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HALL SUBGROUPS OF ORDER NOT DIVISIBLE BY 3

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1. Introduction. In [**9**] it is proved that if a finite group G has a Hall π-subgroup and if π does not contain 2, then all Hall π-subgroups of G are conjugate. The proof of this is based upon proving it in the special case when G is a simple group. Further, it is shown that if H is a Hall π-subgroup of a finite simple group and $2 \notin \pi$, then H has a Sylow tower. An examination of the proof of this shows that a crucial point in the argument is that a Weyl group has very few Hall subgroups other than the Sylow subgroups. Indeed, if H is a Hall subgroup of a Weyl group and the order of H is divisible by at least two distinct primes, then H must have even order; one use of the assumption that $2 \notin \pi$ in the result quoted above is to assert that, when G is a simple group of Chevalley type, a Hall π-subgroup of the Weyl group of G must be a Sylow subgroup. Since if H is as above, it is also true that $|H|$ is divisible by 3 as well as by 2, it is tempting to consider Hall π-subgroups of a finite simple group where now 2 may belong to π but we exclude 3. The main result of this paper deals with the case when the simple group is either of the groups $A_n(q)$ or $C_n(q)$; specifically, we prove the following:

Proposition. *Let S be either $A_n(q)$ or $C_n(q)$. Assume S has a Hall π-subgroup with $3 \notin \pi$. Then all Hall π-subgroups of S are conjugate in S and a Hall π-subgroup of S has a Sylow tower. Further, if $S \leq G \leq \mathrm{Aut}(S)$, then G has a Hall π-subgroup, all Hall π-subgroups of G are conjugate in G, and a Hall π-subgroup of G is solvable.*

It should be noted that the results in this paper make use of the fact (an immediate consequence of the classification of finite simple groups) that the Suzuki groups are the only non-Abelian finite simple groups of order not divisible by 3. Using this, the above proposition is straightforward to prove when S is $A_n(q)$. When S is the Symplectic group $C_n(q)$, however, the argument is more difficult and also requires

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a result of Glauberman [**5**] concerning normal 2-complements of groups which do not involve S_4 as well as some recent work of mine [**10**] on large Abelian subgroups of wreath products.

Conjecture 1. *The above result holds true if S is allowed to be any of the classical simple groups, i.e., if S is one of $A_n(q)$, $B_n(q)$, $C_n(q)$, $D_n(q)$, ${}^2A_n(q)$ or ${}^2D_n(q)$.*

Note that the proposition implies that if S has a Hall π-subgroup with $3 \notin \pi$, then $\mathrm{Aut}\,(S)$ also has a Hall π-subgroup. The significance of this is that if G is a finite group and $\mathrm{Aut}\,(S)$ has a Hall π-subgroup for each composition factor S of G, then G has a Hall π-subgroup [**7**, Corollary 3.6].

Conjecture 2. *Let G be a finite group and π a set of primes with $3 \notin \pi$. Then G has a Hall π-subgroup if and only if each composition factor of G has a Hall π-subgroup.*

It is not true that if a simple group G has a Hall π-subgroup with $3 \notin \pi$, then all Hall π-subgroups of G are conjugate. For suppose G is the group ${}^2G_2(27)$ and $\pi = \{2, 7\}$. Then $|G|_\pi = 56$ and G has 2 conjugacy classes of subgroups of order 56 [**19**]. The Hall π-subgroups of G have Sylow towers but in one class the Sylow 2-subgroup is normal, whereas in the other class the Sylow 7-subgroup is normal. It is also possible for a Hall π-subgroup of a simple group to not be solvable when $3 \notin \pi$. If G is one of the Suzuki groups ${}^2B_2(2^{2n+1})$ and $H = G$, then H is a Hall subgroup of G, 3 does not divide $|H|$, and H certainly does not have a Sylow tower. I conjecture that this is the only case where this occurs.

Conjecture 3. *If H is a Hall π-subgroup of the finite simple group G and $3 \notin \pi$, then either H has a Sylow tower or $H = G =$ a Suzuki group.*

It is easy to show that if the third conjecture is true, so is the second. Also, it follows from [**9**] that all 3 of the above conjectures are true if $2 \notin \pi$. Thus, these conjectures really are concerned with the situation

$2 \in \pi$ and $3 \notin \pi$. Conjecture 3 above is true in the special case when G is a Chevalley group and π contains the characteristic of the underlying field (see Theorem 3.1). It should be mentioned that if π contains both 2 and 3, it is quite possible for each composition factor of a group G to have a Hall π-subgroup and yet for G to not have a Hall π-subgroup (see, for example, [**11**]).

The results leading to the above proposition also lead to necessary and sufficient conditions for the groups $GL_n(q)$ and $Sp_{2n}(q)$ to have a Hall π-subgroup, at least when q is odd and π does not contain 3. In the case of $GL_n(q)$, once it is shown that a Hall π-subgroup of $GL_n(q)$ is solvable, the results of [**13**] are applicable and lead at once to the desired result. In the case of $Sp_{2n}(q)$, necessary and sufficient conditions for the existence of a Hall π-subgroup were known if $2 \notin \pi$ ([**13, 6**]), but not when $2 \in \pi$ and $3 \notin \pi$.

2. Notation and preliminary results. The groups considered in this paper are assumed finite. We follow the notation of [**3**] for describing the linear and Chevalley groups. The symmetric group of degree n is denoted by S_n. For a general reference on the Chevalley groups, see [**15, 1**].

Throughout this paper, π is a set of primes and π' is the set of all primes not belonging to π. Following [**12**] we say that the group G satisfies E_π if G has a Hall π-subgroup; G satisfies C_π if G satisfies E_π and all Hall π-subgroups of G are conjugate; and G satisfies D_π if G satisfies C_π and every π-subgroup of G is contained in a Hall π-subgroup of G.

The largest normal π-subgroup of the group G is $O_\pi(G)$. The group G is said to be π-closed if $O_\pi(G)$ contains all π-elements of G. In particular, G is $2'$-closed if and only if G has a normal 2-complement. If n is a positive integer, then n_π is the largest divisor of n all of whose prime factors belong to π. In particular, n_p is the largest power of the prime p which divides n. The set of primes dividing n is denoted by $\pi(n)$; if G is a group, then $\pi(G) = \pi(|G|)$. The integral part of the real number x is denoted by $[x]$.

If G is a p-group for a prime p and n is a positive integer, then $\Omega_n(G)$ is the subgroup generated by all elements of order dividing p^n. Again with G a p-group, $J(G)$ is the subgroup generated by all Abelian subgroups of maximal order in G.

