ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 2, Spring 1993

THE GENERALIZED GAUSS MAP AND APPLICATIONS

JAIME B. RIPOLL AND MARCOS SEBASTIANI

0. Introduction. In his work on the theory of surfaces Gauss introduced what is called today the (normal) Gauss map of an orientable hypersurface in Euclidean space E^{n+1}. Formerly, the Gauss map was used to compute the intrinsic curvature of a surface of E^3. However, this map became one of the most important tools in Euclidean geometry, being used to prove results in many of its different branches.

The Gauss map in E^{n+1} is determined by the (linear) translations of E^{n+1}. Since translations make sense in a parallelizable manifold (see Section 1), it is possible to define a similar map in a Riemannian parallelizable manifold. Our aim here is first to make this construction, defining a Gauss map for hypersurfaces in a Riemannian parallelizable manifold and studying its general properties. We then apply these results generalizing some classical results on Differential Geometry. In particular, we obtain a generalization of the Gauss-Bonnet formula and a generalization of a result of R. Langevin about curvature and complex singularities. We also obtain an application to the study of convexity of hypersurfaces. We also consider here the case of immersed manifolds of arbitrary codimension.

Translations in a Riemannian manifold can be obtained, for instance, by fixing a point of the space and taking the parallel translation of the tangent vectors at that point along geodesics. The Gauss map defined this way and in the case that the ambient space has constant sectional curvatures has been studied by J. Weiner in [**9**]. Translations also appear in a Lie group with an invariant metric by taking invariant vector fields.

I. General definitions and results. We recall that a $(n+1)$-dimensional differentiable manifold N is called parallelizable if its tangent bundle is trivial, that is, if there exists a differentiable map $\Gamma : TN \to N \times \mathbf{R}^{n+1}$ which makes commutative the diagram:

Received by the editors on July 19, 1991.

$$\begin{array}{ccc} TN & \xrightarrow{\Gamma} & N \times \mathbf{R}^{n+1} \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi_2} \\ N & \xrightarrow{I} & N \end{array}$$

and such that, given $p \in N$, the map $\Gamma_p := \Gamma(p, \cdot) : T_p(N) \to \mathbf{R}^{n+1}$ is an isomorphism between vector spaces. π_1 and π_2 are the usual projections and I the identity map on N.

The following notations and definitions will be used throughout part I: N will be an $(n+1)$-*dimensional Riemannian parallelizable manifold* together with a map $\Gamma : TN \to N \times \mathbf{R}^{n+1}$ as above and such that $\Gamma_p : T_p(N) \to \mathbf{R}^{n+1}$ is an *isometry*, for any $p \in N$, considering in $\mathbf{R}^{n+1}$ the usual Euclidean inner product. Γ will be called a *translation* in N.

Given $p \in N$ and $X \in T_p(N)$, we define a vector field $\tilde{X}$ in N by setting $\tilde{X}(q) := \Gamma_q^{-1}(\Gamma_p(X))$. Such vector fields will be called *invariant* (or Γ-*invariant*) vector fields.

M will be an m-dimensional Riemannian manifold, $m < n$, isometrically immersed in N. Denote by $N(M)$ the normal bundle of M and by $SN(M)$ the correspondent sphere bundle. Let $p \in M$.

1. The invariant (or Γ-invariant) second fundamental form. We first recall that the second fundamental form B of M is given by

$$B_p(\eta)(X, Y) = \langle \nabla_X \overline{Y}, \eta \rangle$$

where $\langle \quad , \quad \rangle$ is the Riemannian metric of N and ∇ its associated connection, $X, Y \in T_p(M)$, $\eta \in N_p(M)$ and $\overline{Y}$ any extension of Y tangent to M. By setting $A_p(\eta) : T_p(M) \to T_p(M)$, $A_p(\eta)(X) = -(\nabla_X \eta)^P$, where $(\quad)^P$ denotes the orthogonal projection on $T_p(M)$ we have $B_p(\eta)(X, Y) = \langle A_p(\eta)(X), Y \rangle$.

