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# ALGEBRAIC CHARACTERIZATION OF DISTRIBUTIONS OF RAPID GROWTH

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ABSTRACT. In this paper we obtain algebraic characterization of the space $\mathcal{K}'_M$ of distributions which grow no faster than $\exp(M(kx))$, and the space $O'_c(\mathcal{K}'_M : \mathcal{K}'_M)$ of its convolution operators; where $M$ is an index function and $k$ is a positive integer. We show that $\mathcal{K}'_M, O'_c(\mathcal{K}'_M : \mathcal{K}'_M)$ are homeomorphic to the vector spaces of module homeomorphisms $\mathrm{Hom}_{\mathcal{K}_M}(\mathcal{K}_M, O_c)$ and $\mathrm{Hom}_{\mathcal{K}_M}(\mathcal{K}_M, \mathcal{K}_M)$, respectively. The relation between these results and invertibility of convolution operators in $\mathcal{K}'_M$ is being discussed.

**1. Introduction.** Algebraic characterization of the space $\mathcal{D}'$ of Schwartz distributions was obtained by Struble [**7**], he proved that $\mathcal{D}'$ is homeomorphic to the vector space of homeomorphisms from $\mathcal{D}$ into $\mathcal{E}$ over $\mathcal{D}$, when both spaces are provided with the topology of pointwise convergence, where $\mathcal{D}$ and $\mathcal{E}$ are the Schwartz spaces of test functions. Abdullah [**1**] obtained algebraic characterizations of the space $\mathcal{K}'_p$ of distributions which grow no faster than $\exp(k|x|^p)$; $p \geq 1$, $k \geq 0$ and the space $O'_c(\mathcal{K}'_p : \mathcal{K}'_p)$ of its convolution operators. It has been shown that $\mathcal{K}'_p$ is homeomorphic to the vector space $\mathrm{Hom}_{\mathcal{K}_p}(\mathcal{K}_p, O_c(\mathcal{K}'_p : \mathcal{K}'_p)$ of homeomorphism from $\mathcal{K}_p$ into $O_c(\mathcal{K}'_p : \mathcal{K}'_p)$ over $\mathcal{K}_p$ and that $O'_c(\mathcal{K}'_p : \mathcal{K}'_p)$ is homeomorphic to the ring $\mathrm{Hom}_{\mathcal{K}_p}(\mathcal{K}_p, \mathcal{K}_p)$ of homeomorphisms from $\mathcal{K}_p$ into itself over $\mathcal{K}_p$. All the spaces involved were provided with their strong topologies. In this paper we extend the results of [**1**] to the spaces $\mathcal{K}'_M$ and $O'_c(\mathcal{K}'_M : \mathcal{K}'_M)$, where $\mathcal{K}'_M$ is the space of distributions which grow no faster than $\exp(M(kx))$ and $O'_c(\mathcal{K}'_M : \mathcal{K}'_M)$ is the space of its convolution operators (see the next section for definitions). We show that $\mathcal{K}'_M$ is homeomorphic to the vector space $\mathrm{Hom}_{\mathcal{K}_M}(\mathcal{K}_M, O_c(\mathcal{K}'_M : \mathcal{K}'_M))$ of module homeomorphisms from $\mathcal{K}_M$ into $O_c(\mathcal{K}'_M : \mathcal{K}'_M)$ over $\mathcal{K}_M$, and that $O'_c(\mathcal{K}'_M : \mathcal{K}'_M)$ is homeomorphic to the ring $\mathrm{Hom}_{\mathcal{K}_M}(\mathcal{K}_M, \mathcal{K}_M)$ of module homeomorphisms from $\mathcal{K}_M$ into itself over $\mathcal{K}_M$. On the one hand, the topological spaces $\mathcal{K}'_M$ and $O'_c(\mathcal{K}'_M : \mathcal{K}'_M)$ are assigned algebraic structures. And, on the other

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hand, the purely algebraic structures $\mathrm{Hom}_{\mathcal{K}_M}(\mathcal{K}_M, O_c(\mathcal{K}'_M : \mathcal{K}'_M))$ and $\mathrm{Hom}_{\mathcal{K}_M}(\mathcal{K}_M, \mathcal{K}_M)$ are assigned topologies. With the topologies they will be equipped with, both spaces are Montel and bornologic. Moreover, we investigate the relation between the unit elements of $\mathrm{Hom}_{\mathcal{K}_M}(\mathcal{K}_M, \mathcal{K}_M)$ and the invertible convolution operators in $\mathcal{K}'_M$.