The derived subgroup and center of the group G are denoted by G' and $Z(G)$, respectively, while $\mathrm{Aut}\,(G)$ is the automorphism group of G. If H is a subgroup of G, we write $H \leq G$ and use $H < G$ to indicate that H is a proper subgroup of G. Similarly, $H \trianglelefteq G$ denotes that H is a normal subgroup of G while $H \triangleleft G$ implies that H is a proper normal subgroup of G. The normalizer and centralizer of H in G are $N_G(H)$ and $C_G(H)$, respectively.

If r is a prime and s is an integer not divisible by r, then $e(s,r)$ is the smallest positive integer m such that $(s^m - 1)_r > 2$. Hence, if $r > 2$, then $e(s,r)$ is the order of s in the multiplicative group of $GF(r)$; $e(s,2)$ is 1 if $s \equiv 1 \pmod 4$ while $e(s,2) = 2$ if $s \equiv 3 \pmod 4$.

If n is a positive integer, then Z_n denotes a cyclic group of order n. If A is a group and B is a permutation group of degree n, then $A \,\mathrm{wr}\,(B,n)$ is the semi-direct product of M, the direct product of n copies of A, by B where B acts on M by permuting the factors. We refer to M as the base of $A \,\mathrm{wr}\,(B,n)$.

The group G has a Sylow tower if for some ordering of the primes $\{p_1, \dots, p_m\}$ which divide $|G|$, G has a normal series $1 = G_0 < G_1 < \cdots < G_m = G$ such that G_i/G_{i-1} is isomorphic to a Sylow p_i-subgroup of G for $1 \leq i \leq m$. If H is a group with a Sylow tower relative to the same ordering of the primes, we say that G and H have Sylow towers of the same complexion. The p-rank of the group G is denoted by $m_p(G)$ and is the largest positive integer m such that G contains an elementary Abelian p-subgroup of order p^m. If k is a positive integer, then $\mu_k(G)$ is the maximum of $|A|$ where A runs through all Abelian subgroups of G with the property that $x^k = 1$ for all $x \in A$.

The following result is well known and may be found in [**12**].

Lemma 2.1 *If M is a normal subgroup of the group G, then the following are true*:

(a) *If H is a Hall π-subgroup of G, then $H \cap M$ is a Hall π-subgroup of M and HM/M is a Hall π-subgroup of G/M.*

(b) *If M satisfies C_π and G/M satisfies E_π, then G satisfies E_π.*

(c) *If M and G/M both satisfy C_π, then G satisfies C_π.*

Lemma 2.2 *Let r be a prime, s an integer not divisible by r, $e = e(s, r)$, and n a positive integer. Then*

$$(s^n - 1)_r = \begin{cases} (s^e - 1)_r (n/e)_r & \text{if } e \text{ divides } n, \\ 1 & \text{if } e \text{ does not divide } n \text{ and } r > 2, \\ 2 & \text{if } e \text{ does not divide } n \text{ and } r = 2. \end{cases}$$

Proof. This is proved in [**20**] if $r > 2$. Using induction on n, the result is easily extended to the case $r = 2$. □

Theorem 2.3 *If H and K are both Hall π-subgroups of G and if H and K have Sylow towers of the same complexion, then H and K are conjugate in G.*

Proof. This is Theorem A1 of [**12**]. □

Lemma 2.4 *Let $G = A \operatorname{wr}(B, n)$ where B is a subgroup of S_n, and let M be the base subgroup of G. Let p be a prime and k a positive integer. Then the following are true*:

(a) *If $|A|_p > 2$ and P is a Sylow p-subgroup of G, then $J(P) = J(P \cap M) \leq M$.*

(b) *If $\mu_k(A) > 2$, then $\mu_k(G) = \mu_k(A)^n$ and, if H is an Abelian subgroup of G such that $|H| = \mu_k(G)$ and $x^k = 1$ for all $x \in H$, then $H \leq M$.*

(c) *If $|A|_p > 1$, then $m_p(G) = m_p(A)n$.*

Proof. This is proved in [**10**, Theorem 3.4 and Corollary 3.6]. □

Lemma 2.5 *Let t be an odd prime, and let $G = GL_e(q)$ with $(q, t) = 1$ and $e = e(q, t)$. Let T be a Sylow t-subgroup of G, $C = C_G(T)$, and $N = N_G(T)$. Then T is cyclic of order $(q^e - 1)_t$, C is cyclic of order $(q^e - 1)$ and N/C is cyclic of order e.*

Proof. Let $E = GF(q^e)$ and U the additive group of E. For $0 \neq \lambda \in E$, let T_λ be the mapping $u \to u\lambda$ for all $u \in U$, and let

σ be the mapping $u \to u^q$. Then $\{T_\lambda \mid 0 \neq \lambda \in E\}$ is a cyclic subgroup of order $(q^e - 1)$ in $GL_e(q)$ and $\langle \sigma \rangle$ is a subgroup of order e in $GL_e(q)$. Now

$$|T| = |G|_t = (q^e - 1)_t,$$

and so we may assume $T \leq \{T_\lambda \mid 0 \neq \lambda \in E\}$. Since t does not divide $q^k - 1$ for $1 \leq k < e$, C must be isomorphic to the multiplicative group of E. Hence, $C = \{T_\lambda \mid 0 \neq \lambda \in E\}$. Since N must normalize C and since, as is well known, the normalizer of $\{T_\lambda \mid 0 \neq \lambda \in E\}$ in $GL_e(q)$ is $\langle \sigma \rangle \{T_\lambda \mid 0 \neq \lambda \in E\}$, $N = C\langle \sigma \rangle$. Therefore, N/C is cyclic of order e. □

Theorem 2.6 *Let t be an odd prime, and let $G = SP_{2n}(q)$ with $(q, t) = 1$. Assume $e = e(q, t)$ and $n = e/(e, 2)$. Let T be a Sylow t-subgroup of G, $C = C_G(T)$ and $N = N_G(T)$. Then T is cyclic, C is cyclic of order $q^n + (-1)^e$, and N/C is cyclic of order $2n$. If $t > 3$, then N contains a Sylow 2-subgroup of G if and only if $(q, 2) = 1$ and $e(q, 2) = e$.*

Proof. First assume that e is odd. Then $n = e$. If J is the matrix

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

with I the identity of $GL_n(q)$, G may be identified with the group of all $x \in GL_{2n}(q)$ such that $xJx^\dagger = J$ (here $x^\dagger$ is the transpose of x). If $A \in GL_n(q)$, let A^* be the inverse transpose of A. Then

$$\begin{bmatrix} A & 0 \\ 0 & A^* \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & A \\ -A^* & 0 \end{bmatrix}$$

belong to G for all $A \in GL_n(q)$. Now $GL_n(q)$ contains an element B of order $(q^n - 1)_t$. If

$$x = \begin{bmatrix} B & 0 \\ 0 & B^* \end{bmatrix},$$

then $\langle x \rangle$ is a Sylow t-subgroup of G. Hence, we may assume that $T = \langle x \rangle$. If V is the vector space of dimension $2n$ on which G acts, $V = U_1 \oplus U_2$ where U_1 and U_2 are irreducible $GF(q)T$-modules with x represented by B on U_1 and by B^* on U_2.

