ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 4, Fall 1992

DISSIPATIVE DECOMPOSITION OF PARTIAL DIFFERENTIAL EQUATIONS

PETER J. OLVER AND CHEHRZAD SHAKIBAN

ABSTRACT. A general decomposition theorem that allows one to express an arbitrary differential polynomial as a sum of conservative, dissipative and higher order dissipative pieces is proved. The decomposition generalizes the dissipative decomposition of ordinary differential equations, but is no longer unique. The proof relies on the properties of certain generalizations of the standard symmetric polynomials known as multi-symmetric polynomials. The nonuniqueness of the decomposition is a consequence of the syzygies among the power sum multi-symmetric polynomials.

1. Introduction. In a previous paper, [**14**], we proved that any polynomial ordinary differential equation in one independent and one dependent variable can be decomposed into a conservative part, a dissipative part and higher order dissipative pieces. Subject to certain homogeneity requirements, the decomposition is unique; in particular, it determines a unique conservative component of such an equation. The goal of this paper is to investigate to what extent the conservative/dissipative decomposition of nonlinear ordinary differential equations generalizes to partial differential equations. We will prove that an analogous decomposition always exists for polynomial partial differential equations in one dependent variable, but there is no longer a corresponding uniqueness result. The present proof of the decomposition theorem relies on a transform developed by Gel'fand and Dikii [**4**], and the second author, [**15**], which, in analogy with the classical Fourier transform, reduces problems in differential algebra to problems in commutative algebra. In our case, the problem transforms to a result in the theory of "multi-symmetric polynomials," which are certain multi-variable generalizations of the standard symmetric polynomials studied by, among others, Junker [**6, 7, 8**], and MacMahon [**9**] almost a century ago. The proof of the general dissipative decomposition then rests on some basic formulas relating the multi-symmetric analogs of

Received by the editors on September 18, 1990.
Research of the first author supported in part by NSF Grant DMS 89-01600.
Research of the second author supported in part by NSF Grant DMS 89-07578.

power sum and elementary symmetric polynomials. The nonuniqueness of the decomposition in the partial differential equation case is a consequence of the fact that, in contrast to the usual elementary symmetric polynomials, their multi-symmetric analogs no longer freely generate the ring of multi-symmetric polynomials, and there exist nontrivial syzygies among the generators. The general form of these syzygies is poorly understood. A table of dissipative decompositions for some low order and low degree polynomial partial differential equations in one dependent variable and two independent variables is given.

Consider the differential algebra $\mathcal{A}^m = \mathcal{A}\{u, x\}$ which consists of all constant coefficient partial differential polynomials $P[u]$ in one dependent variable, u, and m independent variables, $x = (x^1, x^2, \dots, x^m)$. Thus, $\mathcal{A}^m$ consists of all polynomials $P[u]$ depending on finitely many of the derivative variables

$$u_I = \partial_{i_1}\partial_{i_2}\dots\partial_{i_v}u, \qquad \partial_i = \partial/\partial x^i.$$

Here we employ one of the standard multi-index notations for derivatives, so $I = (i_1, i_2, \dots, i_v)$, $1 \le i_v \le m$, is a symmetric multi-index of order $|I| = v$. We let $\mathcal{A}^m_n \subset \mathcal{A}^m$, $n \ge 0$, denote the subspace consisting of all homogeneous differential polynomials of degree n in the variables u_I.

The most important linear operators on $\mathcal{A}^m$ are the total derivatives and the Euler operator or variational derivative. We let D_k denote the total derivative with respect to x^k and $D^I = D_{i_1}\dots D_{i_v}$ denote the corresponding higher order total derivative. The Euler operator or variational derivative has the usual formula

$$E = \sum_I (-D)^I \frac{\partial}{\partial u_I}, \tag{1.1}$$

cf. [**12**]. Note that the total derivatives preserve homogeneity, whereas the Euler operator reduces it by one:

$$D^I : \mathcal{A}^m_n \to \mathcal{A}^m_n, \qquad E : \mathcal{A}^m_n \to \mathcal{A}^m_{n-1}.$$

A partial differential equation $P[u] = 0$ is called *conservative* if it is the Euler-Lagrange equation for some variational principle, meaning $P = E(L)$ for some *Lagrangian* $L[u]$. Conservative differential polynomials are completely characterized by the Helmholtz conditions using the well-known solution to the inverse problem to the calculus of variations, cf. [**12**, Theorem 5.68].

The main result of this paper is the following decomposition theorem.

Theorem 1. *Let $P \in \mathcal{A}_n^m$ be a homogeneous differential polynomial of degree n in one dependent variable and m independent variables. Then there exist conservative differential polynomials*

$$Q_I = E(L_I) \in \mathcal{A}_n^m, \qquad 0 \le |I| \le n,$$

such that P can be decomposed as

$$P = \sum_{|I|=0}^{n} D^I Q_I = \sum_{|I|=0}^{n} D^I E(L_I). \tag{1.2}$$

This theorem is the direct analog of the dissipative decomposition for ordinary differential equations proved in [**14**]. In that case, $m = 1$, the differential polynomials Q_I were uniquely determined by P. In the present case, $m > 1$, this is no longer true. In fact, the nonuniqueness is related to some interesting combinatorial identities among multivariable analogs of the elementary symmetric polynomials.

For example, according to Theorem 1, any quadratic partial differential equation $P[u] = 0$, depending on $u = u(x, y)$, can always be written in the form

$$\begin{aligned} P &= Q_0 + D_x Q_1 + D_y Q_2 + D_x^2 Q_3 + D_x D_y Q_4 + D_y^2 Q_5 \\ &= E[L_0] + D_x E[L_1] + D_y E[L_2] \\ &\quad + D_x^2 E[L_3] + D_x D_y E[L_4] + D_y^2 E[L_5]. \end{aligned} \tag{1.3}$$

There are six Lagrangians, $L_0, L_1, L_2, L_3, L_4, L_5$, each determined by the corresponding Q_i up to a divergence. The first, L_0, can be identified with the Lagrangian for the conservative component of the problem, while L_1 and L_2 determine the *first order dissipation* and L_3, L_4 and L_5 give *second order dissipation*.

As a specific example, consider the differential polynomial $P = u_y u_{xxx} - u_x u_{xxy}$. One decomposition is given by

$$\begin{aligned} u_y u_{xxx} - u_x u_{xxy} &= (u_y u_{xxx} + 2u_x u_{xxy} + 3u_{xy} u_{xx}) \\ &\quad - (3u_x u_{xxy} + 3u_{xy} u_{xx}) \\ &= D_x(u_y u_{xx} + 2u_x u_{xy}) + D_y(-3u_x u_{xx}) \\ &= D_x E\left[-\frac{1}{2} u_x^2 u_y \right] + D_y E\left[\frac{1}{2} u_x^3 \right]. \end{aligned}$$

In this case, we interpret the differential polynomial P as purely first order dissipative, and we can take the first order dissipation to be given by the Lagrangians

$$L_1 = -\frac{1}{2} u_x^2 u_y \quad \text{and} \quad L_2 = \frac{1}{2} u_x^3,$$

while the rest of the Lagrangians are trivial and can be set to zero. However, the representation (1.3) is not unique since we also have

$$\begin{aligned} u_y u_{xxx} - u_x u_{xxy} &= D_x^2(-2uu_{xy} - u_y u_x) + D_x D_y (2uu_{xx} + u_x^2) \\ &= D_x^2 E[uu_x u_y] + D_x D_y E[-uu_x^2]. \end{aligned}$$

This gives P the interpretation of a second order dissipative equation. The difference between these two interpretations, and how they are related to Rayleigh-type dissipation laws [**14**], remains to be worked out.

Finally, we remark that there is an analogous decomposition theorem for the homogeneous subalgebras $\tilde{\mathcal{A}}_n^m$ of the larger differential algebra $\tilde{\mathcal{A}}^m$ consisting of all partial differential polynomials with smooth, x-dependent coefficients. Indeed, the methods used in [**14**] to go from the constant coefficient ordinary differential polynomial case to the variable coefficient case work without change in the present context. We leave it to the reader to fill in the details.

2. Multi-symmetric polynomials. We assume that the reader is familiar with the elementary theory of symmetric polynomials, namely polynomial functions $f(z_1, \ldots, z_n)$ of a single set of variables which are symmetric under permutations, cf. [**10**]. We will require the generalization of these concepts to "multi-symmetric polynomials," which are, analogously, functions of several sets of variables which are symmetric under simultaneous permutations of the variables in each set. In the case of ordinary symmetric functions, the space of symmetric polynomials is isomorphic to a polynomial ring, and the elementary symmetric polynomials freely generate it. In other words, the elementary symmetric polynomials are algebraically independent, and every symmetric polynomial can be expressed as a unique polynomial function of them. In the case of multi-symmetric polynomials, it is easy to write down the analogs of the elementary symmetric polynomials, and, again, these generate the ring. However, the key difference is that the generators are no longer algebraically independent, since the ring is no longer isomorphic to a polynomial ring. Thus, every multi-symmetric polynomial can be written as a polynomial in the elementary multi-symmetric polynomials, but this polynomial is no longer necessarily unique. For our purposes, it is convenient to also introduce another generating set, the power sum symmetric polynomials and their multi-symmetric analogs. These are easily related to the elementary symmetric polynomials via standard generating function techniques.

