ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 1, Winter 1993

WEAK AND NORM CONVERGENCE ON THE UNIT SPHERE

J.R. TORREGROSA

ABSTRACT. In this paper we prove that the properties (KK) and (K) in a Banach space are stable for the generalized Banach products. We also establish some relationship between these and other properties related with weak and norm convergence on the unit sphere of a Banach space.

1. Notations. We follow standard terminology that can be found in [**1**]. Let $(X, ||\cdot||)$ be a Banach space. B_X denotes its closed unit ball, S_X the unit sphere, X^* the topological dual of X. If (x_n) is a sequence in X, let $\text{sep}\,(x_n) = \inf\{||x_n - x_m||$, for all $n, m \in \mathbf{N}$, $m \neq n\}$. We denote by $\mathcal{F}_f(I)$ the family of the finite subsets of set I. $\mathbf{K}$ denotes the field of real or complex numbers.

2. Introduction. Several classes of Banach spaces have been introduced in the past according to the fulfillment of certain properties related with weak and norm convergence on the unit sphere of a Banach space $(X, ||\cdot||)$. We can mention:

(KK): *Kadec-Klee Property*: If (x_n) is a sequence of elements in X converging weakly to an element x in X such that $||x_n|| \to ||x||$, then (x_n) converges to x in norm, (i.e., for sequences on the unit sphere weak and norm convergence coincide).

(K): *Kadec Property*: The weak and the norm topology coincide on the unit sphere.

(α): *Property* (α) (Rolewicz [**6**]): Given an element $f \in X^*$ such that $||f|| = 1$ and $\varepsilon > 0$, let

$$S(f, \varepsilon) = \{x : x \in B_X, f(x) \geq 1 - \varepsilon\}.$$

The Kuratowski index of noncompactness $\alpha(A)$ of a subset A of X is defined as the infimum of all positive numbers r such that A can be

Received by the editors on November 27, 1990.
AMS *Subject Classification.* 46B20, 46B10, 52A07.
Key words and phrases. Reflexive Banach spaces, locally uniformly rotund.

covered by a finite number of sets of diameter less than r. Then a Banach space $(X, ||\cdot||)$ is said to have *property* (α) if for every $f \in X^*$, $||f|| = 1$,

$$\lim_{\varepsilon \to 0} \alpha(S(f, \varepsilon)) = 0.$$

$(L\alpha)$: *Property* $(L\alpha)$: Given $f \in X^*$, $||f|| = 1$, such that f attains the supremum in $x_0 \in S_X$, then

$$\forall\, \varepsilon > 0, \quad \exists\, \delta = \delta(\varepsilon, f) > 0 \quad \text{such that} \quad \alpha(S(f, \sigma)) < \varepsilon.$$

It is evident that $(\alpha) \Rightarrow (L\alpha)$, and both properties coincide in reflexive spaces.

In [**6**], S. Rolewicz proved that *every Banach space with property* (α) *is reflexive.* D. Kutzarova [**3**] completed this result given a geometric characterization of reflexivity. She proved that a "Banach space X is reflexive if and only if it admits an equivalent norm with property (α)." Later V. Montesinos [**5**] showed that a "Banach space has property (α) if and only if it has property (KK) and it is reflexive."

Obviously, (K) $\Rightarrow$ (KK). The converse is not true, but S.L. Troyanski [**9**] showed that "both properties are equivalent for every separable Banach space which does not contain l^1." Even more, this equivalence is true for a not necessarily separable Banach space.

I.E. Leonard [**4**] studied the behavior of property (KK) in relation to the Banach products of l^p type. With this paper, we expand the Leonard results, and we enlarge on the family of Banach spaces with property (KK), establishing the stability of this property for the spaces introduced by R. Huff [**2**], that generalize "Banach products" for a countable quantity of spaces.

In addition, we characterize property (K) in terms of nets $||\cdot||$-convergence which allows us to obtain for this property analogous results to those mentioned before.

3. Kadec-Klee property. The following simple result relates properties (KK) and $(L\alpha)$.

Proposition 3.1. *Every Banach space* $(X, ||\cdot||)$ *with property* $(L\alpha)$ *has property* (KK).