The *Invariant* (or Γ-*invariant*) second fundamental form $\tilde{B}$ of M is given by

$$\tilde{B}_p(\eta)(X, Y) = \langle \nabla_X \tilde{Y}, \eta \rangle$$

where $\tilde{Y}$ is the invariant vector field such that $\tilde{Y}(p) = Y$. By setting $\tilde{A}_p(\eta) : T_p(M) \to T_p(M)$, $\tilde{A}_p(\eta)(X) = -(\nabla_X \tilde{\eta})^P$, where $\tilde{\eta}$ is the invariant vector field such that $\tilde{\eta}(p) = \eta$, we obtain

$$\tilde{B}_p(\eta)(X, Y) = \langle \tilde{A}_p(\eta)(X), Y \rangle.$$

Let us define $\tilde{K}_p(\eta) = \det(A_p(\eta) - \tilde{A}_p(\eta))$. Assuming that the $N(M)$ is orientable, we define

$$\tilde{K}_p = \frac{1}{c_{k-1}} \int_{SN_p(M)} \tilde{K}_p(\eta)\, d\sigma(\eta)$$

where $d\sigma$ is the volume form of $SN_p(M)$ and c_{k-1} the volume of the $(k-1)$-dimensional unit sphere, $k = n - m$.

This definition is completely similar to the usual definition of Lipschitz-Killing curvature for immersed Riemannian manifolds (see [**2**]). $\tilde{K}$ will be called the *invariant curvature* of M.

2. The "Gauss" map determined by a translation. We define

$$\gamma : SN(M) \to S^{n-1}$$

by

$$\gamma(\eta) = \Gamma_p(\eta)$$

where S^{n-1} is the unit sphere in $\mathbf{R}^n$. We have $d\gamma(\eta) : T_\eta(SN(M)) \to T_{\gamma(\eta)}(S^{n-1})$.

The canonical projection of $SN(M)$ induces a surjective linear map $T_\eta(SN(M)) \to T_p(M)$.

3. Theorem.

$$\langle \Gamma_p^{-1}(d\gamma(\eta)(\hat{X})), Y \rangle = -B_p(\eta)(X, Y) + \tilde{B}_p(\eta)(X, Y)$$

where $\eta \in SN_p(M)$, $X, Y \in T_p(M)$ and $\hat{X}$ is any lift of X to $T_\eta(SN(M))$.

Proof. We observe that any two lifts of X differ by a vector $Z \in \operatorname{Ker} d\gamma(\eta) = T_\eta(SN(M))$. We see that $\Gamma_p^{-1}(d\gamma(\eta)(Z))$ is orthogonal to

$T_p(M)$, so that the first member of the above equality is independent of the lift $\hat{X}$ of X.

We choose a lift of X in the following way. Let $\eta \in T_p(M)$, and let $f : (-\varepsilon, \varepsilon) \to M$ such that $f(0) = p$ and $f'(0) = X$. Let η_t be the parallel transport of η along f. Then define $\hat{X} := D\eta_t/dt(0)$, where D/dt denotes the covariant derivative. Therefore, we have

$$\Gamma_p^{-1}(d\gamma(\eta)(\hat{X})) = \frac{D}{dt}(\Gamma_p^{-1} \circ \Gamma_{f(t)})(\eta_t)(0).$$

