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QUASILINEAR ELLIPTICITY AND JUMPING NONLINEARITIES

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ABSTRACT. Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with smooth boundary. Also let

$$Qu = -D_i(a^{ij}(x,u,Du)D_ju) + b(x,u,Du)u.$$

Under five assumptions on the coefficients of Q (Caratheodory, symmetry, growth, ellipticity, and monotonicity) existence and nonexistence results for weak solutions to the generalized Dirichlet problem

$$Qu(x) = f(x,u) + t\Phi(x) + h(x) \quad \text{for } x \in \Omega;$$
$$u(x) = 0 \quad \text{for } x \in \partial\Omega$$

are established subject to jumping nonlinearity assumptions on $f(x,u)$ where $t \in \mathbf{R}$, $h \in L^\infty(\Omega)$, and $\Phi \in L^\infty(\Omega)$ is positive a.e. on Ω.

1. Introduction and statement of result. This paper will demonstrate some existence and nonexistence results for a quasilinear Dirichlet problem under Ambrosetti-Prodi, Berger-Podolak, Kazdan-Warner type assumptions (see [**1, 2** and **6** Theorems 3.4–3.8]). Let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary denoted by $\partial\Omega$. Unless otherwise noted, all function spaces such as $L^2, W^{1,2}$, and H_0^1 will have domain Ω. We define the quasilinear operator

$$Qu = -D_i(a^{ij}(\cdot,u,Du)D_ju) + b(\cdot,u,Du)u$$

and study the existence and nonexistence of weak solutions to the generalized Dirichlet problem

$$(1.1_t) \qquad \begin{aligned} (Qu)(x) &= f(x,u(x)) + t\Phi(x) + h(x), \qquad x \in \Omega \\ u(x) &= 0, \qquad x \in \partial\Omega. \end{aligned}$$

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(*Note.* The summation convention is employed in the definition of Q and the sequel.) We make the following assumptions.

(Q1) The coefficients of Q; a^{ij}, $i, j = 1, \dots, N$; and b are defined for $(x, z, p) \in \Omega \times \mathbf{R} \times \mathbf{R}^N$ and satisfy the usual Caratheodory conditions of measurability and continuity. (That is, they are measurable in Ω for all $(z, p) \in \mathbf{R} \times \mathbf{R}^N$ and continuous on $\mathbf{R} \times \mathbf{R}^N$ for a.e. $x \in \Omega$.)

(Q2) $a^{ij}(x, z, p) = a^{ji}(x, z, p)$ for $i, j = 1, 2, \dots, N$; for all $(z, p) \in \mathbf{R} \times \mathbf{R}^N$; and for a.e. $x \in \Omega$.

(Q3) The functions $a^{ij}(x, z, p)$ and $b(x, z, p)$ are in $L^\infty(\Omega \times \mathbf{R} \times \mathbf{R}^N)$.

(Q4) There are positive constants α_1, α_2 such that $0 < \alpha_1 |\xi|^2 \leq a^{ij}(x, z, p)\xi_j \xi_i \leq \alpha_2 |\xi|^2$ holds for a.e. $x \in \Omega$, for all $(z, p) \in \mathbf{R} \times \mathbf{R}^N$, and for all $\xi \in \mathbf{R}^N$; thus Q is uniformly elliptic. (*Note.* $|\xi|^2 = \sum_{i=1}^N \xi_i^2$.)

(Q5) $[a^{ij}(x, z, p)p_j - a^{ij}(x, z, p')p'_j](p_i - p'_i) > 0$ for a.e. $x \in \Omega$, for all $z \in \mathbf{R}$, and for all $p, p' \in \mathbf{R}^N$ with $p \neq p'$.

Definition 1.1. A function $u \in H_0^1$ will be called a *weak solution* of (1.1_t) if the form defined on $H_0^1 \times H_0^1 \times H_0^1$ by

$$q(w, u, v) = \int_\Omega a^{ij}(\cdot, w, Dw) D_j u D_i v + b(\cdot, w, Dw) uv$$

satisfies $q(u, u, v) = \int_\Omega [f(\cdot, u) + t\Phi + h]v$, for all $v \in H_0^1$.

In order to formulate the hypotheses on the right hand side of (1.1_t), we first consider a related *linear* elliptic problem. Define $\tilde{H}^1 = \{v \in W^{1,2} : v = u + M \text{ for some } u \in H_0^1 \text{ and } M \in \mathbf{R}\}$. Observe $H_0^1 \subset \tilde{H}^1 \subset L^2$. To each $v \in \tilde{H}^1$ we associate the linear, self-adjoint elliptic operator

$$Q^v u = -D_i(a^{ij}(\cdot, v, Dv) D_j u) + b(\cdot, v, Dv) u.$$

Let $\lambda_1(v)$ and ϕ_1^v be the first eigenvalue and eigenfunction, respectively, corresponding to the Dirichlet problem for Q^v, i.e.,

$$Q^v \phi_1^v = \lambda_1(v) \phi_1^v \quad \text{and} \quad \phi_1^v \in H_0^1. \tag{1.2}$$

It is well known [**5**, p. 214] that ϕ_1^v can be chosen so that it is positive on Ω and satisfies $||\phi_1^v||_{L^2}^2 = 1$. We let

$$\Lambda_1 = \inf\{\lambda_1(v) : v \in \tilde{H}^1\} \quad \text{and} \quad \Lambda_2 = \sup\{\lambda_1(v) : v \in \tilde{H}^1\},$$

and observe that $-\infty < \Lambda_1 \leq \Lambda_2 < \infty$. Indeed, from the variational characterization of $\lambda_1(v)$, hypotheses (Q3) and (Q4) and the Poincaré inequality, $k_1||u||_{L^2} \leq ||Du||_{L^2}$, we obtain for all $v \in \tilde{H}^1$

$$\begin{aligned}\lambda_1(v) &= \inf_{\substack{u\in H_0^1\\ u\neq 0}} \frac{q(v,u,u)}{||u||_{L^2}^2} \geq \inf_{\substack{u\in H_0^1\\ u\neq 0}} \alpha_1 \frac{||Du||_{L^2}^2}{||u||_{L^2}^2} - ||b||_\infty \\ &\geq \alpha_1 k_1^2 - ||b||_\infty > -\infty\end{aligned} \tag{1.3}$$

and

$$\lambda_1(v) = \inf_{\substack{u\in H_0^1\\ u\neq 0}} \frac{q(v,u,u)}{||u||_{L^2}^2} \leq \inf_{\substack{u\in H_0^1\\ u\neq 0}} \alpha_2 \frac{||Du||_{L^2}^2}{||u||_{L^2}^2} + ||b||_\infty = k_2 < \infty.$$

Now we assume the following about the right hand side of (1.1_t).

