ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 1, Winter 1993

A DUAL BASIS FOR THE INTEGER TRANSLATES OF AN EXPONENTIAL BOX SPLINE

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ABSTRACT. Exponential box splines are multivariate compactly supported functions on a regular mesh. Let ϕ be an exponential box spline associated with integer vectors. Then ϕ is piecewise in a space H spanned by exponential polynomials. In this paper we construct a dual basis for the integer translates of ϕ, when these translates are linearly independent. The dual basis is shown to be unique in a certain sense. Our construction is based on a study of the polynomial space F which consists of all polynomials p such that $p(D)\phi$ is a bounded function, where $p(D)$ denotes the partial differential operator induced by p. It turns out that the linear space F is dual to H. Thus, as a by-product, we give a short proof for the formula of the dimension of H.

1. Introduction. As usual, let $\mathbf{N}, \mathbf{Z}, \mathbf{R}$ and $\mathbf{C}$ be the set of nonnegative integers, integers, real and complex numbers, respectively. Let s be a positive integer. Denote by $\mathbf{R}^s$ the s-dimensional Euclidean space. Given two vectors x and y in $\mathbf{R}^s$, we denote by $x \cdot y$ the inner product of them, and by $|x|$ the norm of $x : |x| = \sqrt{x \cdot x}$. The linear span of a collection X of vectors in $\mathbf{R}^s$ will be denoted by span (X). All the continuous complex-valued functions supported on compact sets in $\mathbf{R}^s$ form a linear space over $\mathbf{C}$, which we shall denote by $C_c(\mathbf{R}^s)$. Given $\theta \in \mathbf{C}^s$, the exponential function $x \mapsto \exp(\theta \cdot x)$, $x \in \mathbf{R}^s$, is denoted by e_θ.

We shall use the standard multi-index notation. For $x = (x_1, \dots , x_s) \in \mathbf{R}^s$ and $\alpha = (\alpha_1, \dots , \alpha_s) \in \mathbf{N}^s$, x^α is the monomial given by

$$x^\alpha := x_1^{\alpha_1} \cdots x_s^{\alpha_s}.$$

A polynomial is a complex linear combination of monomials. The total degree of a polynomial p is denoted by $\deg p$. All the polynomials on

Received by the editors on January 30, 1990, and in revised form on April 5, 1991.
AMS (MOS) *Subject Classifications.* Primary 41A63, 41A15, Secondary 41A05, 15A03.
Key words. box splines, exponential box splines, integer translates, dual bases, Poisson's Summation Formula.

$\mathbf{R}^s$ form a linear space over $\mathbf{C}$, which we shall denote by $\pi(\mathbf{R}^s)$, or by π for short. By π_k we denote the subspace of all polynomials of degree at most k. An exponential polynomial is a function of the form $\sum_{\theta\in\Theta} e_\theta p_\theta$, where Θ is a finite subset of $\mathbf{R}^s$, and $p_\theta \in \pi(\mathbf{R}^s)$ for each $\theta \in \Theta$.

For $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbf{N}^s$, let

$$D^\alpha := D_1^{\alpha_1} \cdots D_s^{\alpha_s},$$

where D_j denotes the partial differential operator with respect to the jth argument, $j = 1, \dots, s$. If $p \in \pi(\mathbf{R}^s)$, $p(x) = \sum_\alpha a_\alpha x^\alpha$, then the differential operator $\sum_\alpha a_\alpha D^\alpha$ is denoted by $p(D)$. The kernel of $p(D)$, denoted by $\ker(p(D))$, is the linear space of all infinitely differentiable complex-valued functions f on $\mathbf{R}^s$ such that $p(D)f = 0$.

During the past decade multivariate spline theory has developed rapidly. In particular, the box spline introduced by de Boor and Höllig [**6**] has attracted much attention. Recently, Ron [**16**] initiated the study of exponential box splines, which are a generalization of box splines. Various interesting properties of exponential box splines were developed in [**1**] and [**11**].

Let $X = (x^1, \dots, x^n) \subset \mathbf{R}^s\backslash\{0\}$ and $\mu = (\mu_1, \dots, \mu_n) \in \mathbf{C}^n$, the exponential box spline $C_\mu(\cdot|X)$ is defined to be the linear functional on $C_c(\mathbf{R}^s)$ given by the rule

$$\phi \mapsto \int_{[0,1]^n} e^{-\mu\cdot u}\phi\biggl(\sum_{j=1}^n x^j u_j\biggr)\,du, \qquad \phi \in C_c(\mathbf{R}^s).$$

When $\mu = 0$, $C_\mu(\cdot|X)$ reduces to the box spline $B(\cdot|X)$. It is known that $C_\mu(\cdot|X)$ is a distribution supported on

$$[X] := \biggl\{\sum_{j=1}^n t_j x^j : 0 \le t_j \le 1, \text{ all } j\biggr\}. \tag{1.1}$$

Moreover, if X spans $\mathbf{R}^s$, then $C_\mu(\cdot|X)$ is a piecewise exponential polynomial function.

In the study of univariate splines, de Boor and Fix [**5**] constructed a dual basis for the sequence of B-splines, which has proved to be very useful. Thus, it is desirable to extend the construction of de Boor and Fix to the multivariate case.

Consider the integer translates $C_\mu(\cdot - \beta|X)$, $\beta \in \mathbf{Z}^s$. A sequence of linear functionals $(\lambda_\alpha)_{\alpha\in\mathbf{Z}^s}$ is called a dual basis for these translates if

$$\lambda_\alpha C_\mu(\cdot - \beta|X) = \delta_{\alpha\beta} := \begin{cases} 1, & \text{if } \alpha = \beta; \\ 0, & \text{otherwise.} \end{cases}$$

When $\mu = 0$ (the box spline case), Wang [**20**] (for the case $s = 2$), Dahmen and Micchelli [**10**] and Jia [**14**] (for the general case) have constructed such a dual basis. Dahmen and Micchelli [**11**] also constructed a dual basis for $C_\mu(\cdot - \beta|X)$, $\beta \in \mathbf{Z}^s$. However, the use of Poisson's summation formula in [**10, 11**] was not fully justified. Moreover, the construction given in [**11**] involves a limit process. It is not clear whether such a limit always exists.