If λ is an eigenvalue of B, then the other eigenvalues of B must be the algebraic conjugates $\lambda^q, \lambda^{q^2}, \ldots, \lambda^{q^{n-1}}$. On the other hand, λ^{-1} is an eigenvalue of B^*. If B and B^* were similar matrices, we would have

$$\lambda^{q^i+1} = 1$$

for some i, $0 \leq i < n$. This would imply that t divides $q^i + 1$. Then t divides $q^{2i} - 1$ and so e must divide $2i$. Since we are assuming that e is odd, it follows that e divides i. Then t divides $q^i - 1$. Since $t > 2$, it is impossible for t to divide both $q^i + 1$ and $q^i - 1$. Thus, B and B^* are not similar matrices. It follows that U_1 and U_2 are nonisomorphic $GF(q)T$-modules. This implies that C must fix both U_1 and U_2. Also, every element of N either fixes both U_1 and U_2 or interchanges them.

We now see that C consists of all matrices

$$\begin{bmatrix} A & 0 \\ 0 & A^* \end{bmatrix}$$

with A any matrix in $GL_n(q)$ which commutes with B. It follows from this that C is cyclic of order q^n-1. If N_1 is the subgroup of N consisting of those elements which fix U_1 and U_2, then N_1 consists of all matrices of the above form with A any matrix in $GL_n(q)$ which normalizes $\langle B\rangle$. It follows that N_1/C is cyclic of order n. Now $\langle B\rangle$ and $\langle B^*\rangle$ are both Sylow t-subgroups of $GL_n(q)$. Hence, there exists a $D \in GL_n(q)$ such that $D^{-1}BD = (B^*)^k$ for some k with $(t,k) = 1$. Then if

$$y = \begin{bmatrix} 0 & D \\ -D^* & 0 \end{bmatrix},$$

$y \in G$, $y^{-1}xy = x^k$, and y interchanges U_1 and U_2. Hence, $|N/N_1| = 2$. Therefore, $|N/C| = 2n$. Since N/C is isomorphic to a subgroup of $\mathrm{Aut}\,(T)$ and T is cyclic, N/C is cyclic.

If $t > 3$, then q^n cannot be 2 (since then $|G| = |Sp_2(2)| = 6 \not\equiv 0 \pmod t$). Thus, if q is a power of 2, then $|G|_2 = q^{n^2} > 2 = |N|_2$. Hence, N does not contain a Sylow 2-subgroup of G if $t > 3$ and the characteristic of $GF(q)$ is 2. Assume $(q,2) = 1$. Then $|N|_2 = 2(q^n-1)_2$ which equals 4 if $q \equiv 3 \pmod 4$. Since $|G|_2 \geq 8$, we see that N does not contain a Sylow 2-subgroup of G if $q \equiv 3 \pmod 4$. Assume then that $q \equiv 1 \pmod 4$. Then $e(q,2) = 1$ and $|N|_2 = 2n_2(q-1)_2$ while, from Lemma 2.2,

$$|G|_2 = (q-1)_2^n(2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n)_2.$$

It follows at once that $|G|_2 = |N|_2$ if and only if $n = 1$. Therefore, when $t > 3$ and e is odd, N contains a Sylow 2-subgroup of G if and only if $(q,2) = 1$ and $e(q,2) = e = 1$.

Now suppose that e is even. Then $e = 2n$. Let $F = GF(q)$, $E = GF(q^e)$, U the additive group of E, and p the characteristic of F. If $p = 2$, let $\mu = 1$. If $p \neq 2$, let μ be an element of order $2(q^n-1)_2$ in the multiplicative group of E. In all cases, therefore, $\mu^{q^n} = -\mu$. Now define $[\ ,\ ] : U \times U \to F$ by $[u,v] = \mathrm{trace}_{E/F}(\mu u v^{q^n})$. If we regard U as a vector space of dimension $2n$ over F, then $[\ ,\ ]$ is a nonsingular symplectic form on U. Without loss of generality, we may assume that $[\ ,\ ]$ is the form preserved by G. If λ is a nonzero element of E, let T_λ be the mapping of U given by $u \to u\lambda$. Then $T_\lambda \in G$ if and only if $\lambda^{q^n+1} = 1$. Hence, if $L = \{T_\lambda \mid T_\lambda \in G\}$, L is a cyclic subgroup of order q^n+1 in G. Since $(q^k-1)_t = 1$ for $k < 2n$,

$$|G|_t = (q^{2n}-1)_t = (q^n-1)_t(q^n+1)_t = (q^n+1)_t = |L|_t.$$

Thus, without loss of generality, we may assume $T \leq L$. The centralizer of L in $GL(U)$ is $C_1 = \{T_\lambda \mid 0 \neq \lambda \in E\}$. It follows from this that $C = C_1 \cap G = L$. From Lemma 2.5, the normalizer of T in $GL(U)$ is $C_1\langle\sigma\rangle$ where σ is the map $u \to u^q$ for all $\alpha \in U$.

Now $(\mu^2)^{q^n} = \mu^2$ and so $\mu^2 \in GF(q^n)$. If q is odd, it follows that $\mu^{q-1} \in GF(q^n)$. This implies that there exists a $v \in E$ such that $v^{q^n+1} = \mu^{q-1}$ (if $p = 2$, we simply choose $v = 1$). Define the mapping R on U by $R : u \to vu^q$. Then $C_1\langle\sigma\rangle = C_1\langle R\rangle$ and so $N = G \cap C_1\langle R\rangle$. However, a straightforward calculation reveals that $R \in G$. Hence, $N = \langle R\rangle(G \cap C_1) = RC$. It now follows that N/C is cyclic of order $2n$.