**2. Notations and preliminary results.** By $N^n, R^n$, we denote the sets of $n$-tuples of nonnegative integers and real numbers, respectively. For $\alpha = (\alpha_1, \dots, \alpha_n)$ in $N^n$, we denote by $|\alpha|$ the sum $\alpha_1 + \cdots + \alpha_n$. By $\mathcal{D}$ and $\mathcal{D}'$ we denote Schwartz spaces of test functions and distributions, by $\mathcal{S}$ we denote the space of infinitely differentiable functions rapidly decreasing at infinity and its strong dual $\mathcal{S}'$ is the space of tempered distribution. For any distribution $T$ we denote by $\overset{\vee}{T}$ its image by symmetry with respect to the origin and by $\tau_h T$, $h \in \mathbf{R}^n$, the translation of $T$ by $h$. For $\alpha \in N^n$ we denote by $D^\alpha$ the differential operator $D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$; where $D_j = (1/i)(\partial/\partial x_j)$; $j = 1, 2, \dots, n$. Let $E$ be a locally convex topological vector space and $E'$ its strong dual; for a bounded subset $B$ of $E$ we denote by $B^O$ the polar of $B$, which is the set of all $T$ in $E'$ such that $|\langle T, \varphi \rangle| < 1$ for all $\varphi$ in $B$. For a topological vector space $V$, we denote by $L_b(V)$ the space of all continuous linear maps in $V$.

The spaces $\mathcal{K}_M$, $O_c(\mathcal{K}'_M, \mathcal{K}'_M)$, $\mathcal{K}'_M$ and $O'_c(\mathcal{K}'_M : \mathcal{K}'_M)$ of test functions and distributions are the same as in [**3**]. The index function $M$ is a continuous, increasing and convex function on $[0, \infty)$ with $M(0) = 0$ and $M(\infty) = \infty$. For negative $x$ we define $M(x)$ to be $M(-x)$. For $x = (x_1, \dots, x_n)$ in $R^n$, $n \geq 2$, $M(x)$ is given by $M(x_1) + M(x_2) + \cdots + M(x_n)$. $\mathcal{K}_M$ is the space of all infinitely differentiable functions $\varphi$ such that

$$\omega_k(\varphi) = \sup_{\substack{x \in \mathbf{R}^n \\ |\alpha| \leq k}} e^{M(kx)} |D^\alpha \varphi(x)| < \infty, \qquad k = 0, 1, 2, \dots .$$

$\mathcal{K}_M$ is equipped with the topology generated by the semi-norms $\omega_k$; $k = 0, 1, 2, \dots$ . It has been proved that $\mathcal{K}_M$ is a Frechet nuclear space, moreover it's Montel (hence reflexive), bornologic and is a normal space of distributions. By $\mathcal{K}'_M$ we denote the strong dual of $\mathcal{K}_M$ provided with the topology of uniform convergence on bounded subsets of $\mathcal{K}_M$; $\mathcal{K}'_M$ is the space of distributions which do not grow faster than $\exp(M(kx))$ for some $k \geq 0$. The elements of $\mathcal{K}'_M$ are called distributions of rapid growth. It turns out that $\mathcal{K}'_M$ is bornologic. For $T \in \mathcal{K}'_M$ and $\varphi \in \mathcal{K}_M$ we define the convolution of $T$ and $\varphi$ by the relation $(T \star \varphi)(x) = \langle T_y, \varphi(x-y) \rangle$. We denote by $O_c(\mathcal{K}'_M : \mathcal{K}'_M)$ the union of the spaces $w^k\ \mathcal{S}$; $k = 0, 1, 2, \ldots$, provided with the inductive limit topology, where $w^k$ denotes $\exp(M(kx))$, and $w^{-k}$ denotes $\exp(-M(kx))$. The space $O'_c(\mathcal{K}'_M : \mathcal{K}'_M)$ is defined as the intersection $\cap_{k=0}^{\infty} w^{-k}\mathcal{S}'$ provided with $\tau_p$ the projective limit topology of the spaces $w^{-k}\mathcal{S}'$, as $k \to \infty$. $O_c(\mathcal{K}'_M : \mathcal{K}'_M)$ is the strong dual of $O'_c(\mathcal{K}'_M, \mathcal{K}'_M)$ (see [**3**]). These spaces will be denoted by $O_c$ and $O'_c$ for simplicity. It turns out (see [**3**]) that on $O'_c$ the topology $\tau_p$ coincides with the topology $\tau_b$ (induced by $L_b(\mathcal{K}_M : \mathcal{K}_M)$) of uniform convergence on bounded subsets of $\mathcal{K}_M$. In the case $M(t) = t^p/p$; $p > 1$, the spaces $\mathcal{K}_M$, $\mathcal{K}'_M$ are the spaces $\mathcal{K}_p$ and $\mathcal{K}'_p$ of Sampson and Zielezny [**6**].