In this paper we shall remedy these problems by extending our previous work [**14**] to exponential box splines. We not only construct a dual basis for the integer translates of an exponential box spline, but also demonstrate that such a dual basis is unique in a certain sense. Our approach is constructive; in particular, it does not involve any limit process. Furthermore, our construction is a genuine extension of the dual basis constructed by de Boor and Fix in [**5**].

Our construction of the dual basis is based on a study of the linear space $G(X)$ which consists of all polynomials p such that $p(D)C_\mu(\cdot|X)$ is a bounded function. In the box spline case ($\mu = 0$), the space $G(X)$ already appeared in the paper [**14**], which was received by the editors on July 9, 1987. It was proved there that G is dual to another space $D(X)$ of polynomials associated with the box spline $B(\cdot|X)$. Dyn and Ron in [**12**] proved a more general result that $G(X)$ is dual to $D_\mu(X)$ for any $\mu \in \mathbf{C}^n$ (see Section 4 for the definition of $D_\mu(X)$). We shall reprove this result along a different approach. Our approach has the advantage that it does not rely on the dimension formula of $D_\mu(X)$. Thus, as a by-product, we are able to give a very short proof for the dimension formula of $D_\mu(X)$.

The paper is organized as follows. In Section 2 we introduce the so-called nearly continuous functions and discuss the Poisson summation formula. In Section 3 we extend a part of the Strang-Fix theory [**19**] to the case in which exponential polynomials occur. In Section 4 we introduce the space $G(X)$ and prove that $G(X)$ is dual to $D_\mu(X)$ for any $\mu \in \mathbf{C}^n$. Finally, in Section 5, we complete our construction of the dual basis for the integer translates of an exponential box spline.

2. Poisson's Summation Formula. In this paper the Fourier-Laplace transform of a function ϕ on $\mathbf{R}^s$ is defined as follows:

$$\hat{\phi}(z) := \int_{\mathbf{R}^s} \phi(x) e^{-ix\cdot z}\, dx, \qquad z \in \mathbf{C}^s.$$

Here and in what follows, i stands for the imaginary unit. Restricted on $\mathbf{R}^s$, $\hat{\phi}$ becomes the Fourier transform of ϕ.

The following form of Poisson's Summation Formula may be found in Stein and Weiss [**18**, Chapter 7]. Suppose ϕ and its Fourier transform $\hat{\phi}$ are continuous on $\mathbf{R}^s$, and for some $\delta > 0$,

$$|\phi(x)| \le \text{const}\,(1+|x|)^{-s-\delta}, \qquad x \in \mathbf{R}^s,$$
$$|\hat{\phi}(y)| \le \text{const}\,(1+|y|)^{-s-\delta}, \qquad y \in \mathbf{R}^s.$$

Then

$$\sum_{j\in\mathbf{Z}^s} \phi(j) = \sum_{j\in\mathbf{Z}^s} \hat{\phi}(2\pi j).$$

On the basis of this result, Dahmen and Micchelli in [**9**, Lemma 2.1] gave the following form of Poisson's Summation Formula: Suppose $\phi \in C_c(\mathbf{R}^s)$ and

$$\hat{\phi}(2\pi j) = 0 \qquad \text{for } j \in \mathbf{Z}^s \backslash \{0\}. \tag{2.1}$$

Then

$$\hat{\phi}(0) = \sum_{j\in\mathbf{Z}^s} \phi(j). \tag{2.2}$$

In the following, ϕ may be noncontinuous. Let ϕ be a locally integrable complex-valued function on $\mathbf{R}^s$. The function ϕ is called *nearly continuous* if for any $x \in \mathbf{R}^s$,

$$\phi(x) = \lim_{\varepsilon\to 0} \frac{1}{m(B(x,\varepsilon))} \int_{B(x,\varepsilon)} \phi(y)\, dy, \tag{2.3}$$

where $B(x,\varepsilon)$ denotes the ball of radius ε centered x and m denotes the Lebesgue measure. For a piecewise exponential polynomial function ϕ, the limit on the right of (2.3) always exists, and this limit equals $\phi(x)$ if x is not on the mesh. Thus, we may always assume that a piecewise exponential polynomial function is nearly continuous when redefined by (2.3) at mesh points.

Theorem 2.1. *Suppose that ϕ is nearly continuous and supported on a compact set in $\mathbf{R}^s$. If the Fourier transform $\hat{\phi}$ satisfies* (2.1), *then* (2.2) *holds.*

Proof. For $\varepsilon > 0$, let

$$\phi_\varepsilon(x) := \frac{1}{m(B(x,\varepsilon))}\int_{B(x,\varepsilon)} \phi(y)\,dy.$$

Fix ε for the moment. Then $\phi_\varepsilon \in C_c(\mathbf{R}^s)$. We observe that

$$\begin{aligned}\hat{\phi}_\varepsilon(2\pi j) &= \frac{1}{m(B(0,\varepsilon))}\int_{B(0,\varepsilon)}\int_{\mathbf{R}^s} \phi(x+y)e^{-ix\cdot 2\pi j}\,dx\,dy \\ &= \hat{\phi}(2\pi j)\frac{1}{m(B(0,\varepsilon))}\int_{B(0,\varepsilon)} e^{iy\cdot 2\pi j}\,dy.\end{aligned}$$

Hence,

$$\begin{aligned}\hat{\phi}_\varepsilon(2\pi j) = 0, \qquad j \in \mathbf{Z}^s\backslash\{0\},\\ \hat{\phi}_\varepsilon(0) = \hat{\phi}(0).\end{aligned}$$