Consider the polynomial algebra $\mathbf{C}[Z]$, where $Z = (z_j^k)$, $1 \le k \le m$, $1 \le j \le n$, is an $n \times m$ matrix whose entries are $m \cdot n$ independent unknowns. We use the notation $z_j = (z_j^1, \dots , z_j^m)$ to denote the j^{th} row and $z^k = (z_1^k, \dots , z_n^k)$ to denote the k^{th} column of Z. The symmetric group S_n acts on the matrix Z by permuting its rows and, hence, there is an induced action on $\mathbf{C}[Z]$. We let $\mathcal{P}_n^m = \mathbf{C}[Z]^{S_n}$ denote the subspace of S_n-invariant polynomials. In the case $m = 1$, the ordinary differential equation case, the polynomials in $\mathcal{P}_n^1$ are the ordinary symmetric polynomials in n scalar variables $(z_1, \dots , z_n)$. When $m > 1$, the elements of $\mathcal{P}_n^m$ will be called *multi-symmetric polynomials* as they are invariant under simultaneous permutation of the entries of the columns of Z. Based on the transform, multi-symmetric polynomials have been used effectively in the solution of a number of problems in differential algebra, cf. [**2, 4, 15**]. They were first studied by several mathematicians in the last century; see Junker [**6, 7, 8**] and Netto [**11**, Sections 377–386], for connections with classical invariant theory, and MacMahon [**9**] for connections with combinatorics. They have recently been the subject of renewed interest among combinatorialists, in part due to applications in the cohomology theory of the symmetric group [**1**]; see the work of Gessel [**5**] and Edelman [**3**].

Let

$$\sigma = \frac{1}{n!} \sum_{\pi \in S_n} \pi : \mathbf{C}[Z] \to \mathcal{P}_n^m$$

be the symmetrizing map, which is a projection onto $\mathcal{P}_n^m$. A simple vector space basis of $\mathcal{P}_n^m$ is provided by the monomial symmetric polynomials

$$\begin{aligned} m_{\mathbf{I}}(Z) &= \sigma(z^{\mathbf{I}}) = \sigma(z_1^{I_1} \dots z_n^{I_n}) \\ &= \frac{1}{n!} \sum_{\pi \in S_n} z_{\pi(1)}^{I_1} \dots z_{\pi(n)}^{I_n} \in \mathcal{P}_n^m, \end{aligned} \tag{2.1}$$

where $\mathbf{I} = (I_1, I_2, \dots, I_n)$ is a symmetric "multi-multi-index" whose entries are symmetric multi-indices. Also, given $z_j = (z_j^1, \dots, z_j^m)$, and a symmetric multi-index $I = (i_1, \dots, i_v)$, we define $z_j^I = z_j^{i_1} \dots z_j^{i_v}$.

There are two important "multiplicative" bases or generating sets for $\mathcal{P}_n^m$, being the analogs of the elementary symmetric polynomials and the power sums in the one variable case. Using dummy variables $t = (t^1, t^2, \dots, t^m)$, the generating function for the *elementary multi-symmetric polynomials* is

$$S(t) = \prod_{j=1}^{n} \left(1 + \sum_{i=1}^{m} t^i z_j^i \right) = 1 + \sum_{|I|=1}^{n} e_I(Z) t^I. \tag{2.2}$$

Note that e_I coincides with the monomial symmetric polynomial $m_{\mathbf{I}}$ when $\mathbf{I}$ is the multi-multi-index with singleton entries $I_v = (i_v)$ determined by the entries of I. Define the *power sum multi-symmetric polynomials*

$$p_I(Z) = m_I(Z) = \sum_{j=1}^{n} z_j^I. \tag{2.3}$$

For simplicity, we denote the linear multi-symmetric polynomials (which can be viewed as either monomial, elementary or power sum multi-symmetric polynomials) by

$$\mathfrak{l}_k(Z) = m_k(Z) = e_k(Z) = p_k(Z) = \sum_{j=1}^{n} z_j^k, \qquad k = 1, \dots, m. \tag{2.4}$$

For example, in the case $m = n = 2$, the polynomials in $\mathcal{P}_2^2$ will depend on a 2×2 matrix Z of variables, for which we can use the simplified notation

$$Z = \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix}. \tag{2.5}$$

The linear multi-symmetric functions are

$$p_1 = \mathfrak{l}_1 = z_1 + z_2, \qquad p_2 = \mathfrak{l}_2 = w_1 + w_2. \tag{2.6}$$

There are three quadratic power sum symmetric functions:

$$p_{11} = z_1^2 + z_2^2, \qquad p_{12} = z_1 w_1 + z_2 w_2, \qquad p_{22} = w_1^2 + w_2^2, \tag{2.7}$$

and three quadratic elementary symmetric functions:

$$e_{11} = z_1 w_1, \qquad e_{12} = z_1 w_2 + z_2 w_1, \qquad e_{22} = z_2 w_2.$$

A generating function for the power sums is provided by the formal power series

$$P(t) = \log S(t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{|I|=k} \binom{k}{I} p_I(z) t^I. \tag{2.8}$$

Substituting the formulas for P and S into the relation $S(t) = e^{P(t)}$, we deduce that for $|I| \leq n$, e_I is a polynomial in the power sums p_J for $|J| \leq |I|$ of the form

$$\begin{aligned} e_I &= \Phi_I[p_J] \\ &= (-1)^{k-1} \frac{(k-1)!}{I!} p_I + \cdots + \frac{1}{I!} \mathfrak{l}_I, \qquad |I| = k \leq n, \end{aligned} \tag{2.9}$$

where

$$\mathfrak{l}_I(Z) = \mathfrak{l}_{i_1}(Z) \mathfrak{l}_{i_2}(Z) \ldots \mathfrak{l}_{i_k}(Z), \qquad I = (i_1, i_2, \ldots, i_k). \tag{2.10}$$

On the other hand, for $|I| > n$, the corresponding coefficient of t^I in $S(t)$ is 0, and hence we deduce the syzygy

$$\begin{aligned} 0 &= (-1)^{n-1} \frac{I!}{(k-1)!} \Phi_I[p_J] \\ &= p_I + \cdots + \frac{(-1)^{k-1}}{(k-1)!} \mathfrak{l}_I, \qquad |I| = k > n, \end{aligned} \tag{2.11}$$

for the higher order power sums. A simple induction shows that we can ultimately re-express the higher order power sums in terms of those of order $\leq n$.

Theorem 2. *The elementary multi-symmetric polynomials* $\{e_I : |I| \leq n\}$ *and the power sum multi-symmetric polynomials* $\{p_I : |I| \leq n\}$ *up to order* n*, form generating sets for the space of multi-symmetric polynomials* $\mathcal{P}_n^m$*. Thus, any multi-symmetric polynomial can be written as a polynomial in either the elementary multi-symmetric polynomials, or the power sum multi-symmetric polynomials up to order* n*.*

For a proof of this result, see [**5, 11**]. If $m = 1$, these generating sets are algebraically independent, and the ring of ordinary symmetric polynomials is isomorphic to a polynomial ring. However, for $m > 1$, this is *not* the case, and there are nontrivial syzygies amongst the elementary multi-symmetric polynomials and amongst the power sum multi-symmetric polynomials. See Junker [**6, 7, 8**] for a detailed investigation into the syzygies. The simplest such syzygy is in the case $n = 2$, where, for the power sum multi-symmetric polynomials the general syzygy takes the form

$$2(p_{ij}p_{k\mathfrak{l}} - p_{ik}p_{j\mathfrak{l}}) = p_i p_j p_{k\mathfrak{l}} + p_k p_{\mathfrak{l}} p_{ij} - p_i p_k p_{j\mathfrak{l}} - p_j p_{\mathfrak{l}} p_{ik}. \tag{2.12}$$

Thus, in $\mathcal{P}_2^2$ there is one such syzygy,

$$2(p_{11}p_{22} - p_{12}^2) = p_1^2 p_{22} - 2p_1 p_2 p_{12} + p_2^2 p_{11}, \tag{2.13}$$

and this is the *only* syzygy, which we verified using a Gröbner basis calculation in the computer algebra language MACAULAY [**16**]. For $\mathcal{P}_2^3$ there are six such syzygies, and again these exhaust the possibilities. (Interestingly, the Gröbner basis calculation gives 10 or 11 basic syzygies (depending on which term ordering is used), but the extra 4 or 5 syzygies are all consequences of the first 6.) We conjecture that (2.12) provides a complete list of syzygies for all m when $n = 2$. (Paul Edelman, [**3**], informs us that this conjecture is true.) The higher order cases are considerably more complicated. A similar symbolic calculation in the case of $\mathcal{P}_3^2$ reveals 5 fundamental syzygies, of which the simplest are

$$p_1^2 p_{122} - 2p_1 p_2 p_{112} + p_2^2 p_{111} = 3(p_{11}p_{122} - 2p_{12}p_{112} + p_{22}p_{111}) - 4p_1(p_{11}p_{22} - p_{12}^2) + p_1^3 p_{22} - p_1^2 p_2 p_{12} + p_1 p_2^2 p_{11}, \tag{2.14}$$

and the analogous one obtained by interchanging the indices 1 and 2. Each of these 5 syzygies appears to follow from two general syzygies, like (2.12), valid for all m when $n = 3$. The complete investigation of the syzygies will be the subject of future research.