The following two theorems will be used later in the proofs. Theorem A is due to Pahk (see [**5**, Theorem 2, Chapter 1], the proof of Theorem B is similar to a corresponding result for the special case $M(t) = t^p/p$, $p > 1$, and will be omitted (see [**6**, Theorem 2]).

**Theorem A.** *Let $T$ be any distribution*; *the following statements are equivalent.*

i) *$T$ is in $\mathcal{K}'_M$.*

ii) *$T = D^\alpha[w^k f]$, for some multi-index $\alpha$, a positive integer $k$ and a bounded continuous function $f$.*

iii) *For every $\varphi$ in $\mathcal{D}$ there exists a positive integer $k_1$ so that $(T \star \varphi)(x) = O(w^{k_1})$ as $|x| \to \infty$.*

The following theorem characterizes the elements of $\mathcal{K}'_M$ which are in $O'_c$.

**Theorem B.** *Let $S$ be any element of $\mathcal{K}'_M$*; *the following statements are equivalent.*

(1) *$S$ is in $O'_c$.*

(2) *The distributions $w^k S$, $k = 0, 1, 2, \ldots$, are in $\mathcal{S}'$.*

(3) *For any* $k \geq 0$, *there exists a nonnegative integer* $m$ *such that*

$$S = \sum_{|\alpha| \leq m} D^\alpha f_\alpha$$

*where, for each* $\alpha$, $f_\alpha$ *is a continuous function such that* $w^k f_\alpha \in L^\infty$.

(4) *For any* $k$, *the set of distributions* $\{w^{-k}(h)\tau_h S : h \in \mathbf{R}^n\}$ *is bounded in* $\mathcal{D}'$.

(5) $S \star \varphi$ *is in* $\mathcal{K}_M$ *for all* $\varphi$ *in* $\mathcal{K}_M$ *and the map* $\varphi \to S \star \varphi$ *from* $\mathcal{K}_M$ *into* $\mathcal{K}_M$ *is continuous.*

We remark that the condition $S \star \varphi \in \mathcal{K}_M$ for all $\varphi \in \mathcal{K}_M$ (by itself) implies (3). Condition (3) implies continuity of the map $\varphi \to S \star \varphi$ from $\mathcal{K}_M$ into $\mathcal{K}_M$.

For $S \in O'_c$ and $T \in \mathcal{K}'_M$ we define $S \star T$, the convolution of $S$ and $T$, by $\langle S \star T, \varphi\rangle = \langle T, \check{S} \star \varphi\rangle$; $\varphi \in \mathcal{K}_M$, where $(\check{S} \star \varphi)(x) = \langle S_y, \check{\varphi}(x-y)\rangle$. Let $(T_j)$ be a sequence in $\mathcal{K}'_M$ converging to 0. From property (5) of the above theorem it follows that $S \star B$ is bounded in $\mathcal{K}_M$ for every bounded subset $B$ of $\mathcal{K}_M$. Hence, $\langle S \star T_j, \varphi\rangle = \langle T_j, \check{S} \star \varphi\rangle \to 0$ uniformly in $\varphi \in B$. Since $\mathcal{K}'_M$ is bornological it follows that the map $T \to S \star T$ from $\mathcal{K}'_M$ into $\mathcal{K}'_M$ is continuous. The space $O'_c$ is the space of convolution operators in $\mathcal{K}'_M$.

