ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 4, Fall 1992

AN EMBEDDING THEOREM AND ITS CONSEQUENCES

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ABSTRACT. It is well known that if X is a Tychonoff space and $F \subset C^*(X)$ separates points and closed sets then the evaluation map e_F corresponding to the family F is an embedding. When e_F is an embedding it does not necessarily follow that F separates points and closed sets. In this note we prove a general embedding theorem and then we use the theorem to characterize exactly those $F \subset C^*(X)$ for which e_F are embeddings. In fact, in our characterization, $C^*(X)$ may be replaced by $C(X)$.

Introduction: Let X be a Tychonoff space and $C^*(X)$ be the set of all real valued bounded continuous functions defined on X. In **1**] Ball and Yokura defined $\mathcal{E}(X)$ as the collection of all subsets F of $C^*(X)$ for which the evaluation maps e_F are embeddings. One of our main purposes is to determine the elements of $\mathcal{E}(X)$. If e_F is an embedding, then F generates the T_2-compactification $e_F X$ of X (see [**2**]). This reason leads us to characterize those $F \subset C^*(X)$ whose evaluation map e_F are embeddings. Some characterizations of the members of $\mathcal{E}(X)$ follow from the embedding theorems of Mrowka [**5**, Theorem 2.1] and Engelking [**4**, p. 122, Problem 2.3.D].

The notion "weakly separates points and closed sets" has been introduced for a family of functions from a topological space to each member of a family of topological spaces. We then use the notion to prove a general embedding theorem which is simpler than those of [**5**] and [**4**]. We use this theorem to characterize the elements of $\mathcal{E}(X)$. We conclude the paper by establishing a generalized version of Lemma 3.5 of [**1**].

1. In order to state and prove our embedding theorem we begin by recalling the following:

Let F be a family of functions $f : X \to Y_f$ where X and Y_f for all $f \in F$ are topological spaces. Then the evaluation map corresponding

AMS (MOS) *Subject Classification*: 54C25, 54C30, 54D35
Key words and phrases. Weakly separates points and closed sets
Received by the editors on December 4, 1988 and in revised form on February 12, 1990.

to the family F is $e_F : X \to \mathbf{X}_{f\in F} Y_f$, the cartesian product of spaces $Y_f, f \in F$, which associates with each $x \in X$ the point $e_F(x)$ in $\mathbf{X}_{f\in F} Y_f$ whose fth coordinate is $f(x)$ for each $f \in F$. That is, if $p_f : \mathbf{X}_{f\in F} Y_f \to Y_f$ is the projection map for each $f \in F$, then for each $x \in X, e_F(x)_f = p_f(e_F(x)) = f(x)$.

The family F is said to *separate points* of X if for each x, $y \in X$, $x \neq y$ there exists $f \in F$ such that $f(x) \neq f(y)$.

The family F is said to *separate points and closed sets in* X if for each closed set A in X and each point $x \in X, x \notin A$ there exists $f \in F$ such that $f(x) \notin \overline{f(A)}$.

It is well known (see [**3**]) that if $Y = \mathbf{X}_{f\in F} Y_f$ is the topological product of the spaces Y_f then for each $y \in Y$, $B \subset Y$, $y \in \bar{B}$ if and only if for each finite cover $\{B_1, B_2, \dots , B_n\}$ of B there exists an $i = 1, 2, \dots , n$ such that $p_f(y) = y_f \in \overline{p_f(B_i)} \forall f \in F$.

For our purposes we define the following concept.

Definition 1.1. Let F be a family of functions $f : X \to Y_f$ where X and Y_f for all $f \in F$ are topological spaces. Then the family F is said to *weakly separate points and closed sets in* X if for each closed set A in X and each point $x \in X, x \notin A$ there exists a finite cover $\{A_1, A_2, \dots , A_n\}$ of A such that for each $i = 1, 2, \dots , n$ there exists an $f_i \in F$ satisfying $f_i(x) \notin \overline{f_i(A_i)}$.