3. The transform method. The proof of the main theorem relies on a transform which takes differential polynomials into multi-symmetric polynomials and, thereby, like the Fourier transform, changes problems in the study of differential equations into problems in commutative algebra. A special case of this transform was originally introduced by Gel'fand and Dikii [**4**], and was further developed by the second author [**15**]. It is closely related to the symbolic method of classical invariant theory, cf. [**13**]. We adopt the simplified formulation of the transform given in [**2**] and, especially, [**13**].

Definition 3. The *transform* is the unique linear isomorphism

$$\mathcal{F} : \mathcal{A}_n^m \xrightarrow{\sim} \mathcal{P}_n^m,$$

between the space of homogeneous constant coefficient differential polynomials of degree n in one dependent variable and m independent variables, and the corresponding space of multi-symmetric polynomials of an $n \times m$ matrix of variables Z, with the property that the transform of a differential monomial

$$u_{\mathbf{I}} = u_{I_1} u_{I_2} \dots u_{I_n}, \qquad \mathbf{I} = (I_1, I_2, \dots, I_n),$$

of degree n is the monomial multi-symmetric polynomial

$$\mathcal{F}[u_{\mathbf{I}}] = m_{\mathbf{I}}(Z) = \sigma(z^{\mathbf{I}}) \in \mathcal{P}_n^m, \tag{3.1}$$

cf. (2.1). The elementary fact that $\mathcal{F}$ determines a linear isomorphism is proved in [**15**].

Example 4. For the differential polynomial $P = u_x u_{xxy} \in \mathcal{A}_2^2$, its transform will be a polynomial in the 2×2 matrix of variables, for which we use the simplified notation

$$Z = \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix}.$$

(Note that the z's correspond to x derivatives and the w's to y derivatives.) According to (3.1), we find

$$\mathcal{F}(P) = \sigma(z_1 z_2^2 w_2) = \frac{1}{2}(z_1 z_2^2 w_2 + z_1^2 z_2 w_1).$$

If $P \in \mathcal{A}_n^m$ is any differential polynomial, then we shall denote its transform by $\mathcal{F}[P] = \widehat{P}$. Similarly, if $\psi : \mathcal{A}_n^m \to \mathcal{A}_q^p$ is any linear map, then its transform $\widehat{\psi} : \mathcal{P}_n^m \to \mathcal{P}_q^p$ will be the unique linear map satisfying $\widehat{\psi}(\widehat{P}) = \mathcal{F}[\psi(P)]$ for all $P \in \mathcal{A}_n^m$. In particular, the transforms of the total derivatives and Euler operator are well known [**15, 13**].

Lemma 5. *Given $\widehat{P}(Z) \in \mathcal{A}_n^m$, we have*

(3.2)
$$i)\quad \widehat{D}_j \widehat{P}(Z) = (z_j^1 + z_j^2 + \cdots + z_j^n)\widehat{P}(Z) = \iota_j(Z)\widehat{P}(Z) \in \mathcal{A}_n^m, \qquad j = 1, 2, \ldots, m,$$

(3.3)
$$ii)\quad \widehat{E}\widehat{P}(Z) = n\widehat{P}(z_1, \ldots, z_{n-1}, -z_1 - z_2 - \cdots - z_{n-1}) \in \mathcal{A}_{n-1}^m.$$

(In the second formula (3.3), recall that the z_j's denote the rows of the matrix Z.)

4. Proof of the decomposition theorem. We begin by presenting a new proof of the decomposition theorem in the ordinary differential equation case $m = 1$ using the standard theory of symmetric polynomials. This will make the partial differential equation case, $m > 1$, and the connections with the theory of multi-symmetric polynomials, more readily comprehensible. Furthermore, the uniqueness and nonuniqueness of the decomposition in the two cases will have a clear counterpart in the different algebraic properties of these two types of symmetric polynomials. Here we concentrate on the case of constant coefficient differential polynomials; the extension to variable coefficient differential polynomials is not hard and proceeds along the same lines as the ordinary differential equation case discussed in [**14**].

Given a homogeneous ordinary differential polynomial $P = P(u, u_x, u_{xx}, \dots)$ of degree n, we must prove that we can write P in the form

$$P = \sum_{i=0}^{n} D_x^i Q_i = \sum_{i=0}^{n} D_x^i E(L_i), \tag{4.1}$$

for unique conservative differential polynomials $Q_i = E(L_i)$, $0 \le i \le n$, where each L_i has degree $n+1$. Let us apply the transform to the above decomposition. The transforms of P and the L_i have the form

$$\widehat{P} = \widehat{P}(z_1, z_2, \dots, z_n), \qquad \widehat{L}_i = \widehat{L}_i(z_1, z_2, \dots, z_{n+1}),$$

where the z_j's are scalar variables since $m = 1$. According to Lemma 5,

$$\widehat{D}_x^i \widehat{E}(\widehat{L}_i) = n(z_1 + z_2 + \cdots + z_n)^i \widehat{L}_i(z_1, z_2, \dots, z_n, -z_1 - z_2 - \cdots - z_n).$$

Therefore (4.1) transforms into the following formula

$$\widehat{P}(z_1, \dots, z_n) = n \sum_{i=0}^{n} (z_1 + \cdots + z_n)^i \widehat{L}_i(z_1, \dots, z_n, -z_1 - \cdots - z_n). \tag{4.2}$$

Since the transform is a linear isomorphism, it suffices to prove the polynomial decomposition formula (4.2).

For the proof of (4.2), we introduce a slight modification of the (usual) power sum symmetric polynomials. Using superscripts to denote powers, let

$$p_k(z_1, \dots, z_n) = \sum_{j=1}^{n} z_j^k = z_1^k + z_2^k + \cdots + z_n^k$$

denote the k^{th} power sum of the n variables $z_1, \dots, z_n$, and let

$$\tilde{p}_k(z_1, \dots, z_{n+1}) = \sum_{j=1}^{n+1} z_j^k = z_1^k + z_2^k + \cdots + z_{n+1}^k$$

be the corresponding power sum in $n+1$ variables. (The extra variable z_{n+1} is introduced because the Euler operator drops the degree of differential polynomials by one.) Using (3.3), we define the symmetric polynomials

$$q_k(z_1, z_2, \dots, z_n) = \widehat{E}(\tilde{p}_k) = p_k + (-1)^k p_1^k. \tag{4.3}$$

Note in particular $q_1 = 0$. We use

$$Q(t) = \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} q_k t^k \tag{4.4}$$

as the generating function for the q's. Comparing with (2.8), we find

$$Q(t) = P(t) - \log(1 - p_1 t);$$

hence, $Q(t)$ is related to the generating function (2.2) for the elementary symmetric polynomials according to the formula

$$S(t) = \frac{1}{1 - p_1 t} \cdot \exp Q(t). \tag{4.5}$$

Let

$$\exp Q(t) = 1 + \sum_{k=2}^{\infty} B_k(q_2, \dots, q_k) t^k.$$

(The B_k's are explicitly given in terms of the Bell polynomials, [**10**].) Then, in analogy with (2.6), we deduce relationships of the form

$$\begin{aligned} e_k = {} & B_k(q_2, \dots, q_k) + p_1 B_{k-1}(q_2, \dots, q_{k-1}) + \cdots \\ & + p_1^{k-2} B_2(q_2) + p_1^k, \qquad k \le n, \end{aligned} \tag{4.6}$$

relating the elementary symmetric polynomials to the polynomials $p_1, q_2, \dots, q_k$, and syzygies

$$\begin{aligned} 0 = {} & B_k(q_2, \dots, q_k) + p_1 B_{k-1}(q_2, \dots, q_{k-1}) + \cdots \\ & + p_1^{k-2} B_2(q_2) + p_1^k, \qquad k > n, \end{aligned} \tag{4.7}$$

among the higher order q's. A simple induction using these latter identities will give the key formulae for the higher order powers of p_1 in terms of the q's and the powers of p_1 up to degree n:

$$p_1^k = C_k^0(q_2, \dots, q_k) + p_1 C_k^1(q_2, \dots, q_k) + \cdots + p_1^n C_k^n(q_2, \dots, q_k), \qquad k > n, \tag{4.8}$$

for certain polynomials C_k^i whose precise expressions can be determined, but are not essential for our proof.