Proof. Let (x_n) be a sequence in S_X which converges weakly to $x_0 \in S_X$. Let (y_n) be an arbitrary subsequence of (x_n), and let $f \in X^*$ be such that $||f|| = 1$ and $f(x_0) = 1$. Given $\varepsilon > 0$, let $\delta > 0$ be such that $\alpha(S(f, \delta)) < \varepsilon$. Let n_0 be a positive integer such that, for every $n \geq n_0$, $y_n \in S(f, \delta)$. $S(f, \delta)$ can be covered by a finite number of sets of diameter less than ε. It is now clear that a diagonal procedure allows us to select a Cauchy subsequence (z_n) of (y_n). Then (z_n) is $|| \cdot ||$-convergent to some $z \in X$. But (z_n) converges weakly to x_0. So $z = x_0$, and because (y_n) was an arbitrary subsequence of (x_n), it follows that (x_n) converges to x_0. Thus $(X, || \cdot ||)$ has property (KK). $\square$

The following spaces have been used by R. Huff in [**2**].

Definition 3.2. Let $(Y, ||\cdot||)$ be a Banach space with basis $\{e_i : i \in I\}$ (unconditional if I is noncountable) and such that, for every finite subset J of I,

$$0 \leq \alpha_j \leq \beta_j \qquad \forall\, j \in J \Rightarrow \left\| \sum_{j \in J} \alpha_j e_j \right\| \leq \left\| \sum_{j \in J} \beta_j e_j \right\|.$$

Let $\{X_i : i \in I\}$ be a family of Banach spaces. Let us consider the space

$$Y(X_i : i \in I) = \left\{ x = (x_i)_{i \in I} \in \prod_{i \in I} X_i : \sum_{i \in I} ||x_i|| e_i \in Y \right\},$$

endowed with the norm

$$||x|| = \left\| \sum_{i \in I} ||x_i|| e_i \right\|_Y.$$

The space $Y(X_i : i \in I)$ with this norm is a Banach space.

The following result shows the stability of property (KK) for the spaces $Y(X_i : i \in I)$.

Theorem 3.3. *Let* $(Y, || \cdot ||)$ *be a Banach space with basis* $\{e_i : i \in I\}$ (*unconditional if* I *is noncountable*) *and such that, for every finite* *subset J of I,*

$$0 \leq \alpha_j \leq \beta_j \qquad \forall\, j \in J \Rightarrow \left\| \sum_{j \in J} \alpha_j e_j \right\| \leq \left\| \sum_{j \in J} \beta_j e_j \right\|.$$

Let us suppose that $\{e_j^* : j \in I\}$ is $||\cdot||$-total in Y^*, where $\{e_j^* : j \in I\}$ is the family of coefficients of functionals.

Let $\{X_i : i \in I\}$ be a family of finite dimensional Banach spaces. Then, if $(Y, ||\cdot||)$ has property (KK), $Z = Y(X_i : i \in I)$ also has this property.

Proof. Let (x^n) be a sequence in Z which converges weakly to $x^0 \in Z$ and $(||x^n||) \to ||x^0||$. We shall show that (x^n) converges to x^0 in norm.

First of all, we shall prove that (x_i^n) is $\sigma(X_i, X_i^*)$-convergent to x_i^0, for all $i \in I$, and therefore (x_i^n) is $||\cdot||$-convergent to x_i^0 for all $i \in I$.

Let $i \in I$. Given an element $u_i \in X_i^*$, we define $u : z \to \mathbf{K}$ as

$$\langle x, u \rangle = \langle x_i, u_i \rangle, \qquad \forall\, x = (x_i)_{i \in I} \in Z.$$

It is easy to prove that $u \in Z^*$. Then

$$\lim_{n \to \infty} (\langle x_i^n, u_i \rangle) = \lim_{n \to \infty} (\langle x^n, u \rangle) = \langle x^0, u \rangle = \langle x_i^0, u_i \rangle.$$

Thus (x_i^n) is $\sigma(X_i, X_i^*)$-convergent to x_i^0 for all $i \in I$.