Since $\phi_\varepsilon \in C_c(\mathbf{R}^s)$, (2.2) may apply to ϕ_ε; hence,

$$\hat{\phi}_\varepsilon(0) = \sum_{j\in\mathbf{Z}^s} \phi_\varepsilon(j). \tag{2.4}$$

Since ϕ is nearly continuous,

$$\phi(j) = \lim_{\varepsilon\to 0}\phi_\varepsilon(j), \qquad j \in \mathbf{Z}^s.$$

Letting $\varepsilon \to 0$ in (2.4), we obtain the desired result. □

3. The Strang-Fix conditions. In this section we assume that ϕ is a nearly continuous function supported on a compact set in $\mathbf{R}^s$. Since ϕ is compactly supported, the Fourier-Laplace transform $\hat{\phi}$ of ϕ is an entire function on $\mathbf{C}^s$. The function ϕ gives rise to a semi-discrete convolution operator $\phi *'$:

$$\phi *' : f \mapsto \phi *' f := \sum_{j \in \mathbf{Z}^s} \phi(\cdot - j) f(j),$$

which is well defined for any nearly continuous function f from $\mathbf{R}^s$ to $\mathbf{C}$.

A subspace P of $\pi(\mathbf{R}^s)$ is called D-invariant, if for any $p \in P$, the derivatives $D_j p$ are again in P, $j = 1, \ldots, s$. P is said to be translation invariant if for any $p \in P$ and $x \in \mathbf{R}^s$, the translate $p(\cdot + x)$ is again in P. It is proved in [**3**] that P is D-invariant if and only if it is translation invariant. Furthermore, P is called scale-invariant if for any $p \in P$ and any $r \in \mathbf{R}$, $p(r\cdot)$ is again in P. A mapping T on P is called degree-reducing on P, if for any $p \in P$, Tp is again in P and $\deg(Tp) < \deg p$. We denote by 1 the identity mapping. The following theorem was proved by Strang and Fix [**19**] for the case $P = \pi_k$, by Dahmen and Micchelli [**9**] for the case in which P is D-invariant and scale-invariant, and by de Boor [**3**] for the case in which P is D-invariant.

Theorem 3.1. *Let P be a finite-dimensional D-invariant space of polynomials. Then the mapping $(1 - \phi *')|_P$ is degree-reducing if and only if $\hat{\phi}$ satisfies the following conditions*:

(i) $\hat{\phi}(0) = 1$;

(ii) $p(-iD)\hat{\phi}(2\pi j) = 0$ *for all* $p \in P$ *and* $j \in \mathbf{Z}^s \backslash \{0\}$.

If $P = \pi_{k-1}$, the conditions (i) and (ii) in the above theorem are called the Strang-Fix conditions of order k (see [**19**]).

In [**8**] de Boor and Ron considered the action of the operator $\phi *'$ on spaces of exponential polynomials. Also, see the survey paper [**4**]. Here we take an approach different from theirs. The following theorem is a generalization of Theorem 3.1.

Theorem 3.2. *Let $\theta \in \mathbf{C}^s$, and let P be a D-invariant finite-dimensional space of polynomials. Then $\phi *'$ maps $e_\theta P$ into itself if* *and only if* $\hat{\phi}$ *satisfies*

$$p(-iD)\hat{\phi}(2\pi j - i\theta) = 0 \quad \textit{for all } p \in P \textit{ and } j \in \mathbf{Z}^s\backslash\{0\}. \tag{3.1}$$

Moreover, in this case, one has

$$\phi *' (e_\theta p) = e_\theta p(\cdot - iD)\hat{\phi}(-i\theta) = e_\theta(\hat{\phi}(-i\theta - iD)p), \quad \textit{for all } p \in P. \tag{3.2}$$

Proof. Fix $x \in \mathbf{R}^s$ for the time being. Given $p \in P$, let ψ_x be the function given by

$$\psi_x(y) := e^{\theta \cdot y} p(y)\phi(x - y), \qquad y \in \mathbf{R}^s. \tag{3.3}$$

The Fourier transform of ψ_x can be computed as

$$\hat{\psi}_x(\xi) = e^{(\theta - i\xi)\cdot x} p(x - iD)\hat{\phi}(-\xi - i\theta).$$

Since P is D-invariant, it is also translation invariant. Hence, $p \in P$ implies $p(x + \cdot) \in P$. Thus, if $\hat{\phi}$ satisfies the condition (3.1), then

$$\hat{\psi}_x(2\pi j) = e^{(\theta - i2\pi j)\cdot x} p(x - iD)\hat{\phi}(-2\pi j - i\theta) = 0, \qquad \text{for all } j \in \mathbf{Z}^s\backslash\{0\}.$$

Hence, Theorem 2.1 (the Poisson summation formula) can be applied to ψ_x, so we obtain

$$\sum_{j \in \mathbf{Z}^s} \psi_x(j) = \hat{\psi}_x(0),$$

that is,

$$\phi *' (e_\theta p) = e_\theta p(\cdot - iD)\hat{\phi}(-i\theta). \tag{3.4}$$

By the Taylor theorem, we have

$$p(\cdot - iD) = \sum_\alpha \frac{D^\alpha p}{\alpha!}(-iD)^\alpha.$$

Expand $\hat{\phi}(-i\xi)$ into a power series of $\xi - \theta$:

$$\hat{\phi}(-i\xi) = \sum_{\alpha \in \mathbf{N}^s} a_\alpha (\xi - \theta)^\alpha.$$

Let $[\theta]D^\alpha$ be the linear functional given by $f \mapsto D^\alpha f(\theta)$. Applying $[\theta]D^\alpha$ to both sides of the above equation, we get

$$(-iD)^\alpha \hat{\phi}(-i\theta) = a_\alpha \alpha!.$$

It follows that

$$\begin{aligned} p(\cdot - iD)\hat{\phi}(-i\theta) &= \sum_\alpha \frac{D^\alpha p}{\alpha!}(-iD)^\alpha \hat{\phi}(-i\theta) \\ &= \sum_\alpha a_\alpha D^\alpha p \\ &= \hat{\phi}(-i\theta - iD)p. \end{aligned} \tag{3.5}$$

We view $\hat{\phi}(-i\theta - iD)$ as a formal power series in D, so it is well defined on $\pi(\mathbf{R}^s)$. Note that (3.5) is true for any polynomial p. Since P is D-invariant, the operator $\hat{\phi}(-iD)$ maps P into itself. This proves the sufficiency part of the theorem. Moreover, (3.2) follows from (3.4) and (3.5).