Clearly, since the first n power sums generate the ring of symmetric polynomials $\mathcal{P}_n^1$, the polynomials $p_1, q_2, \dots, q_n$ also generate it. Furthermore, if $\widehat{L} \in \mathcal{P}_{n+1}^1$ is rewritten in terms of the power sums:

$$\widehat{L}(z_1, z_2, \dots, z_n) = \Phi(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_N), \tag{4.9}$$

(actually, we only need to take $N = n+1$, but the argument here works in general) then, using (3.3),

$$\widehat{E}(\widehat{L}) = n\Phi(0, q_2, \dots, q_N). \tag{4.10}$$

Conversely, if the differential polynomial $P \in \mathcal{A}_n^1$ is such that its transform $\widehat{P} \in \mathcal{P}_n^1$ can be rewritten in terms of the q's only:

$$\widehat{P} = \Xi(q_2, \dots, q_N), \tag{4.11}$$

then $P = E(L)$ is conservative, where we can use the Lagrangian with transform

$$\widehat{L} = \frac{1}{n}\Xi(\tilde{p}_2, \dots, \tilde{p}_N). \tag{4.12}$$

To prove (4.2), we begin by re-expressing $\widehat{P}$ in terms of the generators $p_1, q_2, \dots, q_n$, so that $\widehat{P} = \Psi(p_1, q_2, \dots, q_n)$. We then use Taylor's theorem to expand Ψ in its first argument:

$$\Psi(p_1, q_2, \dots, q_n) = \Psi_0(q_2, \dots, q_n) + p_1\Psi_1(q_2, \dots, q_n) + p_1^2\Psi_2(q_2, \dots, q_n) + \cdots + p_1^N\Psi_N(q_2, \dots, q_n). \tag{4.13}$$

(The series terminates since Ψ is a polynomial.) We now substitute the identities (4.8) into the Taylor expansion (4.13) to eliminate the higher order powers of p_1; the result will be an expression of the form

$$\begin{aligned}\Psi(p_1, q_2, \dots, q_n) = \Xi_0(q_2, \dots, q_N) + p_1 \Xi_1(q_2, \dots, q_N) \\ + \cdots + p_1^n \Xi_n(q_2, \dots, q_N).\end{aligned} \tag{4.14}$$

Now according to the above remarks, (4.9–12), each $\Xi_j(q_2, \dots, q_N)$ is conservative, and we can write

$$\Xi_j(q_2, \dots, q_N) = \widehat{E}(\widehat{L}_j) \quad \text{where} \quad \widehat{L}_j = \frac{1}{n} \Xi_j(\tilde{p}_2, \dots, \tilde{p}_N).$$

With this definition of the $\widehat{L}_j$, formula (4.14) coincides with the desired algebraic decomposition (4.2). This completes the proof of Theorem 1 for constant coefficient ordinary differential polynomials.

The proof of uniqueness can also be effected in the transform space, using the fact that the power sums $p_1, q_2, \dots, q_n$ are independent. If we rewrite (4.5) in the form

$$(1 - p_1 t) S(t) = \exp Q(t), \tag{4.15}$$

this yields the formulae

$$\begin{aligned} e_j - p_1 e_{j-1} &= B_j(q_2, \dots, q_n), \qquad j = 2, \dots, n, \\ -p_1 e_n &= B_{n+1}(q_2, \dots, q_{n+1}), \\ 0 &= B_{n+k}(q_2, \dots, q_{n+k}), \qquad k \geq 2. \end{aligned}$$

An easy induction based on the fact that the leading term of the Bell polynomial B_k is $(-1)^k q_k / k$ proves that

$$q_{n+1} = A_0(q_2, \dots, q_n) + p_1 A_1(q_2, \dots, q_n) + \cdots + (-1)^n (n+1) p_1^{n+1}, \tag{4.16}$$

while

$$q_{n+k} = H_k(q_2, \dots, q_n), \tag{4.17}$$

for certain functions $A_0, \dots, A_n$, $H_1, H_2, \dots$, depending on $q_2, \dots, q_n$, whose explicit forms are not required for the proof.

Now, if the decomposition were not unique, we would have an identity of the form

$$\Xi_0(q_2,\dots,q_N)+p_1\Xi_1(q_2,\dots,q_N)+\cdots+p_1^n\Xi_n(q_2,\dots,q_N)=0. \tag{4.18}$$

A slight complication at this point is the fact that the Ξ's may be nonzero since the q's are not all algebraically independent. However, replacing each q_k for $k \geq n+2$ by the function H_k according to (4.17), (which is essentially an "integration by parts" in the transform space) we deduce a corresponding identity of the form

$$\Gamma_0(q_2,\dots,q_{n+1})+p_1\Gamma_1(q_2,\dots,q_{n+1})+\cdots+p_1^n\Gamma_n(q_2,\dots,q_{n+1})=0. \tag{4.19}$$

Uniqueness now amounts to proving that the Γ's must all vanish, since $q_2,\dots,q_{n+1}$ are independent. If this were not the case, using (4.16), we would derive a nontrivial polynomial identity among the power sums $p_1,q_2,\dots,q_n$, since q_{n+1} gets replaced by a polynomial of degree $n+1$ in p_1. This contradiction implies that the identity (4.19) is trivial and hence the decomposition is unique.

Example 6. Consider the quadratic monomial $P = uu_{xx}$. Applying the transform to P, we get

$$\widehat{P}=\frac{1}{2}(z_1^2+z_2^2)=\frac{1}{2}p_2=\frac{1}{2}q_2-\frac{1}{2}p_1^2=\widehat{E}\left[\frac{1}{6}\tilde{p}_2\right]+\widehat{D}_x^2\widehat{E}\left[-\frac{1}{6}\right].$$

Inverting the transform, we find the known decomposition [**14**],

$$uu_{xx}=E\left(\frac{1}{2}u^2u_{xx}\right)+D_x^2E\left(-\frac{1}{6}u^3\right)=E(-uu_x^2)+D_x^2E\left(-\frac{1}{6}u^3\right).$$

The proof of Theorem 1 in the partial differential equation case, $m > 1$, now proceeds along essentially identical lines. First we apply the transform to the desired decomposition (1.2). We have

$$\widehat{P}=\widehat{P}(Z)\in\mathcal{P}_n^m,\qquad \widehat{L}_I=\widehat{L}_I(\tilde{Z})\in\mathcal{P}_{n+1}^m,$$

where Z and $\tilde{Z}$ are, respectively, $n\times m$ and $(n+1)\times m$ matrices of variables. According to Lemma 5,

$$\widehat{D}^I\widehat{E}(\widehat{L}_I)=n\mathfrak{l}_I(Z)\widehat{L}_I(Z,-z_1-z_2-\cdots-z_n),$$

cf. (2.10), (3.2), (3.3). Therefore, the decomposition (1.2) transforms into the following formula

$$\widehat{P}(Z) = n \sum_{|I|=0}^{n} \iota_I(Z)\widehat{L}_I(Z, -z_1 - z_2 - \cdots - z_n). \tag{4.20}$$

As in the ordinary differential equation case, define the modified power sums

$$q_I(Z) = \widehat{E}(p_I(\tilde{Z})) = p_I(Z) + (-1)^{|I|}\iota_I(Z). \tag{4.21}$$

In particular, $q_I = 0$ if $|I| = 1$. Clearly, since the first n power sums generate the ring of multi-symmetric polynomials $\mathcal{P}_n^m$, the collection of polynomials $\iota_k(Z)$, $1 \le k \le m$, $q_I(Z)$, $2 \le |I| \le n$, also serve as generators. Furthermore, if $\widehat{L} \in \mathcal{P}_{n+1}^m$ is rewritten in terms of the power sums:

$$\widehat{L}(\tilde{Z}) = \Phi(p_J(\tilde{Z})),$$

then

$$\widehat{E}(\widehat{L}) = n\Phi(q_J(Z)), \tag{4.22}$$

and, conversely, cf. (3.3).