Now let $\phi : Z \to Y$ be defined by

$$\phi(x) = \phi((x_i)_{i \in I}) = \sum_{i \in I} ||x_i|| e_i, \qquad \forall\, x \in Z.$$

Let e_j^* be a coefficient functional. Then

$$\langle \phi(x^n), e_j^* \rangle = \langle \sum_{i \in I} ||x_i^n|| e_i, e_j^* \rangle = ||x_j^n|| \xrightarrow{n} ||x_j^0|| = \langle \phi(x^0), e_j^* \rangle.$$

Since $\{e_i^* : i \in I\}$ is $||\cdot||$-total in Y^*, we have that $(\phi(x^n))_n$ converges weakly to $\phi(x^0)$. In addition,

$$||\phi(x^n)|| = ||x^n|| \xrightarrow{n} ||x^0|| = ||\phi(x^0)||.$$

Hence, because Y has property (KK), we have that $(\phi(x^n))$ is convergent to $\phi(x^0)$ in norm.

If E Is a subset of I, let $P_E : Y \to Y$ be defined by

$$P_E\left(\sum_{i\in I}\alpha_i e_i\right) = \sum_{i\in E}\alpha_i e_i$$

(in case I is countable and $\{e_i\}$ is not unconditional, we consider only those E's of the form $\{1,2,\dots,n\}$ and $\{n,n+1,\dots\}$). Choose $K>0$ such that $||P_E|| \le K$ for all E. Let $F \in \mathcal{F}_f(I)$.

(1)

$$\begin{aligned}
||x^0 - x^n|| &= \left\|\sum_{i\in I}||x_i^0 - x_i^n||e_i\right\| \\
&= \left\|\sum_{i\in F}||x_i^0 - x_i^n||e_i + \sum_{i\in I\setminus F}||x_i^0 - x_i^n||e_i\right\| \\
&\le \left\|\sum_{i\in F}||x_i^0 - x_i^n||e_i\right\| + \left\|\sum_{i\in I\setminus F}||x_i^0 - x_i^n||e_i\right\| \\
&\le \left\|\sum_{i\in F}||x_i^0 - x_i^n||e_i\right\| + \left\|\sum_{i\in I\setminus F}(||x_i^0|| + ||x_i^n||)e_i\right\| \\
&\le \left\|\sum_{i\in F}||x_i^0 - x_i^n||e_i\right\| + \left\|\sum_{i\in I\setminus F}||x_i^0||e_i\right\| + \left\|\sum_{i\in I\setminus F}||x_i^n||e_i\right\|.
\end{aligned}$$

Let $\varepsilon > 0$, and choose $F \in \mathcal{F}_f(I)$ such that

$$\left\|\sum_{i\in I\setminus F}||x_i^0||e_i\right\| \le \varepsilon/4. \tag{2}$$

In addition, there exists some $n_1 \in \mathbf{N}$ so that

$$\left\|\sum_{i\in F}||x_i^0 - x_i^n||e_i\right\| \le \varepsilon/4, \qquad \forall\, n \ge n_1. \tag{3}$$

On the other hand, there exists some $n_2 \in \mathbf{N}$ so that

$$||\phi(x^n) - \phi(x^0)|| \le \varepsilon/4K, \qquad \forall\, n \ge n_2.$$

Then

$$\begin{aligned}\Big\| \sum_{i\in I\setminus F} ||x_i^n||e_i\Big\| &- \Big\| \sum_{i\in I\setminus F} ||x_i^0||e_i\Big\| \\ &\leq \Big\| \sum_{i\in I\setminus F} ||x_i^n||e_i - \sum_{i\in I\setminus F} ||x_i^0||e_i\Big\| \\ &= ||P_{I\setminus F}(\phi(x^n)-\phi(x^0))|| \\ &\leq K||\phi(x^n)-\phi(x^0)|| \leq \varepsilon/4,\end{aligned}$$

hence

$$(4)\qquad \Big\| \sum_{i\in I\setminus F} ||x_i^n||e_i\Big\| \leq \varepsilon/4+\varepsilon/4=\varepsilon/2, \qquad \forall\, n\geq n_2.$$

Let $n_0=\max\{n_1,n_2\}$. From (1), (2), (3) and (4), we have

$$||x^0-x^n|| \leq \varepsilon/4+\varepsilon/4+\varepsilon/2=\varepsilon, \qquad \forall\, n\geq n_0. \qquad \square$$

For infinite dimensional Banach spaces X_i, we have the following result, whose demonstration follows step by step the proof of Theorem 3.1 in [**4**].