Conversely, suppose that $\phi *' (e_\theta p) \in e_\theta \pi$ for all $p \in P$. Given a polynomial $p \in P$, set

$$g(x, y) := \sum_{k \in \mathbf{Z}^s} \psi_x(y + k), \qquad x, y \in \mathbf{R}^s,$$

where $\psi_x(y)$ is as in (3.3). Then $g(x, \cdot)$ is a 1-periodic function for any fixed x, and $g(\cdot, y) \in e_\theta \pi$ for any fixed y. The latter statement is true because $\phi *' (e_\theta p) \in e_\theta \pi$. Consider the Fourier coefficients $a_j(x)$ of $g(x, \cdot)$:

$$\begin{aligned} a_j(x) :&= \int_{[0,1]^s} g(x, y) e^{-i2\pi j \cdot y}\, dy \\ &= \sum_{k \in \mathbf{Z}^s} \int_{[0,1]^s} \psi_x(y + k) e^{-i2\pi j \cdot y}\, dy \\ &= \int_{\mathbf{R}^s} \psi_x(y) e^{-i2\pi j \cdot y}\, dy \\ &= \hat{\psi}_x(2\pi j) \\ &= e^{(\theta - i2\pi j)\cdot x} p(x - iD)\hat{\phi}(-2\pi j - i\theta). \end{aligned}$$

Note that the image of $e_\theta P$ under the mapping $\phi *'$ is finite-dimensional. Since $g(\cdot, y)$ is in $e_\theta \pi$ for any fixed y, so is a_j for each j. If $j \neq 0$, then $a_j \in e_\theta \pi$ can happen only if

$$p(x - iD)\hat{\phi}(-2\pi j - i\theta) = 0, \qquad \text{for all } x \in \mathbf{R}^s.$$

Choosing $x = 0$ in the above equation, we obtain

$$p(-iD)\hat{\phi}(-2\pi j - i\theta) = 0, \qquad \text{for all } p \in P \text{ and } j \in \mathbf{Z}^s \backslash \{0\},$$

as desired. □

The following two corollaries are easily derived from Theorem 3.2. We note that Corollary 3.3 is a generalization of a result of Ben-Artzi and Ron in [**1**], where the mapping $\phi *'$ was considered for an exponential box spline ϕ. Also the sufficiency part of Corollary 3.3 was first proved by de Boor and Ron in [**8**].

Corollary 3.3. *The mapping $\phi *'$ maps $e_\theta P$ one-to-one and onto itself if and only if $\hat{\phi}$ satisfies the following conditions*:

(i) $\hat{\phi}(-i\theta) \neq 0$,

(ii) $p(-iD)\hat{\phi}(2\pi j - i\theta) = 0$ *for all* $p \in P$ *and* $j \in \mathbf{Z}^s \backslash \{0\}$.

Corollary 3.4. *The mapping $\phi *'$ is an identity on $e_\theta P$ if and only if*

(i) $p(-iD)\hat{\phi}(-i\theta) = p(0)$ *for all* $p \in P$;

(ii) $p(-iD)\hat{\phi}(2\pi j - i\theta) = 0$ *for all* $p \in P$ *and* $j \in \mathbf{Z}^s \backslash \{0\}$.

We note that if $\hat{\phi}(-i\theta) \neq 0$, then the operator $\hat{\phi}(-i\theta - iD)$ is invertible on $\pi(\mathbf{R}^s)$, and its inverse can be found as follows. Expand $1/\hat{\phi}(-i\theta + \xi)$ into a power series in a neighborhood of the origin:

$$1/\hat{\phi}(-i\theta + \xi) = \sum_{\nu \in \mathbf{N}^s} b_\nu \xi^\nu.$$

Then the inverse of $\hat{\phi}(-i\theta - iD)$ is

$$(\hat{\phi}(-i\theta - iD))^{-1} = \sum_{\nu \in \mathbf{N}^s} b_\nu (-iD)^\nu.$$

Let now H be the linear space $\sum_{\theta\in\Theta} e_\theta P_\theta$ where Θ is a finite subset of $\mathbf{R}^s$, and P_θ is a finite-dimensional D-invariant space of polynomials for each $\theta \in \Theta$. Given a linear functional λ on H, consider the operator Q_λ given by

$$Q_\lambda : f \mapsto \sum_{j\in\mathbf{Z}^s} \phi(\cdot - j)\lambda f(\cdot + j).$$

We are interested in the conditions on λ under which Q_λ is an identity on H. This problem has been discussed in [**2, 8**]. In particular, a systematic treatment for quasi-interpolation was given in [**8**]. The following theorem was first proved by Ben-Artzi and Ron in [**2**] under the additional condition that the integer translates of ϕ are linearly independent. It turns out that this condition is *not* necessary.

Theorem 3.5. *Suppose that the following conditions are satisfied:*

(i) $\hat{\phi}(-i\theta) \neq 0$ *for every* $\theta \in \Theta$;

(ii) $p(-iD)\hat{\phi}(-i\theta - 2\pi j) = 0$ *for each* $\theta \in \Theta$ *and any* $p \in P_\theta$ *and* $j \in \mathbf{Z}^s\backslash\{0\}$.

