.g. faithful module. As M is faithful and R is essentially bounded, M is the direct sum of a projective module and singular module. By arguing as we did in the proof of Proposition 9, we see that each member of a minimal primitive cogenerating set must embed in the projective part as a direct summand. Again, each member of a minimal primitive cogenerating set is a fgee, no two of which are isomorphic; and so it follows that any f.g. faithful projective module contains a direct summand isomorphic to a direct sum of the members of a minimal primitive cogenerating set.

There is a class of rings which have all their finitely generated modules behaving very much like finite Abelian groups. Call a ring a CFQF-3 ring if every homomorphic image of R is FQF-3. □

Proposition 16. *Let R be a semi-perfect left* CFQF-3 *ring with radical J such that $\cap J^n = 0$. If M is a f.g. left R-module, then M is isomorphic to $P_1 \oplus P_2 \cdots \oplus P_n$, where each P_i, $i = 1, \dots, n$ is projective over R/A_i with A_i the annihilator of P_i. Moreover, each of the P_i is the direct sum of uniform modules which are either uniserial or injective over R/A_i.*

Proof. In order to prove the proposition, we need to show that every homomorphic image $T = R/A$ of the ring R has the property that $\cap K^n = 0$, where K is the Jacobson radical of T. We claim that K is $(A + J)/A$. To see this, write $R = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n$ with the e_i's primitive orthogonal idempotents. Now suppose x is in K and not in $A + J$. We can write $x = r_1e_1 + r_2e_2 + \cdots + r_ne_n$. Since K is a two-sided ideal, xe_1 is in K as well. If xe_1 is not in J, then K contains Re_1 which is an idempotent left ideal. It follows that e_1 must be in A, and therefore that xe_1 is in A or in J. We can argue in a similar manner for each of the terms xe_i, from which it follows that x was in $A + J$.

We can now proceed to decompose any f.g. module M over R as follows. Let A_1 be the annihilator of M and form $T_1 = R/A_1$ which is left FQF-3. Since M is faithful as a T_1-module, we can find a T_1-projective direct summand P_1 of M so that $M = P_1 \oplus N_2$. Now consider N_2 and let A_2 be the annihilator of N_2. Let T_2 be R/A_2. Since T_2 is left FQF-3, N_2 has a direct summand which is projective over T_2. We can continue in this manner for only a finite number of steps since M is finitely generated and the ring is semi-perfect. The structure of the P_i is given by Proposition 3 of [**7**] since at each stage we can take the P_i to be the minimal faithful module of the ring T_i. □

It should be noted that if R is a left nonsingular, left FQF-3 ring with finite uniform dimension, then Corollary 8 holds and, therefore, so do Corollaries 9–11.

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