We define the space $O_m$ to be the space of all infinitely differentiable functions $f$ such that for every multi-index $\alpha$ there exists a positive integer $k$ such that $D^\alpha f(x) = O(w^k)$ as $|x| \to \infty$. From the definition of $\mathcal{K}_M$ and Leibniz formula, it follows that $f\varphi \in \mathcal{K}_M$ whenever $\varphi \in \mathcal{K}_M$. We provide $O_m$ with the topology $\tau$ generated by the semi-norms

$$\rho_{k,\varphi}(f) = \sup_{\substack{x \in \mathbf{R}^n \\ |\alpha| \leq k}} w^k |D^\alpha (f\varphi)(x)|; \qquad k = 0, 1, 2, \dots, \ \varphi \in \mathcal{K}_M.$$

The following theorem characterizes the elements of $O_m$.

**Theorem C** [**4**, Theorem 7]. *Let* $f \in \mathbf{C}^\infty(\mathbf{R}^n)$. *The following statements are equivalent.*

(1) $f$ *is in* $O_m$.

(2) *The linear mapping* $\varphi \to f\varphi$ *from* $\mathcal{K}_M$ *into itself is continuous.*

(3) *The linear mapping* $T \to fT$ *from* $\mathcal{K}'_M$ *into itself is continuous.*

We remark that the topology $\tau$ of $O_m$ coincides with the topology $\tau_b$ (induced by $L_b(\mathcal{K}_M)$) of uniform convergence on bounded subsets of $\mathcal{K}_M$ and with the topology $\tau_s$ of simple (pointwise) convergence on the elements of $\mathcal{K}_M$ (see [**4**, Theorem 8]).

Denote by $\mathrm{Hom}\,(\mathcal{K}_M, O_c)$ (respectively, $\mathrm{Hom}\,(\mathcal{K}_M)$) the space of all module homeomorphisms from $\mathcal{K}_M$ into $O_c$ (respectively, from $\mathcal{K}_M$ into $\mathcal{K}_M$) where $\mathcal{K}_M$ and $O_c$ are considered as modules over $\mathcal{K}_M$ with convolution as multiplication. Hence, $F$ is in $\mathrm{Hom}\,(\mathcal{K}_M, O_c)$ (or $\mathrm{Hom}\,(\mathcal{K}_M)$) if $F(\varphi \star \psi) = F(\varphi) \star \psi$ for all $\varphi, \psi$ in $\mathcal{K}_M$.

**3. The results.** The first result provides the algebraic characterization of the spaces $\mathcal{K}'_M$ and $O'_c$ as the spaces $\mathrm{Hom}\,(\mathcal{K}_M, O_c)$ and $\mathrm{Hom}\,(\mathcal{K}_M)$, respectively.

**Theorem 1.**

(1) $\mathcal{K}'_M$ *is isomorphic to* $\mathrm{Hom}\,(\mathcal{K}_M, O_c)$.

(2) $O'_c$ *is isomorphic to* $\mathrm{Hom}\,(\mathcal{K}_M)$.

*Proof.* (1) Let $T \in \mathcal{K}'_M$ be given, define the map $F_T$ from $\mathcal{K}_M$ into $O_c$ by $F_T(\varphi) = T \star \varphi$; it is clear that $F_T$ is in $\mathrm{Hom}\,(\mathcal{K}_M, O_c)$. Conversely, for $F$ in $\mathrm{Hom}\,(\mathcal{K}_M, O_c)$, the element $T_F$ in $\mathcal{K}'_M$ such that $F(\varphi) = T_F \star \varphi$ is found as follows. Let $(\psi_n)$ be a sequence in $\mathcal{D}$ so that $\psi_n \to \delta$ in $\mathcal{E}' \subset O'_c$; thus, for any $\varphi$ in $\mathcal{K}_M$, $\psi_n \star \varphi \to \varphi$ in $\mathcal{K}_M$. Hence $F(\varphi) = \delta \star F(\varphi) = \lim_{n\to\infty} \psi_n \star F(\varphi) = \lim_{n\to\infty} F(\psi_n) \star \varphi$ where the convergence is in $O_c$. In particular, this is true when $\varphi$ is in $\mathcal{D}$; thus, the sequence of distributions $(F(\psi_n))$ converges in $\mathcal{D}'$ to a distribution $T_F$. Hence, $T_F \star \varphi = F(\varphi)$ is in $O_c$ for any $\varphi$ in $\mathcal{D}$. By Theorem A, it follows that $T_F \in \mathcal{K}'_M$. One also has $F(\varphi) = T_F \star \varphi$ for all $\varphi$ in $\mathcal{K}_M$. $T_F$ is well defined, i.e., if $T_F$ and $S_F$ are elements of $\mathcal{K}'_M$ which correspond to $F$, then $T_F = S_F$ as elements of $\mathcal{K}'_M$. Indeed, $(T_F - S_F) \star \varphi = 0$ as an element of $O_c$, for all $\varphi$ in $\mathcal{K}_M$. Hence, $\langle T_F - S_F, \varphi \rangle = 0$ for all $\varphi$ in $\mathcal{K}_M$, and $T_F = S_F$. Finally, one can see that the above established correspondences between $\mathcal{K}'_M$ and $\mathrm{Hom}\,(\mathcal{K}_M, O_c)$ are inverses of each other.