To prove (4.20), we begin by noting that, in analogy with (4.8), we can write higher order powers of the linear symmetric polynomials in terms of the powers up to degree n and the polynomials (4.21):

$$\iota_K = \sum_{|I|=0}^{n} \iota_I B_K^I(p_J) = \sum_{|I|=0}^{n} \iota_I C_K^I(q_J), \qquad |K| > n. \tag{4.23}$$

Given P, we re-express its transform $\widehat{P}$ in terms of the basis polynomials:

$$\widehat{P} = \Psi(\iota_k, q_J), \qquad \text{where } k = 1, \dots, m, \qquad 1 \le |J| \le n.$$

We use Taylor's theorem to expand Ψ in its first m arguments and then use (4.23) to substitute for the higher order powers of the ι_k, leading to an expression of the form

$$\Psi(\iota_k, q_J) = \sum_{|I|=0}^{n} \iota_I \Xi_I(q_J) = \sum_{|I|=0}^{n} \widehat{D}^I \Xi_I(q_J). \tag{4.24}$$

We therefore define
$$\widehat{L}_I(\tilde{Z}) = \frac{1}{n}\Xi_I(p_J(\tilde{Z})),$$
so that, according to (4.22), $\Xi_I = \widehat{E}(\widehat{L}_I)$. This completes the demonstration of (4.20) and hence the proof of Theorem 1.

Example 7. Consider the Hessian quadratic differential polynomial
$$P[u] = u_{xx}u_{yy} - u_{xy}^2 \in \mathcal{A}_2^2,$$
of a function $u(x, y)$. Applying the transform to P and using the notation of Example 4, we find
$$\widehat{P} = \frac{1}{2}(z_1^2 w_2^2 + z_2^2 w_1^2) - z_1 w_1 z_2 w_2 \in \mathcal{P}_2^2.$$
It is easy to see that $\widehat{P}$ can be expressed most simply in terms of the power sums as
$$\widehat{P} = \frac{1}{2}(p_{11}p_{22} - p_{12}^2) = \frac{1}{6}(q_{11}q_{22} - q_{12}^2),$$
where
$$\begin{aligned} q_{11} &= p_{11} + p_1^2 = 2(z_1^2 + z_1 z_2 + z_2^2), \\ q_{12} &= p_{12} + p_1 p_2 = 2z_1 w_1 + z_1 w_2 + z_2 w_1 + 2z_2 w_2, \\ q_{22} &= p_{22} + p_2^2 = 2(w_1^2 + w_1 w_2 + w_2^2), \end{aligned} \tag{4.25}$$
cf. (2.6), (2.7). Then, using (4.9) and (4.10), we find
$$\widehat{P} = \frac{1}{18}\widehat{E}(\tilde{p}_{11}\tilde{p}_{22} - \tilde{p}_{12}^2),$$
where
$$\tilde{p}_{11} = z_1^2 + z_2^2 + z_3^2, \qquad \tilde{p}_{12} = z_1 w_1 + z_2 w_2 + z_3 w_3, \qquad \tilde{p}_{22} = w_1^2 + w_2^2 + w_3^2.$$
This latter identity is the transform of
$$u_{xx}u_{yy} - u_{xy}^2 = \frac{1}{3}E(u u_{xx} u_{yy} - u u_{xy}^2), \tag{4.26}$$

which gives one of the dissipative decompositions of P.

A second decomposition is a consequence of the basic multi-symmetric polynomial syzygy (2.13), which implies that we can write the transform of P in the alternative form

$$\begin{aligned}\widehat{P} &= \frac{1}{4}p_1^2p_{22} - \frac{1}{2}p_1p_2p_{12} + \frac{1}{4}p_2^2p_{11}\\ &= \frac{1}{4}p_1^2q_{22} - \frac{1}{2}p_1p_2q_{12} + \frac{1}{4}p_2^2q_{11}.\end{aligned}$$

Thus,

$$\widehat{P} = \frac{1}{12}\widehat{D}_x^2\widehat{E}(\tilde{p}_{22}) - \frac{1}{6}\widehat{D}_x\widehat{D}_y\widehat{E}(\tilde{p}_{12}) + \frac{1}{12}\widehat{D}_y^2\widehat{E}(\tilde{p}_{11}).$$

This is the transform of

(4.27)

$$\begin{aligned}u_{xx}u_{yy} - u_{xy}^2 &= \frac{1}{4}D_x^2E(u^2u_{yy}) - \frac{1}{2}D_xD_yE(u^2u_{xy}) + \frac{1}{4}D_y^2E(u^2u_{xx})\\ &= \frac{1}{2}D_x^2E(-uu_y^2) + D_xD_yE(uu_xu_y) + \frac{1}{2}D_y^2E(-uu_x^2).\end{aligned}$$

Therefore, we can find two distinct decompositions for the Hessian $u_{xx}u_{yy} - u_{xy}^2$ as a consequence of the fundamental syzygy (2.13).

Example 8. There is another type of nonuniqueness, which is connected with a different type of syzygy among the higher order power sums. This is related to the failure of the uniqueness proof, cf. (4.18), which was implemented in the ordinary differential equation case. Consider the quadratic differential polynomial

$$P[u] = u_xu_{xyy} - u_yu_{xxy} \in \mathcal{A}_2^2,$$

in two independent variables x and y. Applying the transform to P, we find

$$\widehat{P} = \frac{1}{2}z_1z_2(w_1^2 + w_2^2) - \frac{1}{2}w_1w_2(z_1^2 + z_2^2) \in \mathcal{P}_2^2.$$

A simple calculation shows that $\widehat{P}$ can be expressed in terms of the power sums as

$$\frac{1}{4}(p_1^2p_{22} - p_2^2p_{11}) = \frac{1}{4}(p_1^2q_{22} - p_2^2q_{11}),$$

cf. (4.25). Thus, using Lemma 5, we find

$$\widehat{P} = \frac{1}{12}\{\widehat{D}_x^2\widehat{E}(\tilde{p}_{22}) - \widehat{D}_y^2\widehat{E}(\tilde{p}_{11})\}.$$

This identity is the transform of the decomposition

$$\begin{aligned} u_x u_{xyy} - u_y u_{xxy} &= \frac{1}{4}\{D_x^2 E(u^2 u_{yy}) + D_y^2 E(-u^2 u_{xx})\} \\ &= \frac{1}{2}\{D_x^2 E(-u u_y^2) + D_y^2 E(u u_x^2)\}. \end{aligned} \tag{4.28}$$

Although the basic syzygy (2.13) plays no role here, nevertheless a second decomposition can be found. Note that we can also write the transform of P in the form

$$\widehat{P} = \frac{1}{2}(p_1 p_{122} - p_2 p_{112}) = \frac{1}{2}(p_1 q_{122} - p_2 q_{112}),$$

where

$$\begin{aligned} p_{112} &= z_1^2 w_1 + z_2^2 w_2, \qquad & p_{122} &= z_1 w_1^2 + z_2 w_2^2, \\ q_{112} &= p_{112} - p_1^2 p_2, \qquad & q_{122} &= p_{122} - p_1 p_2^2. \end{aligned}$$

Thus,

$$\widehat{P} = \frac{1}{6}(\widehat{D}_x\widehat{E}(\tilde{p}_{122}) - \widehat{D}_y\widehat{E}(\tilde{p}_{112})),$$

where

$$\tilde{p}_{112} = z_1^2 w_1 + z_2^2 w_2 + z_3^2 w_3, \qquad \tilde{p}_{122} = z_1 w_1^2 + z_2 w_2^2 + z_3 w_3^2.$$

This is the transform of the identity

$$\begin{aligned} u_x u_{xyy} - u_y u_{xxy} &= \frac{1}{2}(D_x E(u^2 u_{xyy}) - D_y E(u^2 u_{xxy})) \\ &= \frac{1}{2}(D_x E(u_x u_y^2) - D_y E(u_x^2 u_y)). \end{aligned} \tag{4.29}$$

The two distinct decompositions in this case appear as a consequence of the identity

$$p_1 p_{122} - p_2 p_{112} = \frac{1}{2}p_1^2 p_{22} - \frac{1}{2}p_2^2 p_{11} \tag{4.30}$$

connecting the $(n+1)^{\mathrm{st}}$ order power sum multi-symmetric polynomials, which we will call a "syzygy of the second type." It follows from the identities (2.11), which, for the particular polynomials p_{112}, p_{122}, are

$$p_{112} = p_1 p_{12} + \frac{1}{2} p_2 p_{11} - \frac{1}{2} p_1^2 p_2,$$
$$p_{122} = p_2 p_{12} + \frac{1}{2} p_1 p_{22} - \frac{1}{2} p_1 p_2^2.$$

The key point is that the degree of this identity (4.30) in the linear multi-symmetric polynomials p_1, p_2 is 2, which is not greater than $n = 2$. In the usual symmetric polynomial case, $m = 1$, such identities must be of degree at least $n + 1$ in p_1; see (4.19) and the subsequent discussion.