(f1) $f(x,z) \in C^0(\overline{\Omega} \times \mathbf{R})$.

(f2) $\lim_{z\to\pm\infty}(f(x,z)/z)$ exists uniformly for $x \in \overline{\Omega}$. We denote these uniform limits by $f'(x,\pm\infty)$, respectively. Also, there are real constants denoted $f'(\pm\infty)$ such that

$$-\infty < f'(x,-\infty) \leq f'(-\infty) < \Lambda_1 \leq \Lambda_2 < f'(+\infty) \leq f'(x,+\infty) < +\infty$$

holds for all $x \in \overline{\Omega}$.

(R) $t \in \mathbf{R}$, $h \in L^\infty$, and $\Phi \in L^\infty$ where Φ has the additional property that $\Phi > 0$ a.e. on Ω.

The hypothesis (f2) is a generalization of the usual hypothesis that f "jumps" across the first eigenvalue. We are now able to state the main result.

Theorem. *Under the hypotheses* (Q1)–(Q5), (f1), (f2), *and* (R), *there exists a* $T \in \mathbf{R}$ *such that* (1.1_t) *has no weak solutions for* $t > T$ *and at least one weak solution for* $t \leq T$.

Our results were motivated by those of Chabrowski and the theorem we prove here strictly contains some of his results, i.e., Theorems 1 and 2 in [**3**]. (There are many other theorems and results in this interesting paper.) We employ the method of upper and lower solutions described in [**4**] to prove existence. However, we have replaced the use of G-convergence used in [**3**] with a simplified Harnack inequality argument. Furthermore, Chabrowski considers the problem (1.1_t) when the operator Q is simplified by taking $b = 0$ and allowing a^{ij} to depend only on x and u. In this setting the monotonicity hypothesis (Q5) is easily seen to be satisfied by the ellipticity (Q4). Indeed,

$$\begin{aligned}[a^{ij}(x,z)p_j - a^{ij}(x,z)p'_j](p_i - p'_i) &= a^{ij}(x,z)[(p_j - p'_j)(p_i - p'_i)] \\ &\geq \alpha_1 |p - p'|^2 > 0\end{aligned}$$

for a.e. $x \in \Omega$, for all $z \in \mathbf{R}$, and for all $p, p' \in \mathbf{R}^N$ with $p \neq p'$. It is by carefully exploiting the monotonicity assumed in (Q5) that we are able to extend to the case where the coefficients also depend on Du. Moreover, since we define Λ_1 and Λ_2 by taking the infimum and supremum over $v \in \tilde{H}^1$ (instead of $v \in L^2$, as in [**3**]), our interval $[\Lambda_1, \Lambda_2]$ is possibly smaller than that of Chabrowski and hence the class of nonlinearities which satisfy our hypotheses is possibly larger. The results presented here are the first jumping nonlinearity results we know of where the coefficients are allowed to depend on Du.

Also, combining the techniques to be presented here with those of Chabrowski concerning the operator T_t in class (S_+) [**3**, p. 362], a result for the existence of multiple solutions where the coefficients depend on Du can easily be obtained. We leave the details for the interested reader.

2. Example. In the following example we exhibit a general class of nonlinearities which is beyond the scope of [**3**] but which can be treated by our theorem. Let g_1 be any increasing bounded continuous function on $[0, +\infty)$ with $g_1(0) \geq 0$. Extend g_1 to $(-\infty, +\infty)$ so that g_1 is even (i.e., $g_1(-x) = g_1(x)$). Construct functions $g_2, g_3, \dots, g_n$ in the same way as g_1. It is not difficult to verify that for $p \in \mathbf{R}^n$ and $i = 1, 2, \dots, n$ fixed, the function $g_i(p_i)p_i$ is an increasing function of p_i. In particular, then $[g_i(s)s - g_i(t)t](s - t) > 0$ for all $s \neq t$. Suppose $\alpha^{ij}(x) \in L^\infty$ are the coefficients of a linear, strictly elliptic, self-adjoint, partial differential operator. Define $a^{ij}(x, z, p) = \alpha^{ij}(x) + \delta_{ij} g_i(p_i)$ where δ_{ij} is the Kronecker delta. Assume $b(x, z, p)$ is any nonnegative $L^\infty(\Omega \times \mathbf{R} \times \mathbf{R}^N)$ function satisfying the Caratheodory conditions. Then the corresponding operator Q satisfies (Q1)–(Q5).

Clearly (Q1)–(Q3) are valid. Also, by the nonnegativity of g_i, we have (Q4); that is,

$$a^{ij}(x,z,p)\xi_i\xi_j = (\alpha^{ij}(x) + \delta_{ij}g_i(p_i))\xi_i\xi_j \geq \alpha_1|\xi|^2 + g_i(p_i)\xi_i^2 \geq \alpha_1|\xi|^2$$

holds when α_1 is the ellipticity constant for the $\alpha^{ij}(x)$. The upper bound in (Q4) is clear from the boundedness of α^{ij} and each g_i.

It remains to verify the monotonicity condition (Q5). Since the $g_i(p_i)p_i$ are increasing functions, we obtain, by the earlier remarks, for $p, p' \in \mathbf{R}^n$ with $p \neq p'$,

$$\begin{aligned}(a^{ij}(x,z,p)p_j &- a^{ij}(x,z,p')p'_j)(p_i - p'_i)\\ &= \alpha^{ij}(x)(p_j - p'_j)(p_i - p'_i) + (g_i(p_i)p_i - g_i(p'_i)p'_i)(p_i - p'_i)\\ &> \alpha_1|p - p'|^2 > 0.\end{aligned}$$

Some concrete examples of g_i's which satisfy the conditions described above, are as follows. For $t \in \mathbf{R}$ we may choose $g(t) = |t|/\sqrt{1+t^2}$, $g(t) = |t|/(1+|t|)$, $g(t) = ((1+t^2)/(2+t^2))^{1/2}$, etc.