Theorem 3.4. *Let p be with $1\leq p<\infty$, and let $\{X_n : n\in \mathbf{N}\}$ be a countable family of Banach spaces. Let us consider the Banach space $l^p(X_1,X_2,\dots)$ defined by*

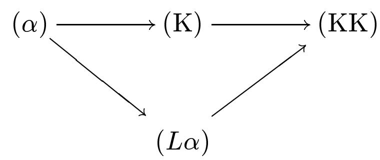
$$l^p(X_1,X_2,\dots)=\{x=(x_n) : x_k\in X_k,\ k=1,2,\dots,\ ||x||_p^p=\sum_{k=1}^{\infty}||x_k||^p<+\infty\}$$

and endowed with $||\cdot||_p$. Then $l^p(X_1,X_2,\dots)$ has property (KK) *if and only if the spaces X_k, $k=1,2,\dots$, have this property.*

4. Kadec property. The following result gives the connection between properties (α) and (K). We omit the simple proof.

Proposition 4.1. *If a Banach space* $(X, ||\cdot||)$ *has property* (α)*, then it has property* (K)*.*

We have



The following examples show that the converse implications are not true.

Example 1. A simple example of a space which is (KK) but not $(L\alpha)$ is the space l^1 endowed with the norm $||\cdot||_1$.

It is clear that l^1 with $||\cdot||_1$ is (KK). We shall prove that it does not have property $(L\alpha)$. Let $f = (1,1,1,\dots) \in S_{l^\infty}$, $\varepsilon_0 = 1/2$ and $\delta > 0$ arbitrarily. Let $x^n := (1-\mu)e_n$, $n \in \mathbf{N}$, where $\{e_n\}$ is the unit vector basis in l^1 and μ has been chosen such that $0 < \mu < \delta$. We have

$$x^n \in S(f,\delta) \qquad \forall\, n \in \mathbf{N},$$

and

$$||x^n - x^m|| = 2(1-\mu), \qquad \forall\, n, m \in \mathbf{N}, n \neq m.$$

So that (x^n) is a sequence in $S(f,\delta)$ such that $\operatorname{sep}(x^n) > \varepsilon_0$, then $\alpha(S(f,\delta)) > \varepsilon_0$.

On the other hand, $(l^1, ||\cdot||_1)$ has property (K) [**9**] and, obviously, it does not have property (α).

Example 2. Let X be a separable Banach space which is not reflexive. By Troyanski's renorming Theorem [**8**], X has an equivalent norm $||\cdot||$ which is locally uniformly rotund (= LUR) (i.e., for any sequence (x_n) and x in X such that $||x_n|| \leq 1$, $n = 1, 2, \dots, ||x|| = 1$, if

$$\lim_{n\to\infty} ||x + x_n|| = 2,$$

then (x_n) converges in norm to x). It is easy to prove that an (LUR)-norm has property $(L\alpha)$. This example allows us to separate properties (α) and $(L\alpha)$.

Example 3. Let C be the space of all convergent sequences with the norm

$$||x|| = \left[\sum_{n=1}^{\infty} \frac{x_n^2 + |x|_n^2}{2^{n+1}}\right]^{1/2}, \qquad \forall\, x = (x_n) \in C,$$

where $|x|_n = \sup\{|x_k| : k \geq n\}$.

S.L. Troyanski [**9**] proved that $(C, ||\cdot||)$ does not have property (K), but it is MLUR (i.e., for any sequence (x_n) and x in X such that

$$\lim_{n\to\infty} [||x + x_n||^2 + ||x - x_n||^2 - 2||x||^2] = 0,$$

then $(||x_n||)$ converges to 0). It is easy to prove that a space MLUR has property (KK).