Then $|N| = 2n(q^n+1)$. Suppose $t > 3$. If $p = 2$ and $|G|_2 = |N|_2$, then $q^{n^2} = 2n_2 \leq 2n$, and it follows that $q = 2$, $n = 1$ and $|G| = |Sp_2(2)| = 6 \not\equiv 0 \pmod t$. Thus N does not contain a Sylow 2-subgroup of G if $p = 2$. Assume then that $p > 2$. If n is even, then $(q^n+1)_2 = 2$ and so $|N|_2 = 4n_2$. Since $q^2 \equiv 1 \pmod 8$, we may use Lemma 2.2 to conclude that

$$|G|_2 = (q^2-1)_2^n n!_2 \geq 8^n n_2 > |N|_2.$$

Hence, N does not contain a Sylow 2-subgroup of G if n is even. Suppose n is odd. Then

$$|G:N|_2 = (q^2-1)_2 \cdots (q^{2n-2}-1)_2(q^n-1)_2/2$$

and it is easy to see that $|G:N|_2 > 1$ if $n \geq 3$. Suppose finally that $n = 1$. Then $e = 2$ and

$$|G:N|_2 = (q-1)_2/2.$$

Therefore, $|G:N|_2 = 1$ if and only if $q \equiv 3 \pmod 4$. Summing up, we see that N contains a Sylow 2-subgroup of G if and only if q is odd and $e(q,2) = 2 = e$. This finishes the proof of the theorem. □

Lemma 2.7 *If x is a real number ≥ 5, then $2^x - x^2 \geq 7$.*

Proof. If $f(x) = 2^x - x^2$, then it is straightforward to show that $f''(x) > 0$ when $x \geq 5$. Then $f'(x) \geq f'(5) > 0$. Hence, $f(x) \geq f(5) = 7$. □

3. The main results.

Theorem 3.1 *Let G be a Chevalley group (normal or twisted), and let p be the characteristic of the underlying field. Let A be a Hall π-subgroup of G and assume that π contains p but not 3. Then G satisfies C_π and either $p = 2$ or $2 \notin \pi$. Further, either $A = G =$ a Suzuki group or A is contained in a Borel subgroup of G.*

Proof. If $2 \notin \pi$, the result follows from [**8**, Theorem 3.2]. Assume then that $2 \in \pi$. If G is a π-group, then G must be a Suzuki group since $3 \notin \pi$ and then the theorem follows immediately. Hence, assume that G is not a π-group. Since $p \neq 3$, G is not of type ${}^2G_2$.

Suppose every Hall π-subgroup of G is contained in some Borel subgroup. Then every pair of Hall π-subgroups have Sylow towers of the same complexion. It follows that G satisfies C_π. Further, if $p \neq 2$ and B is a Borel subgroup of G, then it follows from Propositions 8.6.1 and 14.1.3 of [**1**] that $|G:B| \equiv |W| \pmod 2$ where W is the Weyl group of G. Since $|W|$ is divisible by 2 in all cases, it would follow that B does not contain a Sylow 2-subgroup of G. Since B contains a Hall π-subgroup of G, 2 cannot be contained in π if $p \neq 2$. Thus, the theorem is proved if every Hall π-subgroup of G is contained in a Borel subgroup.

Therefore, there is no loss of generality in assuming that A is not contained in any Borel subgroup of G. Since $p \in \pi$, A contains a Sylow p-subgroup U of G. Let $B = N_G(U)$, and let H be a complement to U in B. Then $B = H(A \cap B)$ and so it follows from [**18**, Proposition 2.5] that $AB = BA > B$ (since $A \not\leq B$). Since $U \trianglelefteq B$, the normal closure of U in AB is contained in A. Since $G > A$, it now follows that $AB \neq G$.

As is shown in [**13**, p. 288], the normal closure of U in AB must contain $\langle X_r, X_{-r}\rangle$ for some fundamental root r. Let $L = \langle X_r, X_{-r}\rangle$. Then $A \geq L$ and $AB \geq LB$. It follows from [**14**, Section 7] and [**15**, p. 183] that L is a homomorphic image of one of the groups $SL_2(q)$, $SL_2(q^2)$, $SL_2(q^3)$, $SU_3(q)$, or ${}^2B_2(q)$. Since 3 does not divide $|A|$, $L \cong {}^2B_2(q)$. Since $G > A$, it follows that $G = {}^2F_4(q)$, $q = 2^{2m+1}$ for some positive integer m, and $p = 2$. Now ${}^2F_4(q)$ only contains 2 distinct proper subgroups which properly contain a given Borel subgroup [**1**, p. 231]. Hence, $A = LB$. This implies that $|A|$ divides $|L|\,|B|$. Since $|{}^2B_2(q)|$ divides $|A|$, we conclude that $(q^2 + 1)$ divides $|A|$. Since $(q^2+1)^2$ divides $|G|$ and since A is a Hall π-subgroup of G, $(q^2+1)^2$ must divide $|A|$. It now follows that $(q^2+1)^2$ divides $|L|\,|B|$. However, $|L| = q^2(q-1)(q^2+1)$ and $|B| = q^{12}(q-1)^2$. Hence, $(q^2+1)^2$ does not divide $|L|\,|B|$, and we have a contradiction. □

Theorem 3.2 *Let G be a subgroup of $GL_n(q)$ with $(3q, |G|) = 1$. If G contains a Sylow 2-subgroup of $SL_n(q)$, then G has an Abelian normal 2-complement.*

Proof. Assume that we have a counterexample with $|G|$ minimal. Let T be a Sylow 2-subgroup of G. If G were solvable, then it follows from [**11**, Theorem 4.6] that G would have a normal Abelian Sylow s-subgroup for every prime $s > 3$. Since 3 does not divide $|G|$, G would have an Abelian normal 2-complement. Hence, G is not solvable.

Now let L be the largest normal solvable subgroup of G, and let K/L be a minimal normal subgroup of G/L. Then K/L is the direct product of isomorphic simple nonabelian groups $\{K_1, \ldots, K_m\}$. Since

3 cannot divide $|K_1|$, K_1 must be a Suzuki group. If T_i is a Sylow 2-subgroup of K_i, then $N_{K_i}(T_i)$ is solvable but not $2'$-closed [**17**]. Hence, since $(T \cap K)L/L$ is a Sylow 2-subgroup of K/L, $N_K((T \cap K)L)$ is solvable but not $2'$-closed. Since T normalizes $N_K((T \cap K)L)$, $TN_K((T \cap K)L)$ is a solvable group containing T which is not $2'$-closed. Now $TN_K((T \cap K)L) \neq G$ since G is not solvable. On the other hand, $TN_K((T \cap K)L)$ satisfies our assumptions for G. Then it follows from the minimality of our counterexample that $TN_K((T \cap K)L)$ has a normal 2-complement. This contradicts our earlier statement, and so the proof is complete. □

Theorem 3.3 *Let $S = A_n(q)$. Assume S satisfies E_π with $3 \notin \pi$. Then S satisfies C_π and a Hall π-subgroup of S has a Sylow tower. Further, if $S \leq G \leq \mathrm{Aut}\,(S)$, then G satisfies C_π and a Hall π-subgroup of G is solvable.*