Obviously, if F separates points and closed sets, then it weakly separates points and closed sets. The converse need not hold (see Remark 1.5).

Theorem 1.2. *Let F be a family of functions $f : X \to Y_f$ where X and Y_f for all $f \in F$ are topological spaces. Then the evaluation map $e_F : X \to \mathbf{X}_{f\in F} Y_f$ is an embedding if and only if the following conditions hold*:

(i) *F separates points of X.*

(ii) *Each member of F is continuous.*

(iii) *F weakly separates points and closed sets in X.*

Proof. Note that e_F is one to one if and only if (i) holds and e_F is continuous if and only if (ii) holds.

Suppose that e_F is an embedding. Let A be a closed set in X and $x \in X - A$. Since e_F is an embedding $e_F(A) = \overline{e_F(A)} \cap e_F(X)$ and hence $e_F(x) \notin \overline{e_F(A)}$. This implies that there exists a finite cover $\{B_1, B_2, \ldots, B_n\}$ of $e_F(A)$ such that for each $i = 1, 2, \ldots, n$ there exists an $f_i \in F$ satisfying

$$P_{f_i}(e_F(x)) \notin \overline{p_{f_i}(B_i)}.$$

Note that $f_i(e_F^{-1}(B_i)) = (p_{f_i} \circ e_F)(e_F^{-1}(B_i)) \subset p_{f_i}(B_i)$. Thus $f_i(x) \notin \overline{f_i(e_F^{-1}(B_i))} \forall i = 1, 2, \ldots, n$. Set $A_i = e_F^{-1}(B_i)) \forall i = 1, 2, \ldots, n$. Then $\{A_1, A_2, \ldots, A_n\}$ covers A and $\forall i = 1, 2, \ldots, n, f_i(x) \notin \overline{f_i(A_i)}$, consequently (iii) holds.

Thus, (i), (ii), and (iii) are necessary for e_F to be an embedding.

Conversely, suppose that (i), (ii), and (iii) hold. Then obviously e_F is one to one and continuous. Thus, to show that e_F is an embedding, one needs to check only that e_F is a closed map from X onto the subspace $e_F(X)$.

Let A be a closed set in X. Then obviously

$$e_F(A) = e_F(\bar{A}) \subset \overline{e_F(A)} \cap e_F(X).$$

Let $x \notin A$. By (iii) there exists a finite cover $\{A_1, A_2, \ldots, A_n\}$ of A such that for each $i = 1, 2, \ldots, n$ there exists an $f_i \in F$ satisfying

$$f_i(x) \notin \overline{f_i(A_i)}$$

or

$$p_{f_i}(e_F(x)) \notin \overline{p_{f_i}(e_F(A_i))}.$$

Since $\{e_F(A_1), e_F(A_2), \ldots, e_F(A_n)\}$ covers $e_F(A)$ it follows by the definition of product topology that $e_F(x) \notin \overline{e_F(A)}$ and hence

$$e_F(A) \supset \overline{e_F(A)} \cap e_F(X).$$

Thus

$$e_F(A) = \overline{e_F(A)} \cap e_F(X).$$

Consequently, e_F is a closed map from X onto the subspace $e_F(X)$.

To characterize the elements of $\mathcal{E}(X)$ we need the following lemma.

Lemma 1.3. *If F is a family of functions $f : X \to Y_f$ where X is a T_1-space and Y_f for all $f \in F$ are topological spaces then Condition* (iii) *of Theorem* 1.2 *implies Condition* (i) *of the same theorem.*

Proof is simple and hence omitted.

As usual, for each $f \in C^*(X)$, I_f will denote a closed interval in $\mathbf{R}$ containing$f(X)$. For each subset F of $C^*(X)$, we let $\mathbf{P}_F$ denote the product of the space $\{I_f : f \in F\}$ and $e_F : X \to \mathbf{P}_F$ the evaluation map of F. If e_F is an embedding, then the closure of $e_F(X)$ in $\mathbf{P}_F$ is a T_2-compactification of X and X is necessarily a Tychonoff space. We shall denote by $\mathcal{E}(X)$, the set of all $F \subset C^*(X)$ for which e_F is an embedding. $\mathcal{S}(X)$ will stand for the collection $\{F \subset C^*(X) : F$ separates points and closed sets$\}$. It is well known that $\mathcal{S}(X) \subset \mathcal{E}(X)$ and in general the inclusion is proper (see [**7**, Example 1, p. 483]).