Let $X_1, \dots, X_m$ be an orthonormal basis of $T_p(M)$ and set $X_{m+1} = \eta$. We extend this basis to an orthonormal basis $X_1, \dots, X_m, \dots, X_n$ of $T_p(N)$. Let $\tilde{X}_j$ be the invariant vector field such that $\tilde{X}_j(p) = X_j$, and let $X_j(t)$ be the parallel transport of X_j along f, $1 \le j \le n$. Then

$$\eta_t = c_1(t)X_1(t) + \cdots + c_m(t)X_{m+1}(t)$$

with

$$c_1(0) = \cdots = c_m(0) = 0, \qquad c_{m+1}(0) = 1.$$

Therefore

$$\Gamma_p^{-1}(d\gamma(\eta)(\hat{X})) = \sum_{j=1}^{m+1} c_j'(0)X_j + \frac{D}{dt}(\Gamma_p^{-1} \circ \Gamma_{f(t)})(X_{m+1}(t))(0)$$

thus

$$\begin{aligned}&\langle \Gamma_p^{-1}(d\gamma(\eta)(\hat{X})), Y\rangle \\ &\qquad = \sum_{j=1}^{m+1} c_j'(0)\langle X_j, Y\rangle + \frac{D}{dt}(\langle(\Gamma_p^{-1} \circ \Gamma_{f(t)})(X_{m+1}(t))(0), Y\rangle).\end{aligned}$$

But $A_p(\eta)(X, Y) = -(D_{\eta_t}/dt)(0) = -\sum_{j=1}^{m+1} c_j'(0)X_j$, so that

$$-B_p(\eta)(X, Y) = \langle A_p(\eta)(X), Y\rangle = -\sum_{j=1}^{m} c_j'(0)\langle X_j, Y\rangle.$$

On the other hand, we can write

$$\tilde{X}_{m+1}(f(t)) = \sum_{j=1}^{n} b_j(t) X_j(t), \qquad b_j(0) = \delta_{j,m+1}.$$

Applying $\Gamma_p^{-1} \circ \Gamma_{f(t)}$ in both sides of this equality, we obtain

$$X_{m+1} = \sum_{j=1}^{n} b_j(t)(\Gamma_p^{-1} \circ \Gamma_{f(t)})(X_j(t)).$$

Taking the derivative

$$0 = \sum_{j=1}^{n} b_j'(0) X_j + \frac{D}{dt}(\Gamma_p^{-1} \circ \Gamma_{f(t)})(X_{m+1}(t))(0)$$

so that

$$\tilde{A}_p(\eta)(X) = -\frac{D\tilde{X}_{m+1}(f(t))}{dt}(0) = -\sum_{j=1}^{m} b_j'(0) X_j.$$

Therefore,

$$\langle \frac{D}{dt}(\Gamma_p^{-1} \circ \Gamma_{f(t)})(X_{m+1}(t))(0), Y\rangle = \langle \tilde{A}_p(X), Y\rangle = \tilde{B}_p(\eta)(X,Y),$$

and the theorem results from these formulae. □

4. Corollary. *Assume that M and N are orientable and m is even. Let dv be the volume form of S^{n-1} and $d\mu$ the volume form of M. Then $\tilde{K}_p d\mu(p)$, $p \in M$, is the integral along the fibers (see [**7**]) of $SN(M) \to M$ of $\gamma^*(dv)$ divided by c_{k-1}.*

Proof. We have the following diagram, exact and commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_\eta(SN_p(M)) & \longrightarrow & T_\eta(SN(M)) & \longrightarrow & T_p(M) & \longrightarrow & 0 \\
 & & \downarrow q & & \downarrow d\gamma(\eta) & & \downarrow -A_p(\eta)+\tilde{A}_p(\eta) & & \\
0 & \longrightarrow & T_{\gamma(\eta)}(S^{k-1}) & \longrightarrow & T_{\gamma(\eta)}(S^{n-1}) & \xrightarrow[P\circ\Gamma_p^{-1}]{} & T_p(M) & \longrightarrow & 0
\end{array}$$