Proof. If $2 \notin \pi$, this follows from [**9**]. If π contains the characteristic of $GF(q)$, the result follows from Theorem 3.1. Hence, we assume that π contains 2 but not the characteristic of the underlying field. The previous theorem now implies that a Hall π-subgroup of S has an Abelian normal 2-complement. Then every Hall π-subgroup of S has a Sylow tower of the same complexion, and so S satisfies C_π. Since G/S is solvable, G/S satisfies C_π. It now follows that G satisfies C_π and every Hall π-subgroup of G is solvable. □

Theorem 3.4 *Let $G = GL_n(q)$ with $n > 1$ and $(2, q) = 1$. Let $\pi \subseteq \pi(G)$ with $2 \in \pi$ but $3 \notin \pi$. Let $\tau = \pi - \{2\}$. Then G satisfies E_π if and only if all of the following are true*:

(i) *π does not contain the characteristic of $GF(q)$.*

(ii) *$e(q, t) = e(q, 2)$ for all $t \in \tau$.*

(iii) *$t > n$ for all $t \in \tau$.*

Proof. If G satisfies E_π, then it follows from Theorem 3.1 that π does not contain the characteristic of $GF(q)$. Then, by Theorem 3.2, a Hall π-subgroup of G is solvable. Thus, G would satisfy $E_{2,t}$ for all $t \in \tau$. It now follows from Theorems 2.2.2 and 2.2.4 of [**13**] that (ii) and (iii) hold.

Conversely, if (i), (ii) and (iii) are all true, let $e = e(q, 2)$, $m = [n/e]$ and $d = n - me$. Then $GL_e(q)$ contains a subgroup isomorphic to the multiplicative group of $GF(q^e)$ extended by the field automorphisms of $GF(q^e)$ over $GF(q)$. This subgroup contains a Hall π-subgroup T of $GL_e(q)$. Then if T_0 is a Sylow 2-subgroup of $GL_d(q)$ (either $T_0 = 1$ or $e = 2$, $d = 1$ and $T_0 \cong Z_2$) and P is a Sylow 2-subgroup of S_m, then G contains the direct product of T_0 and $(T \operatorname{wr}(P, m))$. This subgroup has the right order and so it is a Hall π-subgroup of G. □

Theorem 3.5 *Let G be a subgroup of $Sp_{2n}(q)$ with $(q, |G|) = 1$. If G contains a Sylow 2-subgroup of $Sp_{2n}(q)$, then G has a normal 2-complement.*

Proof. Assume a counterexample with $(|G| + n)$ minimal. Let $F = GF(q)$, Q a Sylow 2-subgroup of $Sp_2(q)$ and P_0 a Sylow 2-subgroup of S_n. Then $Q \operatorname{wr}(P_0, n)$ is a Sylow 2-subgroup P of $Sp_{2n}(q)$ [**2**]. If V is the symplectic space on which $Sp_{2n}(q)$ operates, then V is the orthogonal direct sum

$$V = V_1 \perp \cdots \perp V_n$$

where V_i is a nonsingular subspace of V of dimension 2 for $1 \leq i \leq n$, P_0 faithfully permutes the subspaces $\{V_i, \dots, V_n\}$, and if M is the base subgroup of P, then $M = Q_1 \times \cdots \times Q_n$ where Q_i is a Sylow 2-subgroup of $Sp(V_i)$. Now $Sp_2(q) \cong SL_2(q)$, and so Q and Q_i are generalized quaternion of order $(q^2 - 1)_2$. Then either $J(Q) = Q$ (if $|Q| = 8$) or $J(Q)$ is a cyclic subgroup of index 2 in Q (if $|Q| > 8$). Since Q_i has no noncyclic Abelian subgroups, $\mu_4(Q_i) = 4$. Let z_i be the unique involution in Q_i. Then $Z(M) = \langle z_1, \dots, z_n \rangle = \Omega_1(M)$. Note that z_i is represented by -1 on V_i and by 1 on V_j if $j \neq i$. Hence, $V_i = \operatorname{kernel}(z_i + 1)$. If x is a nonidentity element of Q_i, then $z_i \in \langle x \rangle$ and so $C_V(x) = C_V(z_i) = \langle V_j | j \neq i \rangle$. We now proceed via a series of steps.

(1)

(a) *If $G > H \geq P$, then H has a normal 2-complement.*

(b) *G does not permute the subspaces $\{V_1, \dots, V_n\}$ among themselves.*

(c) $n \geq 2$.

(d) *$N_G(J(P))$ has a normal 2-complement.*

(e) $Z(P) \leq Z(G)$.

Proof. Part (a) follows from the minimality of our counterexample. Now suppose that G did permute $\{V_1, \dots, V_n\}$. The permutation group induced by G on $\{V_1, \dots, V_n\}$ would be a $3'$-subgroup of S_n and would contain P_0, a full Sylow 2-subgroup of S_n. It would follow from [**4**] that this permutation group equals P_0. Hence every $2'$-element of G would fix each V_i. If K is the subgroup consisting of all elements of G which fix each V_i, then $K \trianglelefteq G$ and G/K is a 2-group. Since G does not have a normal 2-complement, neither does K. Let $K_i = C_K(V_i)$. Then K is the subdirect product of the groups $\{K/K_i \mid 1 \le i \le n\}$. Since K is not $2'$-closed, neither is K/K_i for some i. Without loss of generality, we may assume that K/K_1 is not $2'$-closed. Now $K/K_i \le Sp(V_1) \cong SL_2(q)$. Hence, we have a $3'$-subgroup of $SL_2(q)$ which is not $2'$-closed. Since K/K_1 contains a Sylow 2-subgroup of $SL_2(q)$, we have a contradiction to Theorem 3.2. Thus, (b) is proved. This immediately implies (c) since if $n = 1$, then $V = V_1$ and so (b) would have to be true.