Then the operator Q_λ *is an identity on* $H = \sum_{\theta\in\Theta} e_\theta P_\theta$ *if and only if* λ *satisfies the following condition:*

$$\lambda(e_\theta p) = (\hat{\phi}(-i\theta - iD))^{-1}p(0), \qquad \textit{for all } \theta \in \Theta \textit{ and } p \in P_\theta.$$

Proof. Given a linear functional λ on H, there corresponds to each θ a polynomial q_θ (see [**7**]) such that

$$q_\theta(D)p(0) = \lambda(e_\theta p), \qquad \text{for all } p \in P_\theta. \tag{3.6}$$

It follows from (3.2) that

$$\begin{aligned} Q_\lambda(e_\theta p) &= \phi *' (e_\theta q_\theta(D)p) \\ &= e_\theta(\hat{\phi}(-i\theta - iD)q_\theta(D)p). \end{aligned}$$

Hence, Q_λ is an identity on $e_\theta P_\theta$ if and only if

$$q_\theta(D)\hat{\phi}(-i\theta - iD)p = p, \qquad \text{for all } p \in P_\theta. \tag{3.7}$$

Since $\hat{\phi}(-i\theta) \neq 0$, the operator $\hat{\phi}(-i\theta - iD)$ is invertible on P_θ. In (3.7), replacing p by $(\hat{\phi}(-i\theta - iD))^{-1}p$, we see that (3.7) is equivalent to

$$q_\theta(D)p = (\hat{\phi}(-i\theta - iD))^{-1}p, \qquad \text{for all } p \in P_\theta. \tag{3.8}$$

But P_θ is translation invariant, hence (3.8) is equivalent to

$$q_\theta(D)p(0) = (\hat{\phi}(-i\theta - iD))^{-1}p(0), \qquad \text{for all } p \in P_\theta. \tag{3.9}$$

Combining (3.9) with (3.6), we obtain the desired result. □

We close this section by proving a result on the commutativity of semi-discrete convolution (cf. de Boor [**3**]).

Theorem 3.6. *Under the condition* (ii) *of Theorem* 3.5,

$$\phi *' f = f *' \phi, \qquad \textit{for all } f \in H.$$

Proof. It suffices to prove the commutativity for $f = e_\theta p$, where $\theta \in \Theta$ and $p \in P_\theta$. Since ϕ is compactly supported, $f *' \phi \in e_\theta P_\theta$. Moreover, by the assumption, Theorem 3.2 can be applied to ϕ, so $\phi *' f \in e_\theta P_\theta$. Hence, $\phi *' f = e_\theta q_1$, and $f *' \phi = e_\theta q_2$ for some $q_1, q_2 \in P_\theta$. But the restrictions of $\phi *' f$ and $f *' \phi$ on $\mathbf{Z}^s$ coincide. It follows that $q_1 = q_2$ on $\mathbf{Z}^s$, and therefore $q_1 = q_2$, as desired. □

4. Dual spaces. Let E and F be two linear spaces over the complex field $\mathbf{C}$. A bilinear function

$$\langle \quad , \quad \rangle : (x, y) \mapsto \langle x, y \rangle$$

from $E \times F$ to $\mathbf{C}$ is called a scalar product if

(i) $\langle x, y \rangle = 0$ for all $y \in F$ implies $x = 0$;

(ii) $\langle x, y \rangle = 0$ for all $x \in E$ implies $y = 0$.

If there is a scalar product between E and F, then E and F are said to be dual spaces with respect to this scalar product. Two dual spaces have the same dimension. If E and F are dual spaces with respect to $\langle\ ,\ \rangle$, then for any linear functional λ on F, there exists a unique element $x \in E$ such that

$$\lambda(y) = \langle x, y\rangle, \qquad \text{for all } y \in F.$$

Let $X = (x^1, \ldots, x^n) \subset \mathbf{R}^s\backslash\{0\}$ and $\mu = (\mu_1, \ldots, \mu_n) \in \mathbf{C}^n$. Write μ_v for μ_j if $v = x^j$. For $V \subseteq X$, let $p_{\mu,V}$ be the polynomial given by

$$p_{\mu,V} := \prod_{v\in V} p_{\mu_v,v} := \prod_{v\in V}(p_v + \mu_v),$$

where p_v is the linear function given by $p_v(x) := x \cdot v$, $x \in \mathbf{R}^s$. Let

$$D_\mu(X) := \bigcap\{\ker(p_{\mu,V}(D)) : V \subseteq X, \operatorname{span}(X\backslash V) \neq \mathbf{R}^s\}.$$

Then $D_\mu(X)$ is a linear space of exponential polynomials. When $\mu = 0$, we write p_V for $p_{\mu,V}$, and correspondingly write $D(X)$ for $D_\mu(X)$.

If X spans $\mathbf{R}^s$, then $C_\mu(\cdot|X)$ is bounded and piecewise in $D_\mu(X)$. More precisely, we let $\mathbf{H}(X)$ be the collection of all $(s-1)$-dimensional subspaces H which are spanned by elements of X. Furthermore, we set

$$c(X) := \bigcup_{H\in\mathbf{H}(X)} H + \mathbf{Z}^s. \tag{4.1}$$

Then on each connected component of $\mathbf{R}^s\backslash c(X)$, $C_\mu(\cdot|X)$ agrees with some element of $D_\mu(X)$.

Let us now introduce two subspaces $F_\mu(X)$ and $G_\mu(X)$ of $\pi(\mathbf{R}^s)$:

$$\begin{aligned} F_\mu(X) &:= \operatorname{span}\{p_{\mu,V} : V \subseteq X, \operatorname{span}(X\backslash V) = \mathbf{R}^s\}, \\ G_\mu(X) &:= \{p \in \pi(\mathbf{R}^s) : p(D)C_\mu(\cdot|X) \in L^\infty\}. \end{aligned}$$

When $\mu = 0$, we write $F(X)$ for $F_\mu(X)$, and $G(X)$ for $G_\mu(X)$, respectively.