The second rectangle is commutative, according to Theorem 3.1 and Section 2, and q is an isomorphism preserving the metric. We recall that P is the orthogonal projection. Let $Z_1, \dots, Z_{n-1}$ be a basis of $T_\eta(SN(M))$ such that $Z_1, \dots, Z_{k-1}$ is an orthonormal basis of $T_\eta(SN_p(M))$ and the image $Z'_k, \dots, Z'_{n-1}$ is an orthonormal basis of $T_p(M)$. Then, there exists an orthonormal basis $q(Z_1), \dots, q(Z_{k-1}), Y_k, \dots, Y_{n-1}$ of $T_{\gamma(\eta)}(S^{n-1})$ such that $(P \circ \Gamma_p^{-1})(Y_j) = Z'_j$, $k \leq j \leq n-1$. Then

$$\begin{aligned}\gamma^*(dv(\eta))(Z_1, \dots, Z_{n-1}) &= \det(-A_p(\eta) + \tilde{A}_p(\eta)) \\ &= \det(-B_p(\eta) + \tilde{B}_p(\eta)) \\ &= \tilde{K}_p(\eta)\end{aligned}$$

since m is even.

We conclude that the integral of $\gamma^*(dv)$ on the fibers of $SN(M) \to M$ is a form λ such that

$$\lambda(p)(Z'_k, \dots, Z'_{n-1}) = \int_{SN_p(M)} \tilde{K}_p(\eta)\, d\gamma(\eta).$$

Hence, $\lambda = c_{k-1} \tilde{K}_p \, d\mu(p)$. □

II. Applications.

1. The Gauss-Bonnet theorem. We obtain here a generalization of the well-known Gauss-Bonnet formula. This generalization depends on the following result:

Lemma 1.1. *Let M, N be differentiable manifolds with boundary, compact oriented, with the same dimension, N connected. Let $f : (M, \partial M) \to (N, \partial N)$ be a continuous map which admits a lift to a morphism of oriented vector bundles $T(M) \to T(N)$ preserving the direction pointing outwards the boundary. Then the degree of the map $f|_{\partial M} : \partial M \to \partial N$ is equal to $\chi(M) \backslash \chi(N)$.*

Proof. Let c_M and c_N be the respective obstructions to extend to the manifold a vector field without critical points pointing outwards at the boundary. Then $f^*(c_N) = c_M$. On the other hand,

$$(c_M, [M, \partial M]) = \chi(M) \quad \text{and} \quad (c_N, [N, \partial N]) = \chi(N)$$

(where the bracket means fundamental class in homology and (,) the duality between homology and cohomology), as we can see by taking a Morse function which is constant at the boundary. Therefore,

$$\chi(M) = (f^*(c_N), [M, \partial M]) = (c_N, f_*[M, \partial M]).$$

Since N is connected, $f_*([M, \partial M]) = m[N, \partial N]$, $m \in Z$. Then

$$\chi(M) = (c_N, f_*[M, \partial M]) = m\chi(N)$$

and $m = \chi(M)/\chi(N)$. But $f_*([M, \partial M]) = m[N, \partial N]$ implies $f_*[\partial M] = m[\partial N]$, and the lemma results from this. □

Theorem 1.2. *With the same hypothesis and notations of Corollary* 4*, and assuming* M *is compact, we have*

$$\int_M \tilde{K}_p \, d\mu(p) = \frac{c_{n-1}}{c_{k-1}} \chi(M).$$

Proof. From Fubini's theorem and Corollary 4, we have

$$\int_M \tilde{K}_p \, d\mu(p) = \frac{1}{c_{k-1}} \int_{SN(M)} \gamma^*(dv) = \frac{\deg(\gamma)}{c_{k-1}} \int_{S^{n-1}} dv = \frac{c_{n-1}}{c_{k-1}} \deg(\gamma).$$

In order to compute $\deg(\gamma)$ we observe that γ can be extended to a map

$$\gamma : BN(M) \to B^n$$

where $BN(M)$ is the bundle of balls of $N(M)$ and B^n is the unit ball on $\mathbf{R}^n$. We have an isomorphism

$$T(BN(M)) \simeq \pi^*(T(M)) \oplus \pi^*(N(M))$$

where $\pi : BN(M) \to M$ is the canonical projection.