By Lemma 2.4, $J(P) = J(M) = J(Q_1) \times \cdots \times J(Q_n)$. For each i, either V_i is an irreducible $FJ(Q_i)$-module or it is the sum of 2 faithful nonisomorphic irreducible $FJ(Q_i)$-modules. Let $N = N_G(J(P))$. Certainly $N \ge P$ and so, by part (a), N has a normal 2-complement if $N \ne G$. Assume then that $N = G$. Then G must permute the homogeneous $FJ(P)$-modules amongst themselves, and so the homogeneous $FJ(P)$-modules are not $\{V_1, \dots, V_n\}$. It follows that $|Q| > 8$ and V_i is the direct sum of 2 nonisomorphic faithful irreducible $FJ(Q_i)$-modules U_i and W_i, and the homogeneous $FJ(P)$-modules are $\{U_i, W_i \mid 1 \le i \le n\}$. Suppose $x \in G$ and $1 \le i \le n$. Then $U_i x$ must be a U_j or a W_j for some j. In either case, $C_{J(P)}(U_i x) = L_j$ where $L_j = \langle J(Q_k) \mid 1 \le k \le n, k \ne j \rangle$. Since $C_{J(P)}(U_i) = L_i$, we see that $x^{-1} L_i x = L_j$. But $V_i = C_V(L_i)$ and $V_j = C_V(L_j)$. It follows that $V_i x = V_j$. Hence, G permutes $\{V_1, \dots, V_n\}$ after all. This contradiction proves that $N \ne G$, and so N has a normal 2-complement. □

Now if $C_G(Z(P))$ also had a normal 2-complement, then, since 3 does not divide $|G|$, G would have a normal 2-complement [**5**, Corollary 5] contrary to hypothesis. Hence, $C_G(Z(P))$ does not have a normal 2-complement. Since $C_G(Z(P)) \ge P$, it follows from (a) that

$C_G(Z(P)) = G$. Hence, $Z(P) \leq Z(G)$.

(2)
(a) *G is a $2, s$-group for some prime $s > 3$.*
(b) *$G = PS$ where S is a Sylow s-subgroup of G.*
(c) *$n = 2^t$ for some positive integer t.*
(d) *V is an absolutely irreducible FP-module.*

Proof. First suppose that G is not solvable. Let L be the maximal normal solvable subgroup of G and let K/L be a minimal normal subgroup of G/L. Then K/L is the direct product of isomorphic simple non-Abelian groups $T_1, \ldots, T_m$. Since $|G| \not\equiv 0 \pmod 3$, T_i must be a Suzuki group. Then if R_i is a Sylow 2-subgroup of T_i, $N_{T_i}(R_i)$ is solvable but is not $2'$-closed [**17**]. It follows from this that $N_K((P \cap K)L)/L$ is solvable but is not $2'$-closed. Since P normalizes $N_K((P \cap K)L)$, $PN_K((P \cap K)L)$ is a solvable subgroup of G which is not $2'$-closed. From (1a), we conclude that $G = PN_K((P \cap K)L)$, and so G is solvable after all.

Since G is not $2'$-closed, $|O_{2'22'}(G)/O_{2'2}(G)| > 1$. Let s be a prime dividing $|O_{2'22'}(G)/O_{2'2}(G)|$. Certainly, $s > 3$. Now $O_{2'22'}(G)/O_{2'2}(G)$ acts faithfully on $O_{2'2}(G)/O_{2'}(G)$ and $G = N_G(O_{2'2}(G) \cap P)O_{2'}(G)$. If H is a Hall $2, s$-subgroup of $N_G(O_{2'2}(G) \cap P)$ containing P (such an H must exist since G is solvable and P is contained in $N_G(O_{2'2}(G) \cap P)$), then H cannot be s-closed. Therefore, since H is a $2, s$-group, H does not have a normal 2-complement. Since $H \geq P$, it follows from (1a) that $H = G$. Hence, G is a $2, s$-group.

The permutation group P_0 is transitive if and only if n is a power of 2. Assume P_0 is not transitive. Then, without loss of generality, we may assume that $\{V_1, \ldots, V_k\}$ is an orbit of P_0 and $k < n$. Let $U = V_1 \oplus \cdots \oplus V_k$ and $W = V_{k+1} \oplus \cdots \oplus V_n$. Then U and W are nonsingular symplectic subspaces of V. Let $a = z_1 z_2 \cdots z_k$; then, since P_0 must permute $\{z_1, \ldots, z_k\}$, $a \in Z(P) \leq Z(G)$. But a is represented by -1 on U and by 1 on W. Hence $U = \text{kernel}\,(a+1)$ and $W = C_V(a)$. Since $a \in Z(G)$, G must fix both U and W. Then G is the subdirect product of the groups $G/C_G(U)$ and $G/C_G(W)$. Both $G/C_G(U)$ and $G/C_G(W)$ satisfy the original hypothesis. Since U and W have smaller dimension than n, it would follow that both $G/C_G(U)$ and $G/C_G(W)$ have normal 2-complements. Since this would imply that G has a normal 2-complement, we would have a contradiction. Hence, P_0 is transitive and so $n = 2^t$ for some integer $t \geq 1$ (since $n \geq 2$).

Since $\dim(V_i) = 2$ and Q_i is non-Abelian, V_i is an absolutely irreducible FQ_i-module. Furthermore, $\{V_1, \ldots, V_n\}$ are mutually inequivalent nontrivial FM-modules. Thus, any nonzero subspace of V fixed by P must contain V_i for some i. Since P_0 is transitive, such a subspace must be V itself. Extending the field F doesn't affect this argument, and so V is an absolutely irreducible FP-module. □

(3) *G is a primitive subgroup of $GL(V)$.*

Proof. Suppose this is not the case. Then $V = U_1 \oplus \cdots \oplus U_r$ with $r > 1$ and G permutes the subspaces $\{U_1, \ldots, U_r\}$. Since P is an irreducible subgroup of $GL(V)$, P must permute $\{U_1, \ldots, U_r\}$, transitively. Thus, $\dim(U_1) = \cdots = \dim(U_r) = d$ for some d and $rd = 2n = 2^{t+1}$. Hence, both r and d are powers of 2.

Suppose that $Z(M)$ does not fix each U_i. Without loss of generality, we may assume that $U_1 z_1 \neq U_1$. Now $z_1 = x_1^2$ for some $x_1 \in Q_1$. Then $\langle x_1 \rangle$ must permute U_1 in some cycle of length 4. Without loss of generality, we may assume that

$$U_1 x = U_2, \qquad U_2 x_1 = U_3, \qquad U_3 x_1 = U_4, \quad \text{and} \quad U_4 x_1 = U_1.$$

Let u be any nonzero vector in U_1. Then, since $z_1^2 = 1$,

$$u_1(z_1 + 1) \in C_V(z_1) = V_2 \perp \cdots \perp V_n = C_V(x_1).$$

Therefore, $u_1(z_1+1)x_1 = u_1(z_1+1)$. However, $u_1(z_1+1)$ is a nonzero element of $U_1 \oplus U_3$ while $u_1(z_1+1)x_1 \in U_2 \oplus U_4$. This contradiction shows that $Z(M)$ must fix each U_i.