For a polynomial p and an exponential polynomial f, we define

$$\langle p, f\rangle := p(D)f(0). \tag{4.2}$$

In [**14**], the author introduced two spaces $G(X)$ and $F(X)$ and proved that $G(X) = F(X)$ and $G(X)$ is dual to $D(X)$ with respect to the bilinear function $\langle \quad , \quad \rangle$ given in (4.2). It was pointed out in [**11**] that the space $F(X)$ had been also investigated by H. Hakopian.

It was observed by Dyn and Ron [**12**] that $F_\mu(X) = F(X)$ for all μ. Indeed, for any $V \subseteq X$, $p_{\mu,V}$ is a linear combination of p_Y, $Y \subseteq V$, because $p_{\mu_v,v} = p_v + \mu_v$. Conversely, since $p_v = p_{\mu_v,v} - \mu_v$, p_V is also a linear combination of $p_{\mu,Y}$, $Y \subseteq V$.

Dyn and Ron proved in [**12**] that for any $\mu \in \mathbf{C}^n$, $F(X)$ is dual to $D_\mu(X)$ with respect to $\langle \quad , \quad \rangle$. We may apply the technique used in [**14**] to give a new proof of their result. In contrast to the proof given in [**12**], our proof does *not* rely on the dimension formula of $D_\mu(X)$. Instead, our approach enables us to give a very short proof for the dimension formula of $D_\mu(X)$ (see Theorem 4.2 below). This should be compared with the fact that both the proofs given in [**1, 11**] for the dimension formula of $D_\mu(X)$ are very complicated. Also see [**15**] for some topics related to the spaces $F(X)$ and $D_\mu(X)$.

We observe that $F(X) = F_\mu(X)$ is a subspace of $G_\mu(X)$. This comes from the differentiation formula for exponential box splines given in [**16**, Theorem 2.2 (a)]. If $V \subseteq X$ and $\operatorname{span}(X\backslash V) = \mathbf{R}^s$, then $p_{\mu,V}(D)C_\mu(\cdot|X) \in L^\infty$ by the differentiation formula mentioned above; hence $p_{\mu,V} \in G_\mu(X)$. This shows that $F(X)$ is a subspace of $G_\mu(X)$.

Theorem 4.1. *For any $\mu \in \mathbf{C}^n$, the space $G_\mu(X)$ is dual to $D_\mu(X)$ with respect to the bilinear function $\langle \quad , \quad \rangle$. Moreover, $G_\mu(X) = F(X)$ for all $\mu \in \mathbf{C}^n$.*

Proof. If X does not span $\mathbf{R}^s$, then $G_\mu(X)$, $F(X)$ and $D_\mu(X)$ are all trivial, so there is nothing to prove. Thus, we may assume that $\operatorname{span}(X) = \mathbf{R}^s$ in what follows.

First, suppose that $p \in G_\mu(X)$ satisfies $\langle p, f\rangle = 0$ for all $f \in D_\mu(X)$. Since $D_\mu(X)$ is translation invariant, this implies that $p(D)f = 0$ for all $f \in D_\mu(X)$. Let A be a connected component of $\mathbf{R}^s\backslash c(X)$. The restriction of $C_\mu(\cdot|X)$ on A is a function in $D_\mu(X)$; hence, $p(D)C_\mu(\cdot|X) = 0$ on A. This shows that $p(D)C_\mu(\cdot|X)$ is a distribution supported on

$c(X)$. On the other hand, by the very definition of $G_\mu(X)$,

$$p(D)C_\mu(\cdot|X) \in L^\infty.$$

Hence, $p(D)C_\mu(\cdot|X) = 0$. It follows that $p = 0$ because $C_\mu(\cdot|X)$ is compactly supported.

Secondly, suppose that $f \in D_\mu(X)$ satisfies $\langle p, f\rangle = 0$ for all $p \in F(X)$. We want to prove $f = 0$. This will be done by induction on $\#X$, the number of elements in X. If $\#X = s$, then $D_\mu(X)$ is spanned by an exponential function e_θ for some $\theta \in \mathbf{C}^s$. But constants are in $F(X)$; this implies that $f(0) = 0$ for $f \in D_\mu(X)$. Hence, $f = 0$, as desired. Suppose inductively that our claim has been proved for any X' with $\#X' < \#X$ and we want to establish it for X. Consider $p_{\mu_v,v}(D)f$, where $v \in X$. We have

$$p_{\mu,V}(D)p_{\mu_v,v}(D)f(0)=0 \quad \text{for any } V \subseteq X\backslash v \text{ with span}\,(X\backslash v\backslash V)=\mathbf{R}^s.$$

Hence, by the induction hypothesis, $p_{\mu_v,v}(D)f = 0$. This, together with the fact that $f(0) = 0$ implies $f = 0$, since X contains a basis for $\mathbf{R}^s$.

By what has been proved, we have

$$\dim\,(G_\mu(X)) \le \dim\,(D_\mu(X)) \le \dim\,(F(X)).$$

But $F(X)$ is a subspace of $G_\mu(X)$, hence, $F(X) = G_\mu(X)$ for all $\mu \in \mathbf{C}^n$. We conclude that $G_\mu(X)$ is dual to $D_\mu(X)$ with respect to $\langle\ \ ,\ \ \rangle$. □

Now Theorem 4.1 tells us that $\dim\,(D_\mu(X))$ does not depend on μ. One would like to identify those simple cases for which $\dim\,(D_\mu(X))$ can be easily computed. This has already been done by Ben-Artzi and Ron in [**1**]. Let us describe their approach. Denote by $\mathbf{B}(X)$ the collection of all bases for $\mathbf{R}^s$ contained in X. Given $Z \in \mathbf{B}(X)$, let θ_Z be the unique element in $\mathbf{C}^s$ such that

$$v \cdot \theta_Z + \mu_v = 0, \qquad \text{for all } v \in Z.$$

Given an exponential box spline $C_\mu(\cdot|X)$, the pair (X,μ) is called its defining set. A defining set (X,μ) is called *simple*, if for any two different bases Y and Z contained in X, the corresponding θ_Y and θ_Z are different. If (X,μ) is simple, then

$$D_\mu(X) = \operatorname{span}\{e_{\theta_Z} : Z \in \mathbf{B}(X)\}.$$

It follows that for a simple defining set (X,μ),

$$\dim(D_\mu(X)) = \#\mathbf{B}(X).$$

Given $X \subset \mathbf{R}^s\backslash\{0\}$, it is easily seen that there exists $\mu \in \mathbf{C}^n$ such that (X,μ) is simple (see [**12**]). Thus we have reproved the following dimension formula, which was first proved in [**1,11**].