We define a morphism above γ

$$\pi^*(T(M)) \oplus \pi^*(N(M)) \to T(B^n) = B^n \times \mathbf{R}^n$$

by

$$(X, Y) \mapsto (\Gamma_p(\eta), \Gamma_p(X + Y))$$

where $p \in M$, $\eta \in BN_p(M)$, $X \in T_p(M)$ and $Y \in N_p(M)$.

Therefore, one can apply Lemma 4.1 to obtain

$$\deg(\gamma) = \chi(BN(M))/\chi(B^n) = \chi(M)$$

and this proves the theorem. □

2. Invariant curvature and complex singularities. We assume in this section that N is an $(n+1)$-dimensional complex manifold with an Hermitian metric $\langle \ , \ \rangle_c$. Similarly to the real case, a complex translation is determined by a map $\Gamma^c : TN \to N \times C^{n+1}$ which makes commutative the corresponding complex diagram of page 2 and such that, given $p \in N$, $\Gamma_p^c : T_p(N) \to C^{n+1}$ is an isomorphism between Hermitian vector spaces. Let us denote by CP^n the complex projective space of complex lines of C^{n+1}.

Let M be a complex submanifold of N. Then the Gauss map $\gamma_c : M \to CP^n$ of M determined by Γ^c is defined by

$$\gamma_c(p) = \Gamma_p(T_p(M)^\perp).$$

Let $f : N \to C$ be an analytic map with an isolated critical point at $p_0 \in N$ such that $f(p_0) = 0$. Given $H \in CP^n$, H determines a polar curve P_H by the condition:

$$p \in P_H \Longleftrightarrow T_p(f^{-1}(t)) = (\Gamma_p^c)^{-1}(H)^\perp$$

where $t \in C$ is such that $f(p) = t$.

We can define from $\langle \ , \ \rangle_c$ a Riemannian metric $\langle \ , \ \rangle$ in N by setting $\langle \ , \ \rangle := \mathrm{re}(\langle \ , \ \rangle_c)$. The complex translation Γ^c determines a real translation Γ by setting $\Gamma := \Gamma^c + i\Gamma^c$, $i^2 = -1$.

We prove here the following generalization of the Theorem of [**3**]:

Theorem 2.1. *Let $f : N \to C$ be an analytic map with an isolated critical point at p_0 with $f(p_0) = 0$. Then the following formula holds*:

$$\lim_{\varepsilon \to 0} \lim_{t \to 0} \int_{M_t \cap B_\varepsilon} \tilde{K}_p \omega(p) = c_n(\mu^{n+1} - \mu^n)$$

where $M_t = f^{-1}(t)$, B_ε *is the geodesic ball centered at* p_0 *with radius* ε, $\tilde{K}$ *is the invariant curvature of* M_t *determined by* Γ (*see* I.1), ω *the volume form of* M_t *in the induced metric,* μ^{n+1} *the Milnor number of* f *at* p_0, *and* μ^n *the Milnor number of* f *restricted to a generic complex codimension* 1 *submanifold of* N *through* p_0 *at* p_0.

Proof. Let $\gamma : SN(M_t) \to S^{2n}$ be the Gauss map of M_t determined by Γ (see I.1) and $\gamma_c : M_t \to CP^n$ the Gauss map determined by Γ^c. It is not difficult to prove that the following diagram is commutative:

$$\begin{array}{ccc} SN(M_t) & \xrightarrow{\gamma} & S^{2n} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle \pi} \\ M_t & \xrightarrow{\gamma_c} & CP^n \end{array}$$

where π is the projection of the Hopf fibration. Hence,

$$\int_{M_t \cap B_\varepsilon} \tilde{K}_p \omega(p) = \frac{1}{c_2} \int_{SN(M_t \cap B_\varepsilon)} \gamma^*(\sigma) = \frac{1}{c_2} \int_{SN(M_t \cap B_\varepsilon)} \gamma_c^*(\lambda)$$

where λ is the volume form of CP^n.