3. Preliminaries. In what follows, we shall assume without loss of generality that

$$b(x,z,p) \geq 0 \quad \text{for a.e. } x \in \Omega \quad \text{and} \quad \forall\,(z,p) \in \mathbf{R} \times \mathbf{R}^N.$$

If this is not the case, we can add the quantity γu to both sides of (1.1_t) with $\gamma \geq ||b||_\infty$ and obtain the equation

$$\tilde{Q}u = -D_i(a^{ij}(\cdot,u,Du)D_ju) + \tilde{b}(\cdot,u,Du)u = \tilde{f}(\cdot,u) + t\Phi + h$$

where $\tilde{b}(\cdot,u,Du) = b(\cdot,u,Du) + \gamma$ and $\tilde{f}(\cdot,u) = f(\cdot,u) + \gamma u$. This new equation still satisfies (Q1) through (Q5), (f1) and (R). To verify hypothesis (f2), we observe that $\tilde{\Lambda}_i$ associated with $\tilde{Q}$ satisfies $\tilde{\Lambda}_i = \Lambda_i + \gamma$ for $i = 1, 2$ and $\tilde{f}'(x,\pm\infty) = f'(x,\pm\infty) + \gamma$; thus f satisfies (f2) if and only if $\tilde{f}$ does. Also, if we have $b(x,z,p) \geq 0$, we see from an argument similar to that given in (1.3) that $\Lambda_1 > 0$. Hence, from now on, we assume that $\Lambda_1 > 0$.

The following estimates for f will be useful in the sequel. There are positive constants c_1 and c_2 such that

$$(3.1) \qquad |f(x,z)| \leq c_1|z| + c_2 \qquad \forall\, z \in \mathbf{R} \quad \text{and} \quad \text{for } x \in \overline{\Omega}.$$

Further, there exist δ_1, δ_2 and C, all positive constants, such that $0 < \delta_1 < \Lambda_1 \leq \Lambda_2 < \delta_2$, and both

$$f(x, z) \geq \delta_1 z - C \tag{3.2}$$

and

$$f(x, z) \geq \delta_2 z - C \tag{3.3}$$

hold for $x \in \overline{\Omega}$ and for all $z \in \mathbf{R}$. All three of these estimates are direct consequences of the hypotheses (f1) and (f2).

We complete this section by proving the following results which will be needed in the sequel.

Lemma 3.1. *Suppose* (v_k) *is a sequence of elements in* $\tilde{H}^1$. *Let* $\phi_1^{v_k}$ *be defined by* (1.2). *In particular,* $\phi_1^{v_k} \in H_0^1$ *and* $||\phi_1^{v_k}||_{L^2} = 1$. *Suppose further that* $\phi_1^{v_k} \to \phi_1$ *weakly in* H_0^1. *Then there exists a subsequence* v_{k_j} *of the original sequence such that* $\phi_1^{v_{k_j}} \to \phi_1$ *weakly in* H_0^1, *strongly in* L^2, *and a.e. in* Ω. *Furthermore,* $\phi_1 > 0$ *a.e. on* Ω.

Proof. The existence of the subsequence converging as described follows from standard arguments. We observe that $(\phi_1^{v_k})$ is bounded in H_0^1 norm. Indeed, from (Q4) and (1.2) we have

$$\begin{aligned} ||\phi_1^{v_k}||_{H_0^1}^2 = \left(\int_\Omega |D\phi_1^{v_k}|^2 \right) &\leq \frac{1}{\alpha_1} \int_\Omega a^{ij}(\cdot, v_k, Dv_k) D_j \phi_1^{v_k} D_i \phi_1^{v_k} \\ &\leq \frac{q(v_k, \phi_1^{v_k}, \phi_1^{v_k})}{\alpha_1} = \frac{\lambda_1(v_k)}{\alpha_1} \int_\Omega (\phi_1^{v_k})^2 \leq \frac{\Lambda_2}{\alpha_1} < \infty. \end{aligned} \tag{3.4}$$

We then make use of the fact that a norm bounded set in H_0^1 has a weakly convergent subsequence [**5**, Theorem 5.12], and that H_0^1 is compactly imbedded in L^2 [**5**, Theorem 7.22].

To prove $\phi_1 > 0$ a.e. on Ω we first observe that $\phi_1 \geq 0$ a.e. on Ω since $\phi_1^{v_{k_j}} > 0$ on Ω for all $v_{k_j} \in \tilde{H}^1$ and $\phi_1^{v_{k_j}} \to \phi_1$ a.e. on Ω. Furthermore, L^2 convergence guarantees $||\phi_1||_2 = 1$ since the same is true for each $\phi_1^{v_{k_j}}$. Hence, for some $\varepsilon_0 > 0$, we can find $B \subseteq \overline{B} \subset \Omega$ with $|B| > 0$ and such that $\phi_1(x) \geq \varepsilon_0 > 0$ for all $x \in B$. Here $|\cdot|$ denotes Lebesgue measure.

Using Egorov's theorem, we choose a measurable $E \subseteq \Omega$ such that $|E| < |B|/2$ and $\phi_1^{v_{k_j}} \to \phi_1$ uniformly in $\Omega \backslash E$. Notice

$$|(\Omega \backslash E) \cap B| = |B| - |E \cap B| > \frac{|B|}{2} > 0.$$

Therefore $(\Omega \backslash E) \cap B \neq \varnothing$ follows immediately from $|E| < |B|/2$ since $B \nsubseteq E$. Because convergence is uniform, $\phi_1^{v_{k_j}}(x) \geq \varepsilon_0/2$ for all $x \in (\Omega \backslash E) \cap B$ for large enough j. Let Ω' be an arbitrary compact subset of Ω such that $\Omega' \subseteq (\Omega \backslash E) \cap B$. By Harnack's inequality, [**5**, Corollary 8.21], there exists a constant $C > 0$ which can be taken independent of j such that

$$0 < \frac{\varepsilon_0}{2} \leq \sup_{x \in \Omega'} \phi_1^{v_{k_j}}(x) \leq C \inf_{x \in \Omega'} \phi_1^{v_{k_j}}(x) \leq C \phi_1^{v_{k_j}}(x)$$

holds for each $x \in \Omega'$ and all large j. Thus $\phi_1(x) \geq \varepsilon_0/(2C) > 0$ for a.e. $x \in \Omega'$. Since Ω' is arbitrary, we conclude $\phi_1(x) > 0$ for a.e. $x \in \Omega$. □