5. An algorithm for finding the dissipative decomposition. Given a symmetric multi-index $J = (j_1, j_2, \ldots, j_\mu)$, where $1 \le j_\kappa \le m$, we define $\mathcal{A}_n^J$ to be the subspace of $\mathcal{A}_n^m$ spanned by all monomials $u_{\mathbf{I}}$ where $\mathbf{I} = (I_1, \ldots, I_n)$ is such that the symmetric multi-index obtained by juxtaposition of all the I_v's is the same as J. In other words, the multi-index J represents the collection of all derivatives appearing in the entire monomial. For example, $u^2 u_x u_{xyyy}$, $u_x^2 u_y u_{yy}$ and $u u_x u_{xy} u_{yy}$ are all elements of $\mathcal{A}_4^{xxyyy} \subset \mathcal{A}_4^2$ since there are a total of two derivatives with respect to x and three derivatives with respect to y in each quartic monomial. We can find a decomposition for all the basis elements of $\mathcal{A}_n^J$ simultaneously.

Note that the total derivative D^I corresponding to the multi-index I maps $\mathcal{A}_n^J$ to $\mathcal{A}_n^{J,I}$, where J, I denotes the symmetric multi-index obtained by juxtaposing I and J. Therefore, Theorem 1 can be further refined to give the decomposition

$$\mathcal{A}_n^J = \sum_{\substack{|I|=0 \\ I \subset J}}^{n} D^I E(\mathcal{A}_{n+1}^{J \setminus I}), \tag{5.1}$$

where $J \setminus I$ denotes the multi-index obtained by deleting each entry in I from J. For example, $(xxxyyz)\setminus(xyy) = (xxz)$. In the ordinary differential equation case, the sum in (5.1) is direct whereas, as we have seen, in the partial differential equation case there can be nonzero dependencies among the summands.

We can find the dissipative decompositions of the standard monomial basis elements of $\mathcal{A}_n^J$ by using the following algorithm. The first is to find canonical basis elements for the spaces $E(\mathcal{A}_{n+1}^{J\setminus I})$. Unfortunately, the problem of determining an explicit basis for this space is not so straightforward. A naive approach is just to take the monomial basis for $\mathcal{A}_{n+1}^{J\setminus I}$ and determine which of the associated Euler-Lagrange expressions are linearly independent. Note that two Lagrangians give the same Euler-Lagrange expressions if and only if they differ by a divergence. Using this fact, one can considerably pare down the set of possible monomials in $\mathcal{A}_{n+1}^{J\setminus I}$ that need to be considered. However, an explicit general form for a basis of $E(\mathcal{A}_{n+1}^{J\setminus I})$ appears to be a rather difficult open problem.

Lemma 9. *A basis for the space of the conservative differential polynomials in $\mathcal{A}_n^m$ can be chosen from among the set of differential polynomials $\{E(u_{\mathbf{I}})\}$, where $\mathbf{I} = (I_1, \dots, I_{n+1})$ ranges over all multi-multi-indices satisfying $|I_1| \leq |I_2| \leq \cdots \leq |I_{n+1}|$, and either $|I_n| = |I_{n+1}|$ or $|I_{n-1}| = |I_n| = |I_{n+1}| - 1$.*

The proof of this result rests on an integration by parts argument. In other words, Lemma 9 states that to find a complete set of independent Euler-Lagrange expressions, we need only look at Lagrangians in which either the highest order derivative in each monomial occurs at least quadratically or if it is linear then the next highest order derivatives are at least quadratic. In the ordinary differential equation case, the latter possibility never arises since these terms can always be reduced further; however, the nontrivial Lagrangian $u_x^2 u_{yy}$, which is not a divergence, gives an example of this case in the partial differential equation situation. For example, even though there are 19 elements in the space $\{E(\mathcal{A}_3^{xxxyyy})\}$, a basis for $E(\mathcal{A}_3^{xxxyyy}) \subset \mathcal{A}_2^{xxxyyy}$ can be found from among the Euler-Lagrange expressions

$$E(uu_{xxx}u_{yyy}), \quad E(uu_{xxy}u_{xyy}), \quad E(u_{xx}u_{xy}u_{yy}), \quad E(u_{xy}^3).$$

However, we find that the linear combination

$$uu_{xxx}u_{yyy} - uu_{xxy}u_{xyy} + u_{xx}u_{xy}u_{yy} - u_{xy}^3$$

is a total divergence. Therefore, our basis will be provided by the three differential polynomials

$$\begin{aligned} E(uu_{xxx}u_{yyy}) &= -u_{xxx}u_{yyy} - 3u_{xx}u_{xyyy} - 3u_x u_{xxyyy} \\ &\quad - 3u_{yy}u_{xxxy} - 3u_y u_{xxxyy} - 2uu_{xxxyyy} \\ E(u_{xx}u_{xy}u_{yy}) &= u_{xxx}u_{yyy} + 2u_{xx}u_{xyyy} + 5u_{xyy}u_{xxy} \\ &\quad + 2u_{xy}u_{xxyy} + 2u_{yy}u_{xxxy} \\ E(u_{xy}^3) &= 6u_{xyy}u_{xxy} + 6u_{xy}u_{xxyy}. \end{aligned}$$

In order to determine which of the $D^I E(\mathcal{A}_{n+1}^{J\setminus I})$ to use in the decomposition, we use the existing syzygies among them. Once we have determined canonical basis elements for the relevant subspaces $D^I E(\mathcal{A}_{n+1}^{J\setminus I})$ appearing in the decomposition formula (5.1), we can rewrite any constant coefficient differential polynomial P in $\mathcal{A}_n^J$ in terms of these basis elements. Let M_v, $v = 1, \dots, r$, denote the monomial basis of $\mathcal{A}_n^J$. Further, let P_μ, $\mu = 1, \dots, r$, denote the basis elements formed from the decomposition, i.e., the differential polynomials $D^I E(u_{\mathbf{K}})$, where the $u_{\mathbf{K}}$ are the basis elements of $\mathcal{A}_{n+1}^{J\setminus I}$ given by the algorithm after Lemma 9. By inspection, we then determine the coefficient matrix $C = (c_{\mu v})$ for the basis P_μ in terms of the monomial basis M_v, writing

$$P_\mu = \sum_{v=1}^{r} c_{\mu v} M_v.$$

The inverse matrix $B = C^{-1}$, which exists because we assume that we have a basis, will then provide the dissipative decomposition of all the basis monomials in $\mathcal{A}_n^J$:

$$M_v = \sum_{\mu=1}^{r} b_{v\mu} P_\mu.$$

Example 10. Find the Euler decomposition for the basis elements of $\mathcal{A}_2^{xxyy}$: According to (5.1), we have

$$\begin{aligned} \mathcal{A}_2^{xxyy} &= E(\mathcal{A}_3^{xxyy}) + D_x E(\mathcal{A}_3^{xyy}) + D_y E(\mathcal{A}_3^{xxy}) \\ &\quad + D_x^2 E(\mathcal{A}_3^{yy}) + D_x D_y E(\mathcal{A}_3^{xy}) + D_y^2 E(\mathcal{A}_3^{xx}). \end{aligned} \tag{5.2}$$

The monomial basis for $\mathcal{A}_2^{xxyy}$ is given by

$$M_1 = u_{xx}u_{yy}, \qquad M_2 = u_{xy}^2, \qquad M_3 = u_x u_{xyy},$$
$$M_4 = u_y u_{xxy}, \qquad M_5 = uu_{xxyy}.$$

In order to determine which of the terms in the right hand side of (5.2) could be used in the decomposition, we refer back to Examples 7 and 8. There we deduced the identities

$$E\left[\frac{2}{3}u(u_{xx}u_{yy}-u_{xy}^2)\right] = D_x^2E(-uu_y^2)+D_xD_yE(2uu_xu_y)+D_y^2E(-uu_x^2),$$

and

$$D_xE(u_xu_y^2) - D_yE(u_x^2u_y) = -D_x^2E(uu_y^2) + D_y^2E(uu_x^2),$$

cf. (4.26–29). These two identities tell us that we can omit the summands $D_xD_yE(\mathcal{A}_3^{xy})$, $D_y^2E(\mathcal{A}_3^{yy})$, corresponding to the terms $D_xD_yE(uu_xu_y)$, $D_y^2E(uu_x^2)$, in the decomposition (5.2). Therefore, the relevant terms for the decomposition are

$$E(\mathcal{A}_3^{xxyy}) + D_xE(\mathcal{A}_3^{xyy}) + D_yE(\mathcal{A}_3^{xxy}) + D_x^2E(\mathcal{A}_3^{yy}). \tag{5.3}$$

Canonical basis elements for the subspaces in (5.3) are

$$\begin{aligned}
E(\mathcal{A}_3^{xxyy}):\quad & P_1 = E(uu_{xx}u_{yy}) = 3u_{xx}u_{yy} + 2u_xu_{xyy} \\
& \qquad\qquad + 2u_yu_{xxy} + 2uu_{xxyy} \\
& P_2 = E(uu_{xy}^2) = 3u_{xy}^2 + 2u_xu_{xyy} + 2u_yu_{xxy} + 2uu_{xxyy} \\
D_xE(\mathcal{A}_3^{xyy}):\quad & P_3 = D_xE(u_xu_y^2) = -2u_{xx}u_{yy} - 4u_{xy}^2 - 2u_xu_{xyy} - 4u_yu_{xxy} \\
D_yE(\mathcal{A}_3^{xxy}):\quad & P_4 = D_yE(u_x^2u_y) = -2u_{xx}u_{yy} - 4u_{xy}^2 \\
& \qquad\qquad - 4u_xu_{xyy} - 2u_yu_{xxy} \\
D_x^2E(\mathcal{A}_3^{yy}):\quad & P_5 = D_x^2E(uu_y^2) = -2u_{xx}u_{yy} - 2u_{xy}^2 \\
& \qquad\qquad - 4u_xu_{xyy} - 2u_yu_{xxy} - 2uu_{xxyy}
\end{aligned}$$