We prove now that the last integral above is the intersection index $I(P_H, M_0)$ between M_0 and the polar curve P_H determined by a generic complex hyperplane H of C^{n+1}. Theorem 2.1 follows then from a result of Teissier [**8**, Section 1.4].

As in the Lemma of [**3**], and according to the notations of [**3**], one has:

$$\int_{M_t \cap B_\varepsilon} \gamma_c^*(\lambda) = \int_{CP^n} \tau(M_t \cap B_\varepsilon, H) \lambda(H^\perp)$$

where

$$\tau(M_t \cap B_\varepsilon, H) = \sum_{p \in B_\varepsilon} I(M_t, P_H)_p$$

so that in order to compute the limit for $t \to 0$ and $\varepsilon \to 0$ as in [**3**], we have just to assure that the function $t \to \tau(M_t \cap B_\varepsilon, H)$ is bounded.

Clearly, $\tau(M_t \cap B_\varepsilon, H)$ is finite for $t \neq 0$ and, since t goes to 0, we just have to prove that $\lim_{t \to 0} \tau(M_t \cap B_\varepsilon, H)$ is finite. To do this, let us consider the Nash transformation $N_f \subset B_\varepsilon \times CP^n$ of f restricted to B_ε. Let $\pi : N_f \to B_\varepsilon$ and $\gamma : N_f \to CP^{n-1}$ be the projections $(x, H) \to x$ and $(x, H) \to H^\perp$, respectively. Therefore, it is easy to see that

$$\tau(M_t \cap B_\varepsilon, H) = \text{card}\,(\pi_{-1}(M_t) \cap \gamma^{-1}(H^\perp)), \qquad t \neq 0$$

so that

$$\lim_{t\to 0} \tau(M_t \cap B_\varepsilon, H) = \text{card}\,(\pi^{-1}(M_0) \cap \gamma^{-1}(H^\perp)).$$
$$\pi(\pi^{-1}(M_0 \cap \gamma^{-1}(H^\perp)) = \{p \in M_0 \cap B_\varepsilon | p \neq 0$$

and

$$T_p(M_0) = H\} \cup \{0\}$$

is analytic and compact and hence finite.

It follows that $\pi^{-1}(M_0) \cap \gamma^{-1}(H^\perp)$ is finite since

$$\pi : \pi^{-1}(M_0) \cap \gamma^{-1}(H^\perp) \to \pi(\pi^{-1}(M_0) \cap \gamma^{-1}(H^\perp))$$

is bijective. □

In [**3**] Langevin reobtains a result due to Linda Ness [**4**] about the curvature of algebraic curves of CP^2 converging to an algebraic curve with an isolated singularity (see [**3**, Section III]). We obtain here the following generalization:

Corollary 2.2. *Let N be a complex n-dimensional manifold with an Hermitian metric. Let M be a complex hypersurface of N with an isolated singularity at p. Let M_t be a family of complex hypersurfaces of N converging to M as t goes to infinity. Then*

$$\liminf_{t\to\infty} K_I = -\infty$$

where K_I denotes the intrinsic sectional curvature of M_t.

Proof. Let $p_t \in M_t$ be such that $\lim p_t = p_0$. Then it follows from Theorem 2.1 and from the definition of $\tilde{K}$ that

$$\lim_{t\to\infty} \lambda_{p_t} = \infty$$

where $\lambda_t = \max\{\lambda | \lambda$ is an eigenvalue of $A_{p_t}(\eta)$ for some $\eta \in T_{p_t}(M_t)^{\perp}$, $||\eta|| = 1\}$. If X_t is an eigenvector associated to λ_t then $-\lambda_t$ is an eigenvalue with eigenvector iX_t.