Lemma 3.2. *Suppose (u_n) is a sequence of weak solutions of (1.1_{t_n}) where (t_n) is any bounded sequence of reals. Let $u \in H_0^1$ be such that $u_n \to u$ weakly in H_0^1, strongly in L^2, and a.e. in Ω. Then there exists a subsequence (u_{n_k}) such that $Du_{n_k} \to Du$ a.e. in Ω.*

Proof. We first show that there exists a subsequence (u_{n_k}) such that

$$[a^{ij}(\cdot, u_{n_k}, Du_{n_k}) D_j u_{n_k} - a^{ij}(\cdot, u_{n_k}, Du) D_j u](D_i u_{n_k} - D_i u) \to 0 \tag{3.5}$$

a.e. in Ω as $k \to +\infty$. To demonstrate (3.5), it is enough to show

$$\int_\Omega [a^{ij}(\cdot, u_n, Du_n) D_j u_n - a^{ij}(\cdot, u_n, Du) D_j u](D_i u_n - D_i u) \to 0 \tag{3.6}$$

since the integrand is positive a.e. by (Q5) and L^1 convergence implies a.e. convergence of a subsequence. Using (3.1) and the fact that u_n is a weak solution of (1.1_{t_n}), we obtain

$$\begin{aligned}\left|\int_\Omega a^{ij}(\cdot,u_n,Du_n)D_ju_n(D_iu_n-D_iu)+b(\cdot,u_n,Du_n)u_n(u_n-u)\right|\\ \le \int_\Omega(|f(\cdot,u_n)|+|t_n\Phi+h|)|u_n-u|\\ \le \int_\Omega(c_1|u_n|+c_2+|t_n\Phi+h|)|u_n-u|\\ \le K||u_n-u||_{L^2}.\end{aligned}$$

Note that K can be taken independent of n since $||u_n||_{L^2}$ is uniformly bounded and (t_n) is a bounded sequence. Thus, L^2 convergence of (u_n) implies $q(u_n,u_n,u_n-u)\to 0$. Similarly, $\int_\Omega b(\cdot,u_n,Du_n)u_n(u_n-u)\to 0$ since $b\in L^\infty(\Omega\times\mathbf{R}\times\mathbf{R}^N)$. Hence,

$$\int_\Omega a^{ij}(\cdot,u_n,Du_n)D_ju_nD_i(u_n-u)\to 0.$$

Since $u_n\to u$ weakly in H_0^1, we can rewrite this as

$$\begin{aligned}(3.7)\quad &\int_\Omega[a^{ij}(\cdot,u_n,Du_n)D_ju_n-a^{ij}(\cdot,u_n,Du)D_ju]D_i(u_n-u)\\ &+\int_\Omega[a^{ij}(\cdot,u_n,Du)-a^{ij}(\cdot,u,Du)]D_juD_i(u_n-u)\to 0.\end{aligned}$$

Note that the second term of (3.7) approaches 0 as $n\to+\infty$ since $a^{ij}(\cdot,u_n,Du)D_ju\to a^{ij}(\cdot,u,Du)D_ju$ in L^2 (by (Q1), (Q3) and dominated convergence) and $||D_iu_n||_{L^2}$ is uniformly bounded. Thus (3.6) holds and by the preceding remarks, so does (3.5).

Now choose $\Omega'\subset\Omega$ with $|\Omega'|=|\Omega|$ and such that (3.5), (Q4) and (Q5) hold for every $x'\in\Omega'$. Also choose Ω' so that $u_{n_k}(x')$, $u(x')$, $Du_{n_k}(x')$, and $Du(x')$ are finite valued for all $x'\in\Omega'$ and so that $u_{n_k}\to u$ everywhere in Ω'. We first show that for $x'\in\Omega'$,

$\limsup |Du_{n_k}(x')| < \infty$. From (Q4) and (3.5), we have, on Ω',

$$\begin{aligned}\alpha_1|Du_{n_k}|^2 &\le a^{ij}(\cdot, u_{n_k}, Du_{n_k})D_j u_{n_k} D_i u_{n_k} \\ &= [a^{ij}(\cdot, u_{n_k}, Du_{n_k})D_j u_{n_k} \\ &\quad - a^{ij}(\cdot, u_{n_k}, Du)D_j u]D_i(u_{n_k} - u) \\ &\quad + a^{ij}(\cdot, u_{n_k}, Du_{n_k})D_j u_{n_k} D_i u \\ &\quad + a^{ij}(\cdot, u_{n_k}, Du)D_j u D_i(u_{n_k} - u) \\ &\le o(1) + ||a^{ij}||_\infty |D_j u_{n_k}||D_i u| \\ &\quad + ||a^{ij}||_\infty |D_j u|(|D_i u_{n_k}| + |D_i u|) \\ &\le o(1) + K_1|Du_{n_k}||Du| + K_2|Du|^2.\end{aligned}$$

This shows that $\limsup |Du_{n_k}(x')| < \infty$ for all $x' \in \Omega'$.

Suppose the conclusion of the lemma is false. Passing to a further subsequence if necessary, we consequently have an $x' \in \Omega'$ such that $Du_{n_{k_l}}(x') \to \xi' \in \mathbf{R}^N$ and $Du(x') = \xi \in \mathbf{R}^N$ with $\xi' \ne \xi$. If we evaluate (3.5) at this particular x' and let $l \to +\infty$, we find

$$[a^{ij}(x', u(x'), \xi')\xi'_j - a^{ij}(x', u(x'), \xi)\xi_j](\xi'_i - \xi_i) = 0.$$

This fact contradicts the monotonicity hypothesis given in (Q5) and thus the lemma is established. □

Lemma 3.3. *Let* $\gamma = \inf_{v \in \tilde{H}^1} \int_\Omega \Phi \phi_1^v$ *where* $\phi_1^v \in H_0^1$ *with* $||\phi_1^v||_{L^2} = 1$ *is defined by* (1.2) *and* $\Phi \in L^\infty$ *satisfies* $\Phi > 0$ *a.e. on* Ω. *Then* $\gamma > 0$.