Therefore, the coefficient matrix is

$$C = \begin{pmatrix} 3 & 0 & 2 & 2 & 2 \\ 0 & 3 & 2 & 2 & 2 \\ -2 & -4 & -2 & -4 & 0 \\ -2 & -4 & -4 & -2 & 0 \\ -2 & -2 & -4 & -2 & -2 \end{pmatrix}$$

and

$$B = C^{-1} = \begin{pmatrix} \frac{7}{9} & \frac{2}{9} & \frac{1}{3} & -\frac{2}{3} & 1 \\ \frac{4}{9} & \frac{5}{9} & \frac{1}{3} & -\frac{2}{3} & 1 \\ -\frac{5}{9} & -\frac{4}{9} & -\frac{1}{6} & \frac{1}{3} & -1 \\ -\frac{5}{9} & -\frac{4}{9} & -\frac{2}{3} & \frac{5}{6} & -1 \\ \frac{4}{9} & \frac{5}{9} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{2} \end{pmatrix}.$$

We thus find a dissipative decomposition for the basis monomials of $\mathcal{A}_2^{xxyy}$ to be

$$\begin{aligned} u_{xx}u_{yy} &= E(\frac{7}{9}uu_{xx}u_{yy} + \frac{2}{9}uu_{xy}^2) + D_xE(\frac{1}{3}u_xu_y^2) \\ &\quad + D_yE(-\frac{2}{3}u_x^2u_y) + D_x^2E(uu_y^2) \\ u_{xy}^2 &= E(\frac{4}{9}uu_{xx}u_{yy} + \frac{5}{9}uu_{xy}^2) + D_xE(\frac{1}{3}u_xu_y^2) \\ &\quad + D_yE(-\frac{2}{3}u_x^2u_y) + D_x^2E(uu_y^2) \\ u_xu_{xyy} &= E(-\frac{5}{9}uu_{xx}u_{yy} - \frac{4}{9}uu_{xy}^2) + D_xE(-\frac{1}{6}u_xu_y^2) \\ &\quad + D_yE(\frac{1}{3}u_x^2u_y) + D_x^2E(-uu_y^2) \\ u_yu_{xxy} &= E(-\frac{5}{9}uu_{xx}u_{yy} - \frac{4}{9}uu_{xy}^2) + D_xE(-\frac{2}{3}u_xu_y^2) \\ &\quad + D_yE(\frac{5}{6}u_x^2u_y) + D_x^2E(-uu_y^2) \\ uu_{xxy} &= E(\frac{4}{9}uu_{xx}u_{yy} + \frac{5}{9}uu_{xy}^2) + D_xE(\frac{1}{3}u_xu_y^2) \\ &\quad + D_yE(-\frac{1}{6}u_x^2u_y) + D_x^2E(\frac{1}{2}uu_y^2). \end{aligned}$$

Tables of constant coefficient dissipative decompositions for basis monomials for $m = 2$, $n = 2, 3$, and $|J| \le 4$ appear below. We have omitted the ordinary differential monomials (i.e., those that only involve x derivatives or only involve y derivatives) since their decompositions can be found in [**14**]. Also, to get the remaining terms, just interchange x and y in the given decompositions; e.g., the xyy cases are found from the given xxy cases. The computations were done using the symbolic manipulation language MATHEMATICA. Programs and further tables are available from the authors upon request.

Acknowledgments. We would like to thank Paul Edelman for showing us his recent work on multi-symmetric functions, and for helping with the symbolic computations using MACAULAY. We thank the Institute for Mathematics and its Applications (I.M.A.) for the use of its workstations for some of the computational parts of this work. We would also like to thank Amy Fitzer, University of St. Thomas student, for doing the symbolic computations in MATHEMATICA for the tables. Finally, we thank an anonymous referee for alerting us to some of the older references on multi-symmetric functions.

TABLE 1. Dissipative decompositions: $m = 2$, $n = 2$

J	
xy	$uu_{xy} = E(-uu_xu_y) + D_xD_yE(-\frac{1}{6}u^3)$
	$u_xu_y = E(uu_xu_y) + D_xD_yE(\frac{1}{3}u^3)$
xxy	$uu_{xxy} = E(\frac{1}{3}u_x^2u_y) + D_xE(-\frac{1}{3}uu_xu_y) + D_yE(-\frac{1}{6}uu_x^2)$
	$u_yu_{xx} = E(-\frac{1}{6}u_x^2u_y) + D_xE(\frac{2}{3}uu_xu_y) + D_yE(-\frac{2}{3}uu_x^2)$
	$u_xu_{xy} = E(-\frac{1}{6}u_x^2u_y) + D_xE(-\frac{1}{3}uu_xu_y) + D_yE(\frac{1}{3}uu_x^2)$
$xxxy$	$uu_{xxxy} = E(uu_{xx}u_{xy}) + D_xE(\frac{1}{4}u_x^2u_y) + D_yE(-\frac{1}{12}u_x^3) + D_x^2E(\frac{1}{2}uu_xu_y)$
	$u_yu_{xxx} = E(-uu_{xx}u_{xy}) + D_xE(-\frac{1}{2}u_x^2u_y) + D_yE(\frac{2}{3}u_x^3) + D_x^2E(-uu_xu_y)$
	$u_xu_{xxy} = E(-uu_{xx}u_{xy}) + D_yE(\frac{1}{6}u_x^3) + D_x^2E(-uu_xu_y)$
	$u_{xx}u_{xy} = E(uu_{xx}u_{xy}) + D_yE(-\frac{1}{3}u_x^3) + D_x^2E(uu_xu_y)$
$xxyy$	$u_{xx}u_{yy} = E(\frac{7}{9}uu_{xx}u_{yy} + \frac{2}{9}uu_{xy}^2) + D_xE(\frac{1}{3}u_xu_y^2) + D_yE(-\frac{2}{3}u_x^2u_y) + D_x^2E(uu_y^2)$
	$u_{xy}^2 = E(\frac{4}{9}uu_{xx}u_{yy} + \frac{5}{9}uu_{xy}^2) + D_xE(\frac{1}{3}u_xu_y^2) + D_yE(-\frac{2}{3}u_x^2u_y) + D_x^2E(uu_y^2)$
	$u_xu_{xyy} = E(-\frac{5}{9}uu_{xx}u_{yy} - \frac{4}{9}uu_{xy}^2) + D_xE(-\frac{1}{6}u_xu_y^2) + D_yE(\frac{1}{3}u_x^2u_y) + D_x^2E(-uu_y^2)$
	$u_yu_{xxy} = E(-\frac{5}{9}uu_{xx}u_{yy} - \frac{4}{9}uu_{xy}^2) + D_xE(-\frac{2}{3}u_xu_y^2) + D_yE(\frac{5}{6}u_x^2u_y) + D_x^2E(-uu_y^2)$

TABLE 2. Dissipative decompositions: $m = 2$, $n = 3$.