Next suppose $d = 1$. Then $r = 2n$. Since $(s, r) = 1$ (since s is a prime > 3 and r is a power of 2), S, a Sylow s-subgroup of G, must fix some U_i. Without loss of generality, we may assume that $U_1 S = U_1$. Since $C_V(Z(M)) = \{0\}$ (since V is an irreducible FP-module), there must be some i, $1 \leq i \leq n$, such that z_i does not act trivially on U_1. Relabelling, if necessary, we may assume that z_1 does not act trivially on U_1. Since z_1 has order 2 and since $d = 1$, we see that $uz_1 = -u$ for all $u \in U_1$. Now z_1 must be represented by 1 or -1 on each U_i and $\dim(\text{kernel}(z_1+1)) = \dim(V_1) = 2$. It follows that z_1 is represented by -1 on exactly one U_i with $2 \leq i \leq n$. There is no loss of generality in assuming that z_1 is represented by -1 on U_2 and by 1 on U_k for $3 \leq k \leq r$. Then

$$U_1 \oplus U_2 = \text{kernel}(z_1 = 1) = V_1,$$

and also

$$U_3 \oplus \cdots \oplus U_r = C_V(z_1) = V_2 \oplus \cdots \oplus V_n.$$

Then U_1 is orthogonal to U_j for all $j \in \{1, 3, 4, \dots, r\}$. It follows that U_1 cannot be orthogonal to U_2. Thus $U_1 \oplus U_3 \oplus \cdots \oplus U_r$ is the subspace of all vectors in V orthogonal to U_1. Since S fixes U_1, S must fix $U_1 \oplus U_3 \oplus \cdots \oplus U_r$. Since S permutes $\{U_1, \dots, U_r\}$, we see that S must fix U_2. But then S fixes $U_1 \oplus U_2 = V_1$. Then the images of V_1 under $G = SP$ are just the images of V_1 under P, i.e., the images of V_1 under G are $\{V_1, \dots, V_n\}$. This contradicts (1b) and so $d > 1$.

Now assume M does not fix each U_i. Then Q_1 cannot fix each U_i since M is the normal closure of Q_1 in P. Pick x_1 to be an element of smallest order in Q_1 which does not fix each U_i. Without loss of generality, we may assume that $U_1 x_1 = U_2$. From $C_V(z_1) = C_V(x_1)$, we conclude that $C_{U_1}(z_1) = \{0\}$. It follows from this that $u_1 z_1 = -u_1$ for all $u_1 \in U_1$. Now $U_2 x_1 \neq U_2$, and so a similar argument shows that $u_2 z = -u_2$ for all $u_2 \in U_2$. But then

$$U_1 \oplus U_2 \subseteq \text{kernel}(z_1 + 1) = V_1.$$

Since $\dim(V_1) = 2$ and $\dim(U_1 \oplus U_2) = 2d > 2$, we have a contradiction. Hence, M fixes each U_i.

Now $V_1, \dots, V_n$ are mutually inequivalent irreducible FM-modules. Therefore, these are the only irreducible FM-submodules of V. Since M fixes U_1, U_1 must be the sum of a subset of $\{V_1, \dots, V_n\}$. Without loss of generality, we may assume that $U_1 = V_1 \oplus \cdots \oplus V_k$ with $1 \leq k < n$. If $k = 1$, then $U_1 = V_1$ and it follows that G permutes $\{V_1, \dots, V_n\}$ in contradiction to (1b). Thus, $k > 1$. Then $d = \dim(U_1) = 2k$ and $2n = dr = 2kr$. Since P transitively permutes $\{U_1, \dots, U_r\}$ and since M fixes each U_i, P_0 must transitively permute $\{U_1, \dots, U_r\}$. Since $U_1 = V_1 \oplus \cdots \oplus V_k$ and $V = U_1 \oplus \cdots \oplus U_r = V_1 \oplus \cdots \oplus V_n$, it follows that $\{V_1, \ldots, V_k\}$ must be a block of imprimitivity for P_0. Since P_0 is a Sylow 2-subgroup of S_n, we conclude that $P_0 = P_1 \operatorname{wr}(P_2, r)$, where P_1 is a Sylow 2-subgroup of S_k, P_2 is a Sylow 2-subgroup of S_r, and P_2 is the group of permutations induced by P_0 on $\{U_1, \ldots, U_r\}$.

Therefore, the group of all permutations of $\{U_1, \ldots, U_r\}$ induced by G contains a full Sylow 2-subgroup of S_r. Since $|G| \not\equiv 0 \pmod 3$, it follows from [**4**] that this permutation group is a 2-group. Hence, S fixes each U_i. Since U_i is a direct sum of some of the V_j's, each U_i is a nonsingular symplectic subspace of V.

Let $L = N_G(U_1) \cap \cdots \cap N_G(U_r)$. Then $L \triangleleft G$ and G/L is a 2-group. Since G does not have a normal 2-complement, neither does L. But L is the subdirect product of the groups.

$$\{L/C_L(U_i) \mid 1 \leq i \leq r\}.$$

Thus, for some i, $L/C_L(U_i)$ does not have a normal 2-complement. Now $Q_1 \ldots Q_k P_1$ is a Sylow 2-subgroup of $Sp(U_1)$. It follows that $L/C_L(U_i)$ is a subgroup of $Sp(U_i)$ and $L/C_L(U_i)$ contains a full Sylow 2-subgroup of $Sp(U_i)$. Since $\dim(U_i) < \dim(V)$, the minimality of our counterexample implies that $L/C_L(U_i)$ has a normal 2-complement for all i. This contradiction finishes the proof of step 3. □

(4) $$|S| = s.$$

Proof. Let T be a maximal solvable subgroup of $GL_{2n}(q)$ such that $T \geq G$. Since G is primitive, so is T. Certainly, $T \geq Z = Z(GL_{2n}(q))$. Let A be a maximal normal Abelian subgroup of T and let $C = C_T(A)$. Then there are positive integers m and d such that $md = 2n = 2^{t+1}$, A is cyclic of order $q^m - 1$, T/C is cyclic of order dividing m, and C is isomorphic to a subgroup of $GL_d(q^m)$ [**16**, Chapter V].

Now T cannot be $2'$-closed since G isn't. Since T/C is a 2-group, it follows that C is not $2'$-closed. Certainly then C is not Abelian and so $d > 1$. Since $P \leq T$, Lemma 2.4(c) yields

$$m_2(T) \geq m_2(P) = nm_2(Q) = n.$$

On the other hand, since q is odd,

$$m_2(C) \leq m_2(GL_d(q^m)) = d.$$

Since T/C is cyclic, $m_2(T) \leq d+1$. Thus,

$$d+1 \geq n = md/2.$$

If $m \geq 4$, this would imply that $d+1 \geq 2d$ contrary to $d > 1$. Since m is a power of 2, $m = 1$ or 2.