Proof. Let $(v_k) \subset \tilde{H}^1$ be a minimizing sequence such that $\int_\Omega \Phi \phi_1^{v_k} \to \gamma$. Note by the estimate in (3.4) that $(\phi_1^{v_k})$ is a bounded sequence in H_0^1. Hence there exists a subsequence which we continue to label $\phi_1^{v_k}$, and a function $\phi_1 \in H_0^1$, satisfying the hypotheses and conclusion of Lemma 3.1. Consequently, $\phi_1^{v_k} \to \phi_1$ in L^2 and $\phi_1 > 0$ a.e. on Ω. Hence we obtain $0 < \int_\Omega \Phi \phi_1 = \lim_k \int_\Omega \Phi \phi_1^{v_k} = \gamma$. □

4. Proof of the theorem.

Lemma 4.1. *Suppose* γ *is from Lemma* 3.3. *Let* $|\Omega|$ *denote the Lebesgue measure of* Ω. *Then the problem* (1.1_t) *has no weak solutions* *for* $t > \bar{t} = (1/\gamma)[C|\Omega|^{1/2} + ||h||_{L^2}]$ *where* C *is defined in* (3.2) *and* (3.3).

Proof. Suppose u is a weak solution of (1.1_t). Then $q(u, u, \phi_1^u) = \int_\Omega (f(\cdot, u) + t\Phi + h)\phi_1^u$. But $q(u, u, \phi_1^u) = q(u, \phi_1^u, u)$ by (Q2); therefore, $\int_\Omega (f(\cdot, u) + t\Phi + h)\phi_1^u = \int_\Omega \lambda_1(u)\phi_1^u u$. From (3.2) and (3.3), we obtain

$$\lambda_1(u) \int_\Omega \phi_1^u u \geq \int_\Omega (\delta_1 u - C)\phi_1^u + (t\Phi + h)\phi_1^u,$$

and

$$\lambda_1(u) \int_\Omega \phi_1^u u \geq \int_\Omega (\delta_2 u - C)\phi_1^u + (t\Phi + h)\phi_1^u.$$

Thus,

$$t \int_\Omega \Phi\phi_1^u \leq \int_\Omega (\lambda_1(u) - \delta_1)\phi_1^u u + \int_\Omega (C - h)\phi_1^u, \tag{4.1}$$

and

$$t \int_\Omega \Phi\phi_1^u \leq \int_\Omega (\lambda_1(u) - \delta_2)\phi_1^u u + \int_\Omega (C - h)\phi_1^u. \tag{4.2}$$

From this and the fact that $\delta_1 < \lambda_1(u) < \delta_2$ we can conclude

$$t \int_\Omega \Phi\phi_1^u \leq \int_\Omega (C - h)\phi_1^u, \tag{4.3}$$

because if $\int_\Omega \phi_1^u u \geq 0$ we use (4.2), otherwise we use (4.1). Finally, (4.3) yields the desired conclusion via the Cauchy Schwarz inequality, namely, $t \leq \bar{t} = (1/\gamma)[C|\Omega|^{1/2} + ||h||_{L^2}]$. □

Definition 4.2. A function $u \in W^{1,2}$ is said to be a weak lower solution of (1.1_t) if $u^+ \equiv \max(u, 0)$ is in H_0^1 and

$$q(u, u, \psi) \leq \int_\Omega (f(\cdot, u) + t\Phi + h)\psi \tag{4.4}$$

holds for all $\psi \in H_0^1$ with $\psi \geq 0$ in Ω. Similarly, $v \in W^{1,2}$ is said to be a weak upper solution of (1.1_t) if $v^- \equiv \min(v, 0)$ is in H_0^1 and (4.4) holds, with u replaced by v and with the inequality reversed, for all $\psi \in H_0^1$ with $\psi \geq 0$ in Ω.

To demonstrate the existence of weak solutions of (1.1_t), we employ the following result from [**4**].

Theorem 4.3 (Deuel–Hess). *Suppose* (1.1_t) *satisfies* (Q1)–(Q5), (f1), (f2), *and* (R). *If* (1.1_t) *has a weak lower solution* $u \in W^{1,2}$ *and a weak upper solution* $v \in W^{1,2}$ *with* $u \leq v$ *in* Ω, *then* (1.1_t) *admits a weak solution.*

In the course of showing the existence of weak upper and lower solutions, we must be able to find a weak solution for the equation

$$Qu = \delta_1 u + \tilde{h} \tag{4.5}$$

when $\tilde{h} \in L^2$, and $0 \leq \delta_1 < \Lambda_1$. The standard Schauder fixed point argument normally used for a step of this nature does not seem to work when the coefficients of Q depend on Du. To show that (4.5) does indeed have a weak solution for any $\tilde{h} \in L^2$, we invoke

Theorem 4.4 (Leray–Lions). *For any* $\tilde{h} \in L^2$ *there is at least one weak solution of* (4.5) *in* H_0^1 *provided* Q *satisfies the hypotheses* (Q1)–(Q5), *and* $0 \leq \delta_1 < \Lambda_1$.

This result follows from Theorem 2 in [**7**] once we establish the coercivity condition

$$\frac{q(u,u,u) - \delta_1 \int_\Omega u^2}{||u||_{H_0^1}} \to \infty \quad \text{as} \quad ||u||_{H_0^1} \to \infty. \tag{4.6}$$

Suppose to the contrary that there exists $K_1 \in \mathbf{R}$ and a sequence (u_n) in H_0^1 such that $||u_n||_{H_0^1} \to \infty$ and

$$\frac{q(u_n,u_n,u_n) - \delta_1 \int_\Omega u_n^2}{||u_n||_{H_0^1}} \leq K_1. \tag{4.7}$$

Then one of the following two cases must hold, each of which leads to a contradiction.

Case 1. There exists a subsequence (u_{n_k}) such that $||u_{n_k}||^2_{L^2}/||u_{n_k}||_{H^1_0} \to \infty$. Then dividing

$$(\Lambda_1 - \delta_1)\int_\Omega u^2_{n_k} \le q(u_{n_k}, u_{n_k}, u_{n_k}) - \delta_1 \int_\Omega u^2_{n_k}$$

by $||u_{n_k}||_{H^1_0}$ contradicts (4.7).

Case 2. There exists K_2 such that $||u_n||^2_{L^2}/||u_n||_{H^1_0} < K_2$ for all n. Then (4.7) implies $(q(u_n, u_n, u_n))/||u_n||_{H^1_0} \le K_1 + \delta_1 K_2$.