Theorem 4.2. *For all* $\mu \in \mathbf{C}^n$, $\dim(D_\mu(X)) = \#\mathbf{B}(X)$.

5. A dual basis. In this final section we assume that $X = (x^1,\dots,x^n) \subset \mathbf{R}^s\backslash\{0\}$, $\operatorname{span}(X) = \mathbf{R}^s$ and $\mu = (\mu_1,\dots,\mu_n) \in \mathbf{C}^n$. Let

$$\begin{aligned} X_\theta &:= \{x \in X : x\cdot\theta + \mu_x = 0\}, \qquad \theta \in \mathbf{C}^s,\\ N(\mu,X) &:= \{\theta \in \mathbf{C}^s : \operatorname{span}(X_\theta) = \mathbf{R}^s\}. \end{aligned}$$

Then $e_\theta \in D_\mu(X)$ if and only if $\theta \in N(\mu,X)$. The space $D_\mu(X)$ has the following decomposition (see [**1,11**]):

$$D_\mu(X) = \bigoplus_{\theta\in N(\mu,X)} e_\theta D(X_\theta).$$

It follows from [**11**, (4.8)] that

$$p(-iD)C_\mu(\cdot|X)^\wedge(2\pi j - i\theta) = 0, \qquad \text{for } p \in D(X_\theta) \quad \text{and} \quad j \in \mathbf{Z}^s\backslash\{0\}.$$

For $\tau \in \mathbf{R}^s$, let

$$\phi_\tau := C_\mu(\cdot + \tau|X).$$

Then we have

$$\hat{\phi}_\tau(\xi) = e^{i\tau\cdot\xi}C_\mu(\cdot|X)^\wedge(\xi), \qquad \xi \in \mathbf{C}^s. \tag{5.1}$$

Since $D(X_\theta)$ is D-invariant, by the Leibniz formula for differentiation, we obtain

$$(5.2)\quad p(-iD)\hat{\phi}_\tau(2\pi j - i\theta) = 0 \qquad \text{for } p \in D(X_\theta) \quad \text{and} \quad j \in \mathbf{Z}^s \backslash \{0\}.$$

Therefore, by Theorem 3.2 and its corollaries, we see that $\phi_\tau *'$ is an invertible mapping on $D_\mu(X)$ if and only if

$$(5.3) \qquad C_\mu(\cdot|X)^\wedge(-i\theta) \neq 0 \qquad \text{for all } \theta \in N(\mu, X).$$

Theorem 5.1. *Suppose that $C_\mu(\cdot|X)$ satisfies* (5.3). *Let $q_\tau \in F(X)$. Then the operator $(q_\tau(D)\phi_\tau)*'$ is an identity on $D_\mu(X)$ if and only if q_τ satisfies the following condition*:

(5.4)

$$\langle q_\tau, e_\theta p\rangle = (\hat{\phi}_\tau(-iD - i\theta))^{-1} p(0), \qquad \textit{for } \theta \in N(\mu, X) \textit{ and } p \in D(X_\theta).$$

Such a polynomial q_τ exists and is unique.

Proof. We observe that $q_\tau \in F(X)$ implies that $q_\tau(D)\phi_\tau$ is a nearly continuous function. Since ϕ_τ satisfies (5.2), by Theorem 3.6 we have

$$\phi_\tau *' f = f *' \phi_\tau, \qquad \text{for all } f \in D_\mu(X).$$

It follows that, for any $f \in D_\mu(X)$,

$$\begin{aligned} (q_\tau(D)\phi_\tau) *' f &= q_\tau(D)(\phi_\tau *' f) \\ &= q_\tau(D)(f *' \phi_\tau) \\ &= (q_\tau(D)f) *' \phi_\tau \\ &= \phi_\tau *' (q_\tau(D)f). \end{aligned}$$

Hence, $(q_\tau(D)\phi_\tau)*'$ is an identity on $D_\mu(X)$ if and only if

$$(5.5) \qquad \phi_\tau *' (q_\tau(D)f) = f, \qquad \text{for all } f \in D_\mu(X).$$

Let $\lambda = [0]q_\tau(D)$. Then, by Theorem 3.5, (5.5) is true if and only if

$$\lambda(e_\theta p) = (\hat{\phi}_\tau(-iD - i\theta))^{-1} p(0), \qquad \text{for } \theta \in N(\mu, X) \text{ and } p \in D(X_\theta).$$

This is just (5.4). The existence and uniqueness of q_τ follow from Theorem 4.1. □

From now on we assume that $X \subset \mathbf{Z}^s \backslash \{0\}$. X is called unimodular if

$$|\det Z| = 1, \qquad \text{for any basis } Z \subseteq X. \tag{5.6}$$

It is known that the integer translates of $C_\mu(\cdot|X)$ are linearly independent if and only if (X, μ) satisfies (5.3) and (5.6). When $\mu = 0$, this fact was proved in [**9, 13**]. In general, this was proved in [**11, 17**]. In fact, it was proved in [**11**] that the integer translates of $C_\mu(\cdot|X)$ are locally linearly independent, that is, if A is a nonempty open set of $\mathbf{R}^s$, and if

$$\sum_{\alpha \in \mathbf{Z}^s} a_\alpha C_\mu(\cdot - \alpha|X) = 0 \qquad \text{on } A,$$

then $a_\alpha = 0$, provided that the support of $C_\mu(\cdot - \alpha|X)$ intersects A. Note that the support of $C_\mu(\cdot - \alpha|X)$ is $[X] + \alpha$, where $[X]$ is as given in (1.1).