$\underline{\mathbf{J}}$	
xy	$uu_xu_y = E(\frac{1}{2}u^2u_xu_y) + D_xD_yE(\frac{1}{12}u^4)$
	$u^2u_{xy} = E(-u^2u_xu_y) + D_xD_yE(-\frac{1}{12}u^4)$
xxy	$u^2u_{xxy} = E(uu_x^2u_y) + D_x^2D_yE(\frac{1}{12}u^4)$
	$uu_yu_{xx} = E(-\frac{1}{2}uu_x^2u_y) + D_yE(-\frac{1}{2}u^2u_x^2) + D_x^2D_yE(-\frac{1}{12}u^4)$
	$uu_xu_{xy} = E(-\frac{1}{2}uu_x^2u_y) + D_xE(-\frac{1}{2}u^2u_xu_y) + D_x^2D_yE(-\frac{1}{12}u^4)$
	$u_x^2u_y = E(uu_x^2u_y) + D_xE(u^2u_xu_y) + D_yE(\frac{1}{2}u^2u_x^2) + D_x^2D_yE(\frac{1}{4}u^4)$
$xxxy$	$u^2u_{xxxy} = E(u^2u_{xx}u_{xy} - \frac{7}{12}u_x^3u_y) + D_xE(\frac{1}{4}uu_x^2u_y) + D_yE(\frac{1}{12}uu_x^3) + D_x^2E(\frac{1}{4}u^2u_xu_y) + D_xD_yE(\frac{1}{4}u^2u_x^2)$
	$uu_yu_{xxx} = E(-\frac{1}{2}u^2u_{xx}u_{xy} + \frac{1}{4}u_x^3u_y) + D_xE(-\frac{1}{4}uu_x^2u_y) + D_yE(\frac{5}{12}uu_x^3) + D_x^2E(-\frac{1}{4}u^2u_xu_y) + D_xD_yE(-\frac{1}{4}u^2u_x^2)$
	$uu_xu_{xxy} = E(-\frac{1}{2}u^2u_{xx}u_{xy} + \frac{1}{4}u_x^3u_y) + D_xE(\frac{1}{4}uu_x^2u_y) + D_yE(-\frac{1}{12}uu_x^3) + D_x^2E(-\frac{1}{4}u^2u_xu_y) + D_xD_yE(\frac{1}{4}u^2u_x^2)$
	$uu_{xx}u_{xy} = E(\frac{1}{2}u^2u_{xx}u_{xy} - \frac{1}{12}u_x^3u_y) + D_xE(-\frac{1}{4}uu_x^2u_y) + D_yE(-\frac{1}{12}uu_x^3) + D_x^2E(\frac{1}{4}u^2u_xu_y) + D_xD_yE(\frac{1}{4}u^2u_x^2)$
	$u_xu_yu_{xx} = E(-\frac{1}{12}u_x^3u_y) + D_xE(\frac{1}{4}uu_x^2u_y) + D_yE(-\frac{1}{4}uu_x^3) + D_x^2E(\frac{1}{4}u^2u_xu_y) + D_xD_yE(-\frac{1}{4}u^2u_x^2)$
	$u_x^2u_{xy} = E(-\frac{1}{12}u_x^3u_y) + D_xE(-\frac{1}{4}uu_x^2u_y) + D_yE(\frac{1}{4}uu_x^3) + D_x^2E(-\frac{1}{4}u^2u_xu_y) + D_xD_yE(\frac{1}{4}u^2u_x^2)$

TABLE 2. (Continued)

J

$xxyy$

$$u^2u_{xxyy} = E(u^2u_{xy}^2 + \tfrac{23}{4}uu_xu_yu_{xy} + \tfrac{1}{12}u_x^2u_y^2) + D_xE(\tfrac{1}{6}uu_xu_y^2) + D_yE(\tfrac{1}{6}uu_x^2u_y) + D_x^2E(\tfrac{1}{12}u^2u_y^2) + D_y^2E(\tfrac{1}{12}u^2u_x^2) + D_xD_yE(\tfrac{1}{3}u^2u_xu_y)$$

$$uu_yu_{xxy} = E(-\tfrac{1}{2}u^2u_{xy}^2 - uu_xu_yu_{xy} - \tfrac{1}{12}u_x^2u_y^2) + D_xE(-\tfrac{1}{6}uu_xu_y^2) + D_yE(\tfrac{1}{3}uu_x^2u_y) + D_x^2E(-\tfrac{1}{12}u^2u_y^2) + D_y^2E(-\tfrac{1}{12}u^2u_x^2) + D_xD_yE(-\tfrac{1}{3}u^2u_xu_y)$$

$$uu_{xx}u_{yy} = E(\tfrac{3}{4}u^2u_{xy}^2 + 3uu_xu_yu_{xy} + \tfrac{11}{12}u_x^2u_y^2) + D_xE(-\tfrac{1}{6}uu_xu_y^2) + D_yE(-\tfrac{1}{6}uu_x^2u_y) + D_x^2E(\tfrac{5}{12}u^2u_y^2) + D_y^2E(\tfrac{5}{12}u^2u_x^2) + D_xD_yE(-\tfrac{1}{3}u^2u_xu_y)$$

$$uu_xu_{xyy} = E(-\tfrac{1}{2}u^2u_{xy}^2 - uu_xu_yu_{xy} - \tfrac{1}{12}u_x^2u_y^2) + D_xE(\tfrac{1}{3}uu_xu_y^2) + D_yE(-\tfrac{1}{6}uu_x^2u_y) + D_x^2E(-\tfrac{1}{12}u^2u_y^2) + D_y^2E(-\tfrac{1}{12}u^2u_x^2) + D_xD_yE(-\tfrac{1}{3}u^2u_xu_y)$$

$$uu_{xy}^2 = E(\tfrac{1}{2}u^2u_{xy}^2 - \tfrac{1}{12}u_x^2u_y^2) + D_xE(-\tfrac{1}{6}uu_xu_y^2) + D_yE(-\tfrac{1}{6}uu_x^2u_y) + D_x^2E(-\tfrac{1}{12}u^2u_y^2) + D_y^2E(-\tfrac{1}{12}u^2u_x^2) + D_xD_yE(\tfrac{2}{3}u^2u_xu_y)$$

$$u_y^2u_{xx} = E(-\tfrac{5}{24}u^2u_{xy}^2 - 2uu_xu_yu_{xy} - \tfrac{3}{4}u_x^2u_y^2) + D_xE(\tfrac{1}{2}uu_xu_y^2) + D_yE(-\tfrac{1}{2}uu_x^2u_y) + D_x^2E(-\tfrac{1}{4}u^2u_y^2) + D_y^2E(-\tfrac{3}{4}u^2u_x^2) + D_xD_yE(u^2u_xu_y)$$

$$u_xu_yu_{xy} = E(\tfrac{5}{24}u^2u_{xy}^2 + uu_xu_yu_{xy} + \tfrac{1}{4}u_x^2u_y^2) + D_x^2E(\tfrac{1}{4}u^2u_y^2) + D_y^2E(\tfrac{1}{4}u^2u_x^2) + D_xD_yE(-\tfrac{1}{2}u^2u_xu_y)$$

$$u_x^2u_{yy} = E(-\tfrac{5}{24}u^2u_{xy}^2 - 2uu_xu_yu_{xy} - \tfrac{3}{4}u_x^2u_y^2) + D_xE(-\tfrac{1}{2}uu_xu_y^2) + D_yE(\tfrac{1}{2}uu_x^2u_y) + D_x^2E(-\tfrac{3}{4}u^2u_y^2) + D_y^2E(-\tfrac{1}{4}u^2u_x^2) + D_xD_yE(u^2u_xu_y)$$

REFERENCES

1. A. Adem, J. Maginnis and R.J. Milgram, *Symmetric invariants and cohomology of groups*, Math. Ann. **287** (1990), 391–411.

2. J.M. Ball, J.C. Currie and P.J. Olver, *Null Lagrangians, weak continuity, and variational problems of arbitrary order*, J. Funct. Anal. **41** (1981), 135–174.

3. P. Edelman, personal communication.

4. I.M. Gel'fand and L.A. Dikii, *Asymptotic behavior of the resolvent of Sturm-Liouville equations and the algebra of the Korteweg-de Vries equation*, Russian Math. Surveys **30** (1975), 77–113.

5. I.M. Gessel, *Enumerative applications of symmetric functions*, Actes 17^e Séminaire Lotharingien, Publ. I.M.R.A., 348/S-17, Strasbourg, 1988, 5–21.

6. F. Junker, *Die Relationen, welche zwischen den elementaren symmetrischen Functionen bestehen*, Math. Ann. **38** (1891), 91–114.

7. ———, *Ueber symmetrischen Functionen von mehren Reihen von Veränderlichen*, Math. Ann. **43** (1893), 225–270.

8. ———, *Die symmetrischen Functionen und die Relationen zwischen den Elementarfunctionen derselben*, Math. Ann. **45** (1894), 1–84.

9. P.A. MacMahon, *Combinatory analysis*, Cambridge University Press, Cambridge, 1915, 1916.

10. I.G. Macdonald, *Symmetric functions and Hall polynomials*, Clarendon Press, Oxford, 1979.

11. E. Netto, *Vorlesungen über Algebra*, vol. 2, B.G. Teubner, Leipzig, 1900.

12. P.J. Olver, *Applications of Lie groups to differential equations*, Grad. Texts in Math. **107**, Springer-Verlag, New York, 1986.

13. ———, *Invariant theory and differential equations*, in *Invariant theory* (S.S. Koh, ed.), Lecture Notes Math. **1278**, Springer-Verlag, New York, 1987, 62–80.

14. P.J. Olver and C. Shakiban, *Dissipative decomposition of ordinary differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **109** (1988), 297–317.

15. C. Shakiban, *A resolution of the Euler operator* II, Math. Proc. Cambridge Philos. Soc. **89** (1981), 501–510.

16. M. Stillman, M. Stillman and D. Bayer, *Macaulay user's manual*, preprint, Cornell University, April, 1990.

School of Mathematics, University of Minnesota, Minneapolis, MN, USA 55455

Mathematics Department, University of St. Thomas, St. Paul, MN, USA 55105