By (Q4), we have that

$$\alpha_1 ||u_n||_{H^1_0} \le \frac{q(u_n, u_n, u_n)}{||u_n||_{H^1_0}} \le K_1 + \delta_1 K_2,$$

contradicting the fact that $||u_n||_{H^1_0} \to \infty$. Hence, the coercivity condition (4.6) must hold.

We now proceed to demonstrate the existence of weak lower solutions and weak upper solutions.

Lemma 4.5. *For any real number t, (1.1_t) has a weak lower solution u satisfying $u \le 0$ a.e. in Ω.*

Proof. Choose C so large that

$$t\Phi + h < C \qquad \text{a.e. in } \Omega$$

and (3.2) holds. We let $u \in H^1_0$ be a weak solution of $Qu = \delta_1 u - C + t\Phi + h$. The existence of u follows from Theorem 4.4. We claim u is a weak lower solution of (1.1_t). Indeed,

$$q(u, u, \psi) \le \int_\Omega (f(\cdot, u) + t\Phi + h)\psi$$

follows from (3.2) for all $\psi \in H^1_0$ with $\psi \ge 0$ in Ω. Also, if u^+ is not equal to 0 a.e. in Ω, then from the definition of Λ_1, the fact that $\delta_1 < \Lambda_1$ and [**5**, Lemma 7.6], we have

$$\begin{aligned}\delta_1 ||u^+||^2_{L^2} < q(u, u^+, u^+) &= q(u, u, u^+)\\ &= \int_\Omega (\delta_1 u - C + t\Phi + h)u^+ < \delta_1 ||u^+||^2_{L^2}.\end{aligned}$$

This contradiction shows that $u \leq 0$ a.e. in Ω and completes the proof. □

Lemma 4.6. *There exists a real number t such that* (1.1_t) *has a weak upper solution v with $v \geq 0$ a.e. in Ω.*

Proof. For a fixed $N > 0$, define M_N by $M_N = ||h||_\infty + \sup\{||f(\cdot, s)||_\infty : 0 \leq s \leq N\}$. We observe that if $g \in L^{n+1}$, $n \geq 1$ is fixed, and $w \in H_0^1$ is a weak solution of $Qw = Q^w w = g$, then there exists $C_1 > 0$ and independent of g, such that $\sup_\Omega w(x) \leq C_1 ||g||_{L^{n+1}}$ [**5**, Theorem 8.16]. Define $\delta = (N/C_1 M_N)^{n+1} > 0$. Choose open subsets $\Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2 \subset \overline{\Omega}_2 \subset \Omega$ and satisfying $|\Omega - \Omega_1| \leq \delta$. Define $H \in C^0(\overline{\Omega})$ by the following: $H = M_N$ on $\Omega - \Omega_2$, $H = 0$ on Ω_1, and H extends continuously on $\Omega_2 - \Omega_1$, such that $0 \leq H(x) \leq M_N$ for all $x \in \overline{\Omega}$. Now, using Theorem 4.4, let $v \in H_0^1$ be a weak solution of $Qv = Q^v v = H$. Observe that

$$0 \leq \alpha_1 ||Dv^-||_{L^2}^2 \leq q(v, v, v^-) = \int_\Omega H v^- \leq 0$$

which implies $v \geq 0$ a.e. on Ω. In particular, $v^- \in H_0^1$. Furthermore, if t is sufficiently negative to insure $M_N + t\Phi \leq 0$ a.e. in Ω, then

$$q(v, v, \psi) = \int_\Omega H\psi \geq 0 \geq \int_\Omega (M_N + t\Phi)\psi \tag{4.8}$$

holds for all $\psi \in H_0^1$ satisfying $\psi \geq 0$. Now, since

$$0 \leq v(x) \leq C_1 ||H||_{L^{n+1}} \leq C_1 M_N (|\Omega - \Omega_1|^{\frac{1}{n+1}}) \leq C_1 M_N \delta^{\frac{1}{n+1}} \leq N$$

a.e. in Ω, we can conclude that $h(x) + f(x, v(x)) \leq M_N$ for a.e. $x \in \Omega$, and hence, from the estimate in (4.8),

$$q(v, v, \psi) \geq \int_\Omega (f(\cdot, v) + t\Phi + h)\psi$$

for all $\psi \geq 0$, $\psi \in H_0^1$. Thus, v is indeed a weak upper solution for (1.1_t). □

From Lemmas 4.5 and 4.6 we see that if t_0 is sufficiently negative then (1.1_{t_0}) has a weak lower solution u and a weak upper solution v satisfying $u \leq 0 \leq v$. We are now in a position to apply Theorem 4.3, and we conclude that (1.1_t) has at least one weak solution for t sufficiently negative.

Lemma 4.7. *If there exists a weak solution of* (1.1_t) *for* $t = t_0$*, then* (1.1_t) *has a weak solution for all* $t \leq t_0$*.*

Proof. Let $t < t_0$ be fixed and let $v_0 \in H_0^1$ be a weak solution of (1.1_{t_0}). Choose C so large that

$$t\Phi + h \leq C \qquad \text{a.e. in } \Omega$$

and (3.2) holds. Since $t_0 > t$ and $\Phi > 0$, we have

$$f(\cdot, v_0) + t_0\Phi + h > f(\cdot, v_0) + t\Phi + h \geq \delta_1 v_0 - C + t\Phi + h; \tag{4.9}$$

thus, $q(v_0, v_0, \psi) \geq \int_\Omega (f(\cdot, v_0) + t\Phi + h)\psi$ holds for all $\psi \in H_0^1$ with $\psi \geq 0$. Moreover, since $v_0 \in H_0^1$ implies $v_0^- \in H_0^1$, we conclude v_0 is a weak upper solution of (1.1_t).