(2) To every $S$ in $O'_c$ corresponds the homeomorphism $F_S$, where $F_S(\varphi) = S \star \varphi$ for all $\varphi$ in $\mathcal{K}_M$. From Theorem B, it follows that $F_S$ is in $\mathrm{Hom}\,(\mathcal{K}_M)$. Conversely, given $F$ in $\mathrm{Hom}_{\mathcal{K}_M}(\mathcal{K}_M, \mathcal{K}_M)$, as in the proof of part (1), there exists an $S_F$ in $\mathcal{K}'_M$ such that $F(\varphi) = S_F \star \varphi$ for all $\varphi$ in $\mathcal{K}_M$. Once more, Theorem B implies that $S_F$ is in $O'_c$. Moreover, $S_F$ is well defined. For, if $T_F$ is another element of $O'_c$ which corresponds to $F$, then, as in part (1), $T_F = S_F$ on $\mathcal{K}_M$. Since $\mathcal{K}_M$ is dense in $O_c$, it follows that $T_F = S_F$ on $O_c$. Thus, $O'_c$ and $\mathrm{Hom}\,(\mathcal{K}_M)$ are isomorphic. □

Since $\mathcal{K}'_M$ is a topological space, the isomorphism of part (1) of Theorem 1 induces in a canonical way a unique topology on $\mathrm{Hom}\,(\mathcal{K}_M, O_c)$—the induced topology—with which it becomes a topological isomorphism. A subset $V$ of $\mathrm{Hom}\,(\mathcal{K}_M, O_c)$ is a member of 0-neighborhood base in the induced topology if its inverse image under the isomorphism of Theorem 1 is a member of 0-neighborhood base of the topology of $\mathcal{K}'_M$. Similarly, the isomorphism of part (2) of Theorem 1 induces a natural topology on $\mathrm{Hom}\,(\mathcal{K}_M)$, with which it becomes a topological isomorphism.

**Theorem 2.**

(1) *The topology induced on* $\mathrm{Hom}\,(\mathcal{K}_M, O_c)$ *by the isomorphism* $F \to T_F$ *can be defined by the following base of neighborhoods of* 0:

$$W(B, U) = \{F \in \mathrm{Hom}\,(\mathcal{K}_M, O_c) : F(\varphi) \in U \text{ for every } \varphi \in B\},$$

*where* $B$ *runs through all the bounded subsets of* $\mathcal{K}_M$ *and* $U$ *runs through all neighborhoods of* 0 *in* $O_c$.

(2) *The topology induced on* $\mathrm{Hom}\,(\mathcal{K}_M)$ *by the isomorphisms* $F \to S_F$ *can be defined by the following base of neighborhoods of* 0:

$$V(B, W) = \{F \in \mathrm{Hom}\,(\mathcal{K}_M) : F(\varphi) \text{ is in } W \text{ for all } \varphi \text{ in } B\},$$

*where* $B$ *runs through all bounded subsets of* $\mathcal{K}_M$ *and* $W$ *runs through all neighborhoods of* 0 *in* $\mathcal{K}_M$.