For a Tychonoff space X, below we obtain a necessary and sufficient condition which guarantees that e_F is an embedding for $f \subset C^*(X)$.

Theorem 1.4. *Let X be a Tychonoff space and $F \subset C^*(X)$. Then e_F is an embedding if and only if F weakly separates points and closed sets in X.*

Proof. In view of the facts that each f in F is continuous and X is Tychonoff, the result follows from Lemma 1.3 and Theorem 1.2. ◻

Remark 1.5. In view of Theorem 1.2 and Example 1 of [**7**] it follows that there exists a family of functions which weakly separates points and closed sets but fails to separate points and closed sets. Also it should be noted that Theorem 1.4 remains valid even when $C^*(X)$ is replaced by $C(X)$.

It has been proved in Lemma 3.5 of [**1**] that if $G \in \mathcal{S}(X)$ and F is a subset of $C^*(X)$ such that $\bar{F} \supset G$, then $F \in \mathcal{S}(X)$. In particular,

$\bar{F} \in \mathcal{S}(X)$ implies $F \in \mathcal{S}(X)$, where $\bar{F}$ denotes the closure of F with respect to the topology generated by the sup norm on $C^*(X)$.

We conclude with a generalized version of the above result.

Theorem 1.6. *Let F be a family of continuous functions of a topological space X to a uniform space $(Y,\mathcal{U})$ where Y is equipped with the topology induced by $\mathcal{U}$. Denote by $\tilde{\mathcal{U}}$ the uniformity of uniform convergence in Y^X induced by $\mathcal{U}$. Let $\bar{F}$ be the closure of F in Y^X with the topology generated by $\tilde{\mathcal{U}}$. Then e_F is an embedding if and only if e_F is an embedding.*

Proof. Since $\bar{F} \supset F$, if e_F is an embedding then in view of Theorem 1.2 it trivially follows that $e_{\bar{F}}$ is also an embedding.

Conversely, suppose that $e_{\bar{F}}$ is an embedding. We shall show that F weakly separates points and closed sets in X.

Let A be a closed subset of X and let $x \in X$, $x \notin A$. Since $\bar{F}$ weakly separates points and closed sets, there exists a finite cover $\{A_1, A_2, \dots, A_n\}$ of A and a family $\{g_1, g_2, \dots, g_n\} \subset \bar{F}$, such that $g_i(x) \notin \overline{g_i(A_i)}$ for each $i = 1, 2, \dots, n$. Then there exists $U \in \mathcal{U}$ such that $\{y \in Y : (g_i(x), y) \in U\} \cap g_i(A_i) = \varnothing$ for $i = 1, 2, \dots, n$. Take $W \in \mathcal{U}$ with $W \circ W \circ W \subset U$, and for $i = 1, 2, \dots, n$, choose $f_i \in F$ such that $(g_i(z), f_i(z)) \in W$ for each $z \in X$. Suppose that $f_{i_0}(x) \in \overline{f_{i_0}(A_{i_0})}$ for some $i_0 \in \{1, 2, \dots, n\}$. Then there exists $a \in Ai_0$ such that $(f_{i_0}(x), f_{i_0}(a)) \in W$. Since $(g_{i_0}(x), f_{i_0}(x)) \in W$ and $(f_{i_0}(a), g_{i_0}(a)) \in W$, we obtain that $(g_{i_0}(x), g_{i_0}(a)) \in W \circ W \circ W \subset U$, a contradiction. Hence F weakly separates points and closed sets. Arguing similarly, we show that F separates points of X which completes the proof. □

Acknowledgment. The authors are grateful to the referee for useful comments and suggesting Theorem 1.6.

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