Toward finding a lower solution u_0 with $u_0 \leq v_0$ in Ω, we observe, using (4.9), that v_0 is a weak upper solution of $Q^{v_0}u = \delta_1 u - C + t\Phi + h$. Hence, by [**5**, Theorem 8.15], we see that there exist positive constants C_1 and C_2 such that

$$\sup_\Omega(-v_0) \leq C_1(||v_0^-||_{L^2} + C_2|| - C + t\Phi + h||_{L^{n+1}}) \equiv M < \infty.$$

Thus, $v_0 \geq -M$ where $M \geq 0$. Now solve for $w \in H_0^1$ such that

$$\int_\Omega a^{ij}(\cdot, w - M, Dw)D_j w D_i \psi + b(\cdot, w - M, Dw)w\psi = \int_\Omega \delta_1(w - M)\psi + (-C + t\Phi + h)\psi \tag{4.10}$$

for all $\psi \in H_0^1$. We can find such a w by applying Theorem 4.4 with $\tilde{h} = -\delta_1 M - C + t\Phi + h$. Using $\psi = w^+$ in (4.10) gives

$$\begin{aligned}\Lambda_1 \int_\Omega (w^+)^2 &\le \int_\Omega a^{ij}(\cdot, w - M, D(w-M)) D_j w^+ D_i w^+ \\ &\qquad + b(\cdot, w-M, D(w-M))(w^+)^2 \\ &= \int_\Omega \delta_1 (w - M) w^+ + (-C + t\Phi + h) w^+ \\ &\le \int_\Omega \delta_1 (w-M) w^+ \le \int_\Omega \delta_1 (w^+)^2.\end{aligned}$$

Since $\delta_1 < \Lambda_1$, we conclude $w \le 0$ a.e. in Ω. (*Note.* It is for the first inequality above that we require the space $\tilde{H}^1$.) Define $u_0 = w - M$. Notice that $u_0^+ \in H_0^1$. Also observe that for all $\psi \in H_0^1$ with $\psi \ge 0$, we have

$$\begin{aligned} q(u_0, u_0, \psi) &= \int_\Omega a^{ij}(\cdot, u_0, Du_0) D_j u_0 D_i \psi + b(\cdot, u_0, Du_0) u_0 \psi \\ &\le \int_\Omega a^{ij}(\cdot, u_0, Dw) D_j w D_i \psi + b(\cdot, u_0, Dw) w \psi \\ &\le \int_\Omega (\delta_1 u_0 - C + t\Phi + h)\psi \le \int_\Omega (f(\cdot, u_0) + t\Phi + h)\psi. \end{aligned}$$

Therefore, u_0 is a weak lower solution of (1.1_t). Furthermore, $u_0 \le -M \le v_0$; hence, Theorem 4.3 implies (1.1_t) has a weak solution in H_0^1. □

Recall that there exists a t_0 such that (1.1_{t_0}) has a weak solution. Consequently, from Lemma 4.7 we have that (1.1_t) has a weak solution for all $t \le t_0$. Let

$$T = \sup\{t : (1.1_t) \text{ has a weak solution}\}.$$

The last step, therefore, in our proof of the theorem is to show (1.1_T) has at least one weak solution.

Lemma 4.8. *For T defined as above, (1.1_T) has at least one weak solution.*

Proof. First observe that $T < \infty$ by Lemma 4.1. Let (t_n) be an increasing sequence of real numbers approaching T and (u_n) the corresponding sequence of weak solutions of (1.1_{t_n}) for $n = 1, 2, \dots$. We claim that $||u_n||_{H_0^1}$ is bounded independent of n. Once this is shown, we are done by the following argument.

The boundedness of (u_n) in Sobolev norm implies that there exists a $u \in H_0^1$ and a subsequence, which we continue to label u_n, such that $u_n \to u$ weakly in H_0^1, strongly in L^2 and a.e. in Ω. In particular, $f(\cdot, u_n) \to f(\cdot, u)$ a.e. by (f1). Also, for any $v \in H_0^1$, we see from (3.1) that $(f(\cdot, u_n)v)_{n=1}^{\infty}$ is an absolutely equi-integrable sequence. Hence, from Egorov, $\int_\Omega f(\cdot, u_n)v \to \int_\Omega f(\cdot, u)v$. Thus, using the identity

$$q(u_n, u_n, v) = \int_\Omega f(\cdot, u_n)v + t_n \Phi v + hv, \qquad \forall\, v \in H_0^1,$$

we conclude that $q(u_n, u_n, v) \to \int_\Omega (f(\cdot, u) + T\Phi + h)v$ as $n \to +\infty$ for all $v \in H_0^1$.

We complete the argument by showing that, for a subsequence (u_{n_k}) of (u_n), we have $q(u_{n_k}, u_{n_k}, v) \to q(u, u, v)$ for all $v \in H_0^1$. Suppose $v \in H_0^1$. Observe that

$$\begin{aligned}
&|q(u_{n_k}, u_{n_k}, v) - q(u, u, v)| \\
&\qquad \leq \left| \int_\Omega [a^{ij}(\cdot, u_{n_k}, Du_{n_k}) - a^{ij}(\cdot, u, Du)] D_j u_{n_k} D_i v \right| \\
&\qquad\quad + \left| \int_\Omega a^{ij}(\cdot, u, Du)[D_j u_{n_k} - D_j u] D_i v \right| \\
&\qquad\quad + \left| \int_\Omega [b(\cdot, u_{n_k}, Du_{n_k}) - b(\cdot, u, Du)] u_{n_k} v \right| \\
&\qquad\quad + \left| \int_\Omega b(\cdot, u, Du)(u_{n_k} - u) v \right|.
\end{aligned}$$

The second and fourth terms on the right hand side above approach 0 as $k \to \infty$ because $u_n \to u$ weakly in H_0^1 and strongly in L^2. From Lemma 3.2 we see that there exists a subsequence u_{n_k} such that $Du_{n_k} \to Du$ a.e. in Ω. Hence, from hypotheses (Q1) and (Q3), we conclude that $a^{ij}(\cdot, u_{n_k}, Du_{n_k}) D_i v \to a^{ij}(\cdot, u, Du) D_i v$ in L^2 by dominated convergence. Similarly, $b(\cdot, u_{n_k}, Du_{n_k}) v \to b(\cdot, u, Du) v$ in L^2. Since we are assuming (u_n) is uniformly bounded in H_0^1, we conclude from Hölder's inequality that the first and third integrals on the right hand side above approach 0 as $k \to \infty$. Hence, if $||u_n||_{H_0^1}$ is uniformly bounded, u is a weak solution of (1.1_T).