Denote by $K(P_t)$ and $\overline{K}(P_t)$ the sectional curvatures of M_t and N, respectively, at p_t, determined by the plane P_t generated by X_t and iX_t. Without loss of generality, we may assume that P_t converges to P as t tends to infinity. Then, from the Gauss equation of an isometric immersion, one has

$$K(P_t) = \overline{K}(P_t) - \lambda_t^2$$

hence

$$\lim_{t\to\infty} K(P_t) = \overline{K}(P) - \lim_{t\to\infty} \lambda_t^2 = -\infty. \qquad \square$$

3. Convexity of hypersurfaces. A classical result of differential geometry, known as Hadamard theorem, establishes that a compact hypersurface of a Euclidean space whose Gauss-Kronecker curvature (as it is called the Lipschitz-Killing curvature in codimension 1) is everywhere positive is a convex hypersurface. In particular, the hypersurface is diffeomorphic to a sphere. This result is not true for hypersurfaces in an arbitrary Riemannian manifold. A simple counterexample is the natural isometric embedding of the Riemannian product $S^2 \times S^2$ in S^5. As we remark below, if we consider the curvature $\tilde{K}$ instead of the usual Gauss-Kronecker curvature, Hadamard theorem remains partially true. To see this, let us consider again a real differentiable $(n+1)$-dimensional Riemannian manifold N and let M be a hypersurface of N. Let Γ be a translation in N.

In this case the Gauss map γ of M can be considered as a map from M to the sphere S^n of the same dimension. By this identification, we have $\Gamma_p^{-1} \circ d\gamma(p)$: $T_p(M) \to T_p(M)$ and $\tilde{K}_p = \det(A_p(\eta) - \tilde{A}_p(\eta)) = \det\ (\Gamma_p^{-1} \circ d\gamma(p))$. Therefore, if the invariant curvature $\tilde{K}$ is everywhere different from zero, then γ is a local diffeomorphism. Hence, if M is compact and $\tilde{K} \neq 0$ everywhere, γ is a covering map and, since it goes into the sphere which is simply connected, γ is a global diffeomorphism. In particular, M is diffeomorphic to a sphere.

We have proved:

Proposition 3.1. *Let M be an immersed compact hypersurface of N such that $\tilde{K} \neq 0$ everywhere. Then M is diffeomorphic to a sphere.*

In the Euclidean spaces, the fact that the Gauss map is a diffeomorphism implies that the hypersurface is embedded. This is not true in the general situation of this paper. A counterexample is given in the remark after Theorem 9 of [**6**].

In [**1**], J. Eschenburg introduced a stronger condition than the positiveness of the Gauss-Kronecker curvature for a hypersurface in a Riemannian manifold: a hypersurface M of a Riemannian manifold N is called *ε-convex* if all of its principal curvatures have the same sign and absolute values greater than ε. If a hypersurface is ε-convex for some $\varepsilon > 0$, then its Gauss-Kronecker curvature is necessarily positive (perhaps after a reorientation of M). The converse is clearly false (the same example as above). With this notion, Eschenburg proves: *a compact ε-convex hypersurface, for some $\varepsilon > 0$, of a Riemannian space with nonnegative sectional curvature is the boundary of an immersed disk in the space. In particular, the hypersurface is diffeomorphic to a sphere.* As is pointed out in [**4**], this result is false in a Riemannian space with negative curvature. Counterexamples are given by the boundary of tubular neighborhoods around closed geodesics. Using Proposition 3.1 we prove here the following result. We first introduce some notations. As before, let N be a Riemannian $(n+1)$-dimensional manifold and $\Gamma : TN \to N \times \mathbf{R}^{n+1}$ a translation in N. Given $p \in N$, denote by O the set of orthonormal basis of $T_p(N)$. Given an orthonormal basis $\beta = \{\eta, X_1, \dots, X_n\}$ of $T_p(N)$, denote by U the $(n \times n)$-matrix $(a_{kl}) = (\langle \nabla_{X_k} \tilde{\eta}, X_l \rangle)$. Given $1 \leq i_1 < \dots i_j \leq n$, we denote by $\Gamma_{i_1, \dots, i_j}(p, \beta)$ the determinant of the submatrix $U_{i_1, \dots, i_j}$ of U obtained from U by taking out the i_k^{th} row and the i_k^{th} column, $1 \leq k \leq j$, of U. Set