*Proof.* Since (2) follows immediately from the isomorphisms $F \to S_F$, we only need to prove (1). To show that the topology of $\mathrm{Hom}\,(\mathcal{K}_M, O_c)$ with base of neighborhoods of 0 consisting of the $W(B,U)$'s is weaker than the induced topology, we prove continuity of the map $\Lambda_1$ from $\mathcal{K}'_M$ into $\mathrm{Hom}\,(\mathcal{K}_M, O_c)$ which takes prove continuity of the map $\Lambda_1$ from $\mathcal{K}'_M$ into $\mathrm{Hom}\,(\mathcal{K}_M, O_c)$ which takes $T$ to $F_T$. Without loss of generality, we can assume that $U$ is the polar of $B'$ a bounded subset of $(O'_c, \tau_p)$. From [**3**, Theorem 2] it follows that $B'$ is bounded in $(O'_c, \tau_b)$. Consider the set

$$\begin{aligned} V(B,U) &= \{T_F \in \mathcal{K}'_M : F \in W(B,U)\} \\ &= \{T_F \in \mathcal{K}'_M : T_F \star \varphi \in U \text{ for all } \varphi \in B\}. \end{aligned}$$

It's clear that $V(B,U) = \Lambda_1^{-1}(W(B,U))$ and $V(B,U)$ is the polar of $B' \star B$. The continuity of $\Lambda_1$ will be established provided we show that $B' \star B$ is a bounded subset of $\mathcal{K}_M$. For this, let $U_1$ be a neighborhood of 0 in $\mathcal{K}_M$; we proceed to find $\lambda > 0$ such that $\lambda(B' \star B) \subset U_1$. Consider the set $N(B, U_1) = \{S \in O'_c : S \star \varphi \in U_1$ for all $\varphi$ in $B\}$, $N(B, U_1)$ is a member of 0-neighborhood base for the topology $\tau_b$ of $O'_c$. Since $B'$ is bounded in $O'_c$, there exists a $\lambda > 0$ such that $\lambda B' \subset N(B, U_1)$, i.e., $(\lambda B') \star \varphi \in U_1$ for all $\varphi$ in $B$. Hence, $\lambda(B' \star B) = (\lambda B') \star B = \cup_{\varphi \in B}((\lambda B') \star \varphi) \subset U_1$, i.e., $B' \star B$ is bounded in $\mathcal{K}_M$. Finally, we show that every member of the base of neighborhoods of 0 of the induced topology is $W(B,U)$ for some bounded subset $B$ of $\mathcal{K}_M$ and some $U$ a neighborhood of 0 in $\mathcal{K}_M$. To establish this, we show that the map $\Lambda_2$ from $\mathrm{Hom}\,(\mathcal{K}_M, O_c)$ to $\mathcal{K}'_M$ taking $F$ to $T_F$ is continuous. Let $V(B_1) = B_1^0$, the polar of $B_1$ a bounded subset of $\mathcal{K}_M$ be a member of 0-neighborhood base in $\mathcal{K}'_M$; we find $B_2$ a bounded subset of $\mathcal{K}_M$ and $U_2$ a neighborhood of 0 in $O_c$ such that $\Lambda_2^{-1}(V(B_1)) = W(B_2, U_2) = \{F \in \mathrm{Hom}\,(\mathcal{K}_M, O_c) : F(\varphi) \in U_2$ for all $\varphi$ in $B_2\}$. Take $B_2 = \overset{\vee}{B}_1 = \{\overset{\vee}{\varphi} : \varphi \in B_1\}$ and $U = \{\delta\}^O$, the polar of $\{\delta\}$ which is a bounded subset of $O'_c$. One has

$$\begin{aligned} W(B_2, U_2) &= \{F \in \mathrm{Hom}\,(\mathcal{K}_M, O_c) : F(\overset{\vee}{\varphi}) \in \{\delta\}^O \text{ for all } \varphi \in B_1\}; \\ &= \{F \in \mathrm{Hom}\,(\mathcal{K}_M, O_c)\} : |\langle T_F \star \overset{\vee}{\varphi}, \delta\rangle| < 1 \text{ for all } \varphi \in B_1\}; \\ &= \{F \in \mathrm{Hom}\,(\mathcal{K}_M, O_c) : |\langle T_F, \varphi\rangle| < 1 \text{ for all } \varphi \in B_1\}; \\ &= \Lambda_2^{-1}(V(B_1)). \end{aligned}$$