Suppose $||u_n||_{H_0^1}$ is not uniformly bounded. Then there exists some subsequence (without loss of generality (u_n) itself) such that $||u_n||_{H_0^1} \to +\infty$. We observe from (Q4) and (3.1) that

$$\begin{aligned}\alpha_1 \int_\Omega |Du_n|^2 \le q(u_n, u_n, u_n) &= \int_\Omega f(\cdot, u_n)u_n + (t_n\Phi + h)u_n \\ &\le \int_\Omega c_1|u_n|^2 + c_2|u_n| + (t_n\Phi + h)|u_n| \\ &\le c_1||u_n||_{L^2}^2 + \tilde{c}_2||u_n||_{L^2},\end{aligned}$$

where $\tilde{c}_2 = \sup_n ||c_2 + t_n\Phi + h||_{L^2} < \infty$. Whence $||u_n||_{L^2} \to +\infty$. We also observe that the above estimate shows that $v_n = u_n/||u_n||_{L^2}$ is bounded in H_0^1 norm. Thus there exists a $v \in H_0^1$ and a subsequence, which we take to be the full sequence, such that $v_n \to v$ weakly in H_0^1, strongly in L^2 and a.e. on Ω. Clearly, $||v||_{L^2} = 1$.

We now proceed to show $v = 0$ a.e. This contradiction implies that $||u_n||_{H_0^1}$ is uniformly bounded and completes the proof. Consider $q(u_n, v_n, \phi_1^n)$ where $\phi_1^n \equiv \phi_1^{u_n}$ is the first eigenfunction of Q^{u_n}. Note that

$$q(u_n, v_n, \phi_1^n) = \int_\Omega \frac{f(\cdot, u_n)\phi_1^n + (t_n\Phi + h)\phi_1^n}{||u_n||_{L^2}}. \tag{4.11}$$

But

$$q(u_n, v_n, \phi_1^n) = q(u_n, \phi_1^n, v_n) = \lambda_1(u_n) \int_\Omega \phi_1^n v_n. \tag{4.12}$$

Now since $0 < \Lambda_1 \le \lambda_1(u_n) \le \Lambda_2 < \infty$, we can choose a subsequence, without loss of generality (u_n) itself, such that $\lambda_1(u_n) \to \tilde{\lambda}_1$ as $n \to +\infty$. Furthermore,

$$\alpha_1 \int_\Omega |D\phi_1^n|^2 \le q(u_n, \phi_1^n, \phi_1^n) = \lambda_1(u_n) \int_\Omega (\phi_1^n)^2 \le \Lambda_2.$$

Therefore, (ϕ_1^n) is bounded in H_0^1 norm and there exists a $\phi_1 \in H_0^1$ and a subsequence (once again labelled (u_n)) such that $\phi_1^n \to \phi_1$ weakly in

H_0^1, strongly in L^2, and a.e. in Ω. By Lemma 3.1, $\phi_1 > 0$ a.e. on Ω. Using (4.11) and (4.12), we conclude

$$\int_\Omega \frac{f(\cdot, u_n)\phi_1^n}{||u_n||_{L^2}} + \int_\Omega \frac{(t_n\Phi + h)\phi_1^n}{||u_n||_{L^2}} \to \tilde{\lambda}_1 \int_\Omega \phi_1 v.$$

However, $\int_\Omega (t_n\Phi + h)\phi_1^n/||u_n||_{L^2} \to 0$ so we are left with

$$\int_\Omega \frac{f(\cdot, u_n)\phi_1^n}{||u_n||_{L^2}} \to \tilde{\lambda}_1 \int_\Omega \phi_1 v. \tag{4.13}$$

We write $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ where $\Omega_1 = \{x \in \Omega : v(x) > 0\}$, $\Omega_2 = \{x \in \Omega : v(x) < 0\}$ and $\Omega_3 = \{x \in \Omega : v(x) = 0\}$. On Ω_1, $u_n = ||u_n||_{L^2} v_n \to +\infty$ pointwise as $n \to +\infty$. Thus, by Egorov's theorem and absolute equi-integrability of $((f(\cdot, u_n)/||u_n||_{L^2})\phi_1^n)$ we obtain

$$\int_{\Omega_1} \frac{f(\cdot, u_n)}{||u_n||_{L^2}}\phi_1^n = \int_{\Omega_1} \frac{f(\cdot, u_n)}{u_n} v_n \phi_1^n \to \int_{\Omega_1} f'(\cdot, +\infty) v\phi_1.$$

Similarly,

$$\int_{\Omega_2} \frac{f(\cdot, u_n)}{||u_n||_{L^2}}\phi_1^n \to \int_{\Omega_2} f'(\cdot, -\infty) v\phi_1.$$

Finally, we observe by (3.1) that

$$\begin{aligned}\left| \int_{\Omega_3} \frac{f(\cdot, u_n)\phi_1^n}{||u_n||_{L^2}} \right| &\le \int_{\Omega_3} c_1 |v_n|\, |\phi_1^n| + \int_{\Omega_3} \frac{c_2|\phi_1^n|}{||u_n||_{L^2}} \\ &\le c_1 ||v_n||_{L^2(\Omega_3)} + \frac{c_2|\Omega_3|^{1/2}}{||u_n||_{L^2}}.\end{aligned}$$

Since $v_n \to 0$ in $L^2(\Omega_3)$ and $||u_n||_{L^2} \to \infty$ we conclude that

$$\int_{\Omega_3} \frac{f(\cdot, u_n)\phi_1^n}{||u_n||_{L^2}} \to 0.$$

Therefore

$$\int_\Omega \frac{f(\cdot, u_n)\phi_1^n}{||u_n||_{L^2}} \to \int_\Omega (f'(\cdot, +\infty)v^+ + f'(\cdot, -\infty)v^-)\phi_1.$$

Combining this with (4.13) gives the equation

$$\int_{\Omega_1} (f'(\cdot, +\infty) - \tilde{\lambda}_1) v\phi_1 + \int_{\Omega_2} (f'(\cdot, -\infty) - \tilde{\lambda}_1) v\phi_1 = 0. \tag{4.14}$$

Recall that $\phi_1 > 0$ a.e. in Ω by Lemma 3.1. Also $f'(\cdot, +\infty) - \tilde{\lambda}_1 > 0$ and $f'(\cdot, -\infty) - \tilde{\lambda}_1 < 0$ since $\Lambda_1 \leq \tilde{\lambda}_1 \leq \Lambda_2$ and f satisfies (f2). Thus, both terms in (4.14) are nonnegative. We therefore conclude that $\int_{\Omega_1} v\phi_1 = 0$ and $\int_{\Omega_2} v\phi_1 = 0$ and hence $v = 0$ a.e. Thus, $||u_n||_{H_0^1}$ must be uniformly bounded. This completes the proof of the lemma, and hence the theorem. □

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