Let A be a connected component of $\operatorname{Int}[X] \backslash c(X)$,where $\operatorname{Int}[X]$ denotes the interior of $[X]$ and $c(X)$ is as given in (4.1). Pick up a point $\tau \in A$. For each $\alpha \in \mathbf{Z}^s$, the restriction of $C_\mu(\cdot - \alpha|X)$ on A is a function in $D_\mu(X)$, which we shall denote by f_α. Note that $f_\alpha = 0$ if $[X] + \alpha$ does not intersect A. Suppose now that $C_\mu(\cdot|X)$ satisfies (5.3) and (5.6). Then the set $\{f_\alpha : ([X] + \alpha) \cap A \neq \varnothing\}$ is a linearly independent subset of $D_\mu(X)$. But $F(X)$ is dual to $D_\mu(X)$ with respect to the scalar product $\langle \quad , \quad \rangle$, hence there exists a polynomial $q_\tau \in F(X)$ such that

$$q_\tau(D) f_\alpha(\tau) = \delta_{\alpha 0}, \qquad \alpha \in \mathbf{Z}^s.$$

It follows that

$$[\alpha + \tau] q_\tau(D) C_\mu(\cdot|X) = \delta_{\alpha 0}, \qquad \alpha \in \mathbf{Z}^s.$$

Now we are in a position to state and prove the main result of this paper.

Theorem 5.2. *Suppose that $C_\mu(\cdot|X)$ satisfies* (5.3) *and* (5.6). *Then for any $\tau \in \operatorname{Int}[X]$, there exists a unique polynomial $q_\tau \in F(X)$ such that the functionals λ_α given by*

$$\lambda_\alpha := [\alpha + \tau] q_\tau(D) \tag{5.7}$$

satisfy

$$\lambda_\alpha C_\mu(\cdot - \beta|X) = \delta_{\alpha\beta}, \qquad \textit{for all } \alpha, \beta \in \mathbf{Z}^s. \tag{5.8}$$

Moreover, the mapping $\tau \mapsto q_\tau$ *is continuous from* Int $[X]$ *to* $F(X)$, *where the finite-dimensional linear space* $F(X)$ *is equipped with any norm.*

Proof. Let q_τ be an element of $F(X)$ such that the functionals λ_α given by (5.7) satisfy (5.8). Then for all $\alpha \in \mathbf{Z}^s$,

$$(q_\tau(D)\phi_\tau)(\alpha) = [\alpha + \tau] q_\tau(D) C_\mu(\cdot|X) = \lambda_\alpha C_\mu(\cdot|X) = \delta_{\alpha 0}. \tag{5.9}$$

Let $f \in X_\theta$, where $\theta \in N(\mu, X)$. Then it follows from (5.9) that

$$(q_\tau(D)\phi_\tau) *' f(\beta) = f(\beta), \qquad \text{for all } \beta \in \mathbf{Z}^s.$$

This shows that the operator $(q_\tau(D)\phi_\tau)*'$ is an identity on X_θ for every $\theta \in N(\mu, X)$; hence it is an identity on $D_\mu(X)$. By Theorem 5.1, such a polynomial q_τ is unique. The existence of q_τ has been proved for the case $\tau \notin c(X)$. We want to remove this restriction. For this purpose, we first show that the mapping $\tau \mapsto q_\tau$ is continuous from Int $[X]$ to $F(X)$.

From (5.4) and (5.1), we observe that, for any $f \in D_\mu(X)$,

$$\langle q_\tau, f \rangle = q_\tau(D) f(0)$$

is an exponential polynomial function of τ. Choose a basis $f_1, \dots, f_m$ for $D_\mu(X)$, and a basis $q_1, \dots, q_m$ for $F(X)$ such that they are biorthogonal, i.e.,

$$\langle q_j, f_k \rangle = \delta_{jk}, \qquad 1 \le j, k \le m.$$

Suppose

$$q_\tau = \sum_{j=1}^m a_{j,\tau} q_j.$$

Then $a_{j,\tau} = \langle q_\tau, f_j \rangle$ is an exponential polynomial function of τ. This proves that the mapping $\tau \mapsto q_\tau$ is continuous from Int $[X]$ to $F(X)$.

Finally, let $\tau \in$ Int $[X]$, and let q_τ be the unique polynomial in $F(X)$ satisfying (5.4). For simplicity, we write C_μ for $C_\mu(\cdot|X)$. Let λ_α be given by (5.7). Since the mapping $\tau \mapsto q_\tau$ is continuous on $\mathrm{Int}\,[X]$, we have

$$\begin{aligned}\lambda_\alpha C_\mu(\cdot|X) &= (q_\tau(D)C_\mu)(\alpha+\tau)\\ &= \lim_{\varepsilon\to 0}\frac{1}{m(B(0,\varepsilon))}\int_{B(0,\varepsilon)}(q_\tau(D)C_\mu)(\alpha+\tau+y)\,dy\\ &= \lim_{\varepsilon\to 0}\frac{1}{m(B(0,\varepsilon))}\int_{B(0,\varepsilon)}(q_{\tau+y}(D)C_\mu)(\alpha+\tau+y)\,dy\\ &= \delta_{\alpha 0}.\end{aligned}$$

This implies (5.8), as desired. □

Acknowledgment. This work was done when the author was invited by Professor S. Riemenschneider to the University of Alberta. The research was partially supported by NSERC Grant #A 7687.

The author also wishes to thank Professor W. Dahmen and Professor A. Ron for their suggestions on this paper.

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