This completes the proof of the theorem. □

As a result of theorems (1) and (2), we can provide $\mathcal{K}'_M$ with the topology $\tau_c$, which has as 0-neighborhood base all the sets

$$V(B,W) = \{T \in \mathcal{K}'_M : T^*\varphi \in W \text{ for all } \varphi \in B\},$$

where $B$ runs through all bounded subsets of $\mathcal{K}_M$ and $W$ runs through all neighborhoods of 0 in $O_c$. Thus, we have the following

**Corollary.** *On $\mathcal{K}'_M$, the topologies $\tau_c$ and the strong dual topology are equal.*

From Theorems 1 and 2, it follows that the elements of Hom $(\mathcal{K}_M, O_c)$ and Hom $(\mathcal{K}_M)$ are linear and continuous.

A convolution operator $S \in O'_c$ is said to be invertible if it maps $\mathcal{K}'_M$ onto itself. In [**2**] we proved that a convolution operator $S$ is invertible if and only if its Fourier transform $\hat{S}$ is slowly decreasing, i.e., there exist positive constants $C, N$ and $A$ such that

$$\sup_{\substack{|z| \leq A\Omega^{-1}[\log(2+|\xi|)] \\ z \in \mathbf{C}^n}} |\hat{S}(z+\xi)| \geq C(1+|\xi|)^{-N}; \qquad \xi \in \mathbf{R}^n,$$

where $\Omega^{-1}$ is the inverse function of $\Omega$ which is the Young's dual of $M$. It is natural to ask how invertibility of $S$ is related to invertibility of $F_S$ as an element of the ring of module homeomorphisms Hom $(\mathcal{K}_M)$. Actually, for $F$ in Hom $(\mathcal{K}_M)$, we can talk about three types of invertibility: a) invertibility of $F$ as a ring homeomorphism (i.e., $F$ is bijective), b) invertibility of $F$ in the sense that $S_F$ is invertible as a convolution operator on $\mathcal{K}'_M$, c) invertibility of $F$ as a member of the division ring Hom $(\mathcal{K}_M)$. It is clear that the third type is the weakest. For, if $F_1F_2 = 0$, then $S_{F_1} \star S_{F_2} = 0$; hence, $\hat{S}_{F_1} \cdot \hat{S}_{F_2} = 0$. Since $\hat{S}_{F_1}$ and $\hat{S}_{F_2}$ are entire functions (by the Paley-Wiener theorem, see [**2**, p. 199]) it follows that either $\hat{S}_{F_1} = 0$ or $\hat{S}_{F_2} = 0$, i.e., $F_1 = 0$ or $F_2 = 0$. The second type of invertibility is the most interesting one. It would be interesting to know the field of quotients of Hom $(\mathcal{K}_M)$. This could give new conditions for solvability of convolution equations in $\mathcal{K}'_M$. The following result asserts that the first type of invertibility is much stronger than the second type.

**Theorem 3.** *Let $F$ be any element in* $\mathrm{Hom}\,(\mathcal{K}_M)$. *$F$ is bijective if and only if $S_F \star O'_c = O'_c$.*

*Proof.* Suppose that $F$ is bijective, let $F^{-1}$ denote its inverse. For any $\varphi$ in $\mathcal{K}_M$, there exists $\psi$ in $\mathcal{K}_M$ such that $\varphi = S_F \star \psi$. Hence $F^{-1}(\varphi) = \psi, F^{-1} \in \mathrm{Hom}\,(\mathcal{K}_M)$, and there exists $T$ in $O'_c$ such that $F^{-1}(\varphi) = T \star \varphi$ for every $\varphi$ in $\mathcal{K}_M$. Thus $F^{-1}$ is continuous. Define the linear functional $u$ on $\mathcal{K}_M$ by $u(\varphi) = F^{-1}(\varphi)(0)$. Since $F$ is injective, it follows that $u$ is well defined. If $(\varphi_j)$ is a sequence in $\mathcal{K}_M$ which converges to 0 in $\mathcal{K}_M$, then $F^{-1}(\varphi_j)(0) \to 0$, hence $u(\varphi_j) \to 0$. Thus, $u$ is continuous, i.e., $u \in \mathcal{K}'_M$. Moreover, $S_F \star u = \delta$. Indeed, for any $\varphi$ in $\mathcal{K}_M$ one has (assuming without loss of generality $\check{S} = S_F$),

$$\langle S_F \star u, \varphi\rangle = \langle u, \check{S}_F \star \varphi\rangle = F^{-1}(F(\varphi)(0) = \varphi(0) = \langle \delta, \varphi\rangle.$$