Suppose $m = 2$. Then $d = n$, $|T/C| \leq 2$, and since $q^2 \equiv 1 \pmod 4$, we can use Lemma 2.2 to obtain

$$|C|_2 \leq |GL_n(q^2)|_2 = (q^2-1)_2^n n!_2 = |Q \operatorname{wr}(P_0, n)| = |P|.$$

Also $|Z(C)| \geq |A| = q^2 - 1 > q-1$. Since P is an absolutely irreducible subgroup of $GL_{2n}(q)$,

$$|C_{GL_{2n}q}(P)| = q-1.$$

Therefore, P is not contained in C. Since $P \leq T$ and $|T/C| \leq 2$, we see that $T = PC$ and $|T/C| = 2$. Then

$$|P \cap C| = |P|/2 = |GL_n(q^2)|_2/2.$$

If D is a cyclic group of order $(q^2-1)_2$, then a Sylow 2-subgroup of $GL_n(q^2)$ is isomorphic to $D \operatorname{wr}(P_0, n)$ [**2**] and it has an Abelian subgroup of order $(q^2-1)_2^n$. Since C is isomorphic to a subgroup of $GL_n(q^2)$, it follows that $P \cap C$ has an Abelian subgroup of order $\geq (1/2)(q^2-1)_2^n$. Since an Abelian subgroup of Q has order at most $(q^2-1)_2/2$, we can use Lemma 2.4(b) to conclude that an Abelian subgroup of P has order at most

$$\mu_{|P|}(P) = \mu_{|P|}(Q)^n \leq 2^{-n}(q^2-1)_2^n.$$

It follows that $n \leq 1$ which contradicts (1c). Hence, $m = 1$. Then $d = 2n = 2^{t+1}$.

Therefore, $T = C$ and $A = Z$. By [**16**, Chapter V], T has a normal series

$$1 < Z < B < T$$

where B/Z is Abelian of order $(2n)^2$ and T/B is isomorphic to a subgroup of $Sp_{2(t+1)}(2)$. Since the intersection of Z and $Sp_{2n}(q)$ has order 2, and since B/Z is a 2-group, we see that S is isomorphic to a subgroup of $Sp_{2(t+1)}(2)$. Since $s > 3$ while $|Sp_4(2)| = 2^4 \cdot 3^2 \cdot 5$ and $|Sp_6(2)| = 2^9 \cdot 3^4 \cdot 5 \cdot 7$, we conclude that $|S| = s$ if $t \leq 2$.

Assume then that $t > 2$. Now $|T/B|_2 \leq 2^{(t+1)^2}$ while $|B|_2 = (2n)^2(q-1)_2 = 2^{2(t+1)}(q-1)_2$. Hence, $|T|_2 \leq 2^{(t^2+4t+3)}(q-1)_2$. On the other hand,

$$|T|_2 \geq |P|_2 = (q^2-1)_2^n n!_2.$$

Since $n = 2^t$, $n!_2 = 2^{n-1}$. We now obtain

$$2^{n-1}(q^2-1)_2^n \leq 2^{(t^2+4t+3)}(q-1)_2.$$

Let $(q^2-1)_2 = 2^k$. Then $k \geq 3$ and $(q-1)_2 \leq 2^{k-1}$. Therefore

$$n - 1 + nk \leq t^2 + 4t + 3 + k - 1.$$

Then

$$(t+2)^2 \geq (n-1)(k+1) + 2 \geq 4(n-1) + 2 = 4n - 2 = 2^{t+2} - 2.$$

Since $t+2 \geq 5$, it follows from Lemma 2.7 that $2^{t+2} - (t+2)^2 \geq 7$, and we have a contradiction. Thus, $|S| = s$. (The reader may have noticed that we also have shown that $t = 1$ or 2, i.e., $n = 2$ or 4. However, this doesn't appear to lead to any simplifications in the rest of the proof.) □

(5) $$Z(M) \lhd G.$$

Proof. Since G is not s-closed and $|S| = s$, we see that $O_s(G) = 1$. Also, $O_{2,s}(G) = O_2(G)S$. Then $P/O_2(G)$ is isomorphic to a subgroup of the automorphism groups of S. Hence, $P/O_2(G)$ is cyclic. In particular, $Q_i/(Q_i \cap O_2(G))$ is cyclic. This implies that $Q_i \cap O_2(G)$ must contain some element x_i of order 4. Then $\langle x_i \mid 1 \leq i \leq n \rangle$ is an Abelian group of exponent 4 and order 4^n contained in $O_2(G)$. Now let

$$K = \langle A \mid A \leq O_2(G),\ A' = 1,\ |A| = 4^n,\ x^4 = 1 \text{ for all } x \in A \rangle.$$

Then $x_i \in K$. Since $\mu_4(Q_i) = 4$, Lemma 2.4(b) yields $\mu_4(P) = 4^n$. Then another use of Lemma 2.4(b) implies that if A is an Abelian sugroup of P such that $|P| = 4^n$ and $x^4 = 1$ for all $x \in A$, then $A \leq M$. Therefore, $K \leq M$. Clearly, $K \lhd G$ and so $\Omega_1(K) \lhd G$. Now $z_i \in \langle x_i \rangle$; hence, $z_i \in \Omega_1(K)$. Since $K \leq M$ and $\Omega_1(M) = \langle z_1, \ldots, z_n \rangle$, it follows that $Z(M) = \langle z_1, \ldots, z_n \rangle = \Omega_1(K) \lhd G$. □

(6) *Contradiction.*

Proof. $Z(M)$ is elementary Abelian of order $2^n > 2$ and so $Z(M)$ is not cyclic. But, since G is primitive, any normal Abelian subgroup of G must be cyclic. □

Theorem 3.6 *Let H be a Hall π-subgroup of $Sp_{2n}(q)$ with $(q,2) = 1$. Assume that $2 \in \pi$, $3 \notin \pi$, $\pi \subseteq \pi(Sp_{2n}(q))$, and let $\tau = \pi - \{2\}$. Then π does not contain the characteristic of $GF(q)$, all τ-subgroups of $Sp_{2n}(q)$ are Abelian, $Sp_{2n}(q)$ satisfies D_τ, $Sp_{2n}(q)$ satisfies C_π, and H is solvable with an Abelian normal 2-complement. Also, $e(q,t) = e(q,2)$ and $t > n$ for all $t \in \tau$.*

Proof. By Theorem 3.1, π does not contain the characteristic of $GF(q)$ and so $(q, |H|) = 1$. Then Theorem 3.5 implies that H has a normal 2-complement K. Hence, H is solvable and K is a Hall τ-subgroup of $Sp_{2n}(q)$. Let $F = GF(q)$, $G = SP_{2n}(q)$ and let V be the symplectic space on which G acts. If $\pi = \{2\}$, the theorem is trivial; hence, we assume that τ is nonempty.

Let t be any prime in τ. Then H contains a Hal $2,t$-subgroup L of G. Since H