It's known that (K) and (KK) properties are equivalent in a Banach space $(X, ||\cdot||)$ which does not contain l^1, as we can obtain from an extension of the Rosenthal's criterion [**7**] that every bounded subset of X is weakly sequentially dense in its weak closure.

This equivalence and Montesinos's result [**5**] allows us to establish that:

"In a Banach space $(X, ||\cdot||)$ reflexive, the properties (α), $(L\alpha)$, (K) and (KK) are equivalent."

In order to prove the stability of property (K) for Banach products of l^p type and for spaces $Y(X_i : i \in I)$ we need the following simple characterization of this property

Proposition 4.2. *A Banach space* $(X, ||\cdot||)$ *has property* (K) *if and only if, given* $(x_i : i \in I, \leq)$*, a net in* X *converging weakly to* $x \in X$ *such that* $(||x_i||) \to ||x||$*, then* $(x_i : i \in I, \leq)$ *converges to* x *in norm.*

Proof. Obviously, the condition is sufficient. Now we suppose that X has property (K). Let $(x_i : i \in I, \leq)$ be a net in X which converges weakly to $x \in X$ and $(||x_i||) \to ||x||$.

In the nontrivial case $x \neq 0$, we can suppose $||x_i|| > K > 0$ for all $i \in I$. It is evident that $(x_i/||x_i||)$ is $\sigma(X, X^*)$-convergent to $x/||x||$ in

S_X. Then $(x_i/||x_i||)$ is $||\cdot||$-convergent to $x/||x||$. So (x_i) converges to x in norm. $\square$

The following result is close in spirit to a theorem given by I.E. Leonard [**2**] concerning (KK)-property.

Theorem 4.3. *Let p be with $1 \le p < \infty$, and let $\{X_n : n \in \mathbf{N}\}$ be a countable family of Banach spaces. $Z = l^p(X_1, X_2, \dots)$ with the norm $||\cdot||_p$ has property* (K) *if and only if the spaces X_k, $k = 1, 2, \dots$, have this property.*

Proof. The space X_k is isometric to the closed linear subspace of those points of Z of the form $(0, 0, \dots, x_k, 0 \dots)$. Since property (K) is stable for closed subspaces and isometries, if Z has property (K), then X_k has this property, $k = 1, 2, \dots$.

Conversely, suppose that all spaces X_k have property (K). Let $(x^i : i \in I, \le)$ be a net in Z which converges weakly to $x \in Z$ such that $(||x^i||) \to ||x||$. Obviously, $(x_k^i)_i$ is $\sigma(X_k, X_k^*)$-convergent to x_k, $k = 1, 2, \dots$. We claim that $(||x_k^i||)_i \to ||x_k||$, $k = 1, 2, \dots$, and by property (K) of X_k, $(x_k^i)_i$ is $||\cdot||$-convergent to x_k, $k = 1, 2, \dots$. Then (x^i) is $||\cdot||_p$-convergent to x. By Proposition 4.2, Z has property (K).

To prove the claim, we define $f_i \in \mathbf{R}^{\mathbf{N}}$, $i \in I$, by

$$f_i(k) = ||x_k^i||^p, \qquad k = 1, 2, \dots, \quad i \in I.$$

It is easy to prove that there exists $M > 0$ such that

$$(f_i : i \in I, i \ge i_0) \subset [0, M^p]^{\mathbf{N}},$$

then there exists $(f_j : j \in J)$ a subset of $(f_i : i \in I, i \ge i_0)$ such that (f_j) is convergent to $f \in [0, M^p]^{\mathbf{N}}$ in the product topology, i.e.,

$$(||x_k^j||^p)_j \to f(k), \qquad \forall\, k \in \mathbf{N}.$$

By the weak lower semicontinuity of the norm, $||x_k||^p \le f(k)$, $k = 1, 2, \dots$. We shall prove that $||x_k||^p = f(k)$ for all $k \in \mathbf{N}$.