Finally, to show that $S_F \star O'_c = O'_c$, it suffices to show that if $T$ is in $\mathcal{K}'_M$ with $S_F \star T = \delta$, then $T^* \varphi$ is in $\mathcal{K}_M$ for every $\varphi$ in $\mathcal{K}_M$. Let $\psi = F^{-1}(\varphi)$, then $S_F \star \psi = F(\psi) = \varphi$. Applying $T$ to both sides one gets $T \star (S_F \star \psi) = \psi = T \star \varphi$, hence $T \star \varphi \in \mathcal{K}_M$. Moreover, $F_T = F^{-1}$. Hence, the map $\varphi \to T \star \varphi$ from $\mathcal{K}_M$ into $\mathcal{K}_M$ is continuous, i.e., $T$ is in $O'_c$.

Conversely, if $S \star O'_c = O'_c$, then there exists a $T \in O'_c$ such that $S \star T = T \star S = \delta$. Hence, $F_T = F_S^{-1}$. Moreover, for any $\varphi$ in $\mathcal{K}_M$ one has $\varphi = S \star (T \star \varphi) = S \star \psi$. Thus, $S \star \mathcal{K}_M = \mathcal{K}_M$ and $F_S$ is onto. The continuity of $F_T$ is trivial since $T$ is in $O'_c$. This completes the proof of the theorem. □

We remark that $\mathcal{K}_M$ is also module with function addition and multiplication. In this case we define $\mathrm{Hom}\,(\mathcal{K}_M)$ as the set of all continuous module homeomorphisms of $\mathcal{K}_M$ over $\mathcal{K}_M$, i.e., for all $\varphi, \psi$ in $\mathcal{K}_M$, $F(\varphi + \psi) = F(\varphi) + F(\psi)$, $F(\varphi\psi) = \varphi F(\psi)$, and the map $\varphi \to F(\varphi)$ is continuous. The following result characterizes the elements of $\mathrm{Hom}\,(\mathcal{K}_M)$.

**Theorem 4.** *The homeomorphism $F$ is in* $\mathrm{Hom}\,(\mathcal{K}_M)$ *if and only if there exists an $f$ in $O_m$ such that $F(\varphi) = f\varphi$ for every $\varphi$ in $\mathcal{K}_M$.*

*Proof.* For any $f$ in $O_m$, define the linear map $F_f$ from $\mathcal{K}_M$ into itself by $F_f(\varphi) = f\varphi$. From Theorem C, it follows that $F_f$ is in Hom $(\mathcal{K}_M)$. Conversely, given $F$ in Hom $(\mathcal{K}_M)$, we proceed to find $f \in O_m$ such that $F(\varphi) = f\varphi$ for all $\varphi$ in $\mathcal{K}_M$. Since $\mathcal{K}_M$ is dense in $O_m$ and the constant function 1 is in $O_m$, there exists a sequence $(\psi_j)$ in $\mathcal{K}_M$ which converges to 1 in $O_m$. Hence, for every $\varphi$ in $\mathcal{K}_M$, the sequence $(\psi_j\varphi)$ converges to $\varphi$ in $\mathcal{K}_M$. By continuity of $F$, it follows that the sequence $(F(\psi_j\varphi)) = (F(\psi_j)\varphi)$ converges to $F(\varphi)$ in $\mathcal{K}_M$. Hence, the sequence $(F(\psi_j))$ converges in $O_m$. Let $f(x) = \lim_{j\to\infty} F(\psi_j)(x)$. Thus, for any $\varphi$ in $\mathcal{K}_M$, one has

$$F(\varphi) = F(1, \varphi) = F(\lim_{j\to\infty} \psi_j\varphi) = \lim_{j\to\infty} F(\psi_j\varphi) = (\lim_{j\to\infty} F(\psi_j))\varphi = f\varphi.$$

The isomorphism of Theorem 4 induces in a canonical way a unique topology on Hom $(\mathcal{K}_M)$ with which it becomes a topological isomorphism. This topology can be described in several ways as the remark which follows Theorem C asserts. □

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