$$\Gamma_{i_1, \dots, i_j}(p) := \max\{\Gamma_{i_1, \dots, i_j}(p, \beta) \mid \beta \in O\}$$

and denote

$$\Gamma_{i_1, \dots, i_j} = \sup\{\Gamma_{i_1, \dots, i_j}(p) \mid p \in N\}.$$

We have

Theorem 3.2. *Let M be a compact hypersurface of N. Assume that* *the principal curvatures* $\lambda_1, \ldots, \lambda_n$ *of* M *satisfy*

$$\lambda_{i_1} \ldots \lambda_{i_j} > C \cdot \Gamma_{i_1, \ldots, i_j}$$

for any choice of $1 \leq i_1 < \cdots < i_j \leq n$, *where* $C := \binom{n}{1} + \cdots + \binom{n}{n}$. *Then* M *is diffeomorphic to a sphere. In particular, if* M *is an* ε*-convex with* $\varepsilon^j > C \cdot \Gamma_{i_1, \ldots, i_j}$ *for any choice of* $1 \leq i_1 < \cdots < i_j \leq n$, *then* M *is diffeomorphic to a sphere.*

Proof. Diagonalizing the matrix of $A_p(\eta)$ and using elementary linear algebra, we see that $\tilde{K}(p) = \det(A_p(\eta) - \tilde{A}_p(\eta))$ is positive for any $p \in M$. The theorem results then from Proposition 3.1. □

An extension of Hadamard's theorem for immersions of arbitrary codimension is obtained in [**5**]. By using Theorem II 1.2 for the case of Lie groups, one can obtain the following:

Proposition 3.3. *Let* M *be a compact, connected, oriented surface immersed in a Lie group* G *with a bi-invariant metric. Assume that the Lipschitz-Killing curvature of* M *is everywhere positive. Then* M *is diffeomorphic to* S^2.

Proof. Let us consider the curvature $\tilde{K}$ of M determined by the left translation in G. Since the metric is bi-invariant, we have $\nabla_X Y = (1/2)[X, Y]$, where $[\ ,\ ]$ denotes the Lie bracket and X and Y are left invariant vector fields. It follows that the invariant second fundamental form $\tilde{B}$ is skew-symmetric. Since $\dim(M) = 2$, we have

$$\tilde{K}_p(\eta) \geq K_p(\eta) > 0.$$

Hence by Theorem II 1.2, $\chi(M) > 0$ and M is diffeomorphic to a sphere. □

The Gauss map and invariant second fundamental form associated to a left translation in a Lie group are studied in [**6**] for the case of hypersurfaces.

REFERENCES

1. J. Eschenburg, *Local convexity and nonnegative curvature: Gromov's proof of the sphere theorem*, Invent. Math. **84** (1986), 507–522.

2. R. Langevin, *Courbure, feuilletages et surfaces*, Thèse Université de Paris Sud, (1980).

3. ———, *Courbure et singularités complexes*, Comment. Math. Helv. **54** (1979), 6–16.

4. L. Ness, *Curvature on algebraic curves*, Doctoral Thesis, Harvard, 1975.

5. J. Ripoll, *Characterizations of compact ε-convex immersions*, preprint.

6. ———, *On hypersurfaces of Lie groups*, Illinois J. Math. **35** (1991), 47–56.

7. M. Sebastiani, *Un exposé de la formule de Leray-Norguet*, Bol. Soc. Brasil Mat. **1** (1970), 45–47.

8. B. Teisser, *Varietés polaires. Invariants polaires des singularités d'hypersurfaces*, Invent. Math. **40** (1977), 267–292.

9. J.L. Weiner, *The Gauss map in spaces of constant curvature*, Proc. Amer. Math. Soc. **38** (1973), 157–161.

Universidade Federal do R.G. do Sul, Instituto de Matemática, Av. Bento Gonçalves 9500, 91500 - Porto Alegre - RS, Brazil