Let us suppose instead that there exists $k_0 \in \mathbf{N}$ such that $||x_{k_0}||^p < f(k_0)$. We can choose $\varepsilon > 0$ such that

$$||x_{k_0}||^p < ||x_{k_0}||^p + \varepsilon < f(k_0).$$

Let $N \in \mathbf{N}$ be such that $k_0 < N$ and $\sum_{k=N+1}^{\infty} ||x_k||^p < \varepsilon/2$. We have

$$\sum_{k=1}^{N} ||x_k||^p = ||x||^p - \sum_{k=N+1}^{\infty} ||x_k||^p > ||x||^p - \varepsilon/2.$$

Then

$$\begin{aligned}\lim_j \sum_{k=1}^{N} ||x_k^j||^p &= \lim_j \sum_{k=1}^{N} f_j(k) = \sum_{k=1}^{N} f(k) \\ &> \sum_{k=1}^{N} ||x_k||^p + \varepsilon > ||x||^p + \varepsilon/2.\end{aligned} \tag{1}$$

On the other hand,

$$\sum_{k=1}^{\infty} ||x_k^j||^p \geq \sum_{k=1}^{N} ||x_k^j||^p, \tag{2}$$

i.e.,

$$||x^j||^p \geq \sum_{k=1}^{N} ||x_k^j||^p, \qquad \forall\, j \in J.$$

From (1) and (2), we obtain

$$||x||^p \geq ||x||^p + \varepsilon/2,$$

a contradiction.

Since the previous reasoning is true for an arbitrary subnet of $(f_i : i \in I, i \geq i_0)$, it follows that $(f_i : i \in I, i \geq i_0)$ converges to f in product topology. This concludes the proof. □

Finally, using Proposition 4.2, we obtain the following result whose demonstration follows step by step the proof of Theorem 3.3.

Theorem 4.4. *Let $(Y, ||\cdot||)$ be a Banach space with basis $\{e_i : i \in I\}$ (unconditional if I is noncountable) and such that, for every finite subset J of I,*

$$0 \leq \alpha_j \leq \beta_j, \qquad \forall\, j \in J \Rightarrow \left\| \sum_{j\in J} \alpha_j e_j \right\| \leq \left\| \sum_{j \in J} \beta_j e_j \right\|.$$

Let us suppose that $\{e_j^* : j \in J\}$ *is* $||\cdot||$*-total in* Y^**.*

Let $\{X_i : i \in I\}$ *be a family of finite dimensional Banach spaces. Then, if* $(Y, ||\cdot||)$ *has property* (K), $Y(X_i : i \in I)$ *also has this property.*

Acknowledgment. The author would like to thank Professor V. Montesinos for discussions concerning the topic of this paper.

REFERENCES

1. M.M. Day, *Normed linear spaces*, (3rd Ed), Springer, New York, 1973.

2. R. Huff, *Banach spaces which are nearly uniformly convex*, Rocky Mountain J. Math. **10** (1980), 743–749.

3. D.N. Kutzarova, *A sufficient condition for the drop property*, Compt. Rend. Acad. Bulg. Sci. **39** (1986), 17–19.

4. I.E. Leonard, *Banach sequence spaces*, J. Math. Anal. Appl. **54** (1976), 245–265.

5. V. Montesinos, *Drop property equals reflexivity*, Studia Math. **87** (1987), 93–100.

6. S. Rolewicz, *On drop property*, Studia Math. **85** (1987), 27–35.

7. H. Rosenthal, *Some recent discoveries in the isomorphic theory of Banach spaces*, Bull. Amer. Math. Soc. **84** (1978), 803–831.

8. S.L. Troyanski, *On locally uniformly convex and differentiable norms in certain non-separable Banach spaces*, Studia Math. **37** (1971), 173–180.

9. ———, *On a property of the norm which is close to local uniform rotundity*, Math. Ann. **271** (1985), 305–313.

Departamento de Matemática Aplicada, E.T.S.I. Telecomunicación, Universidad Politécnica de Valencia, C/Vera, s/n, 46071-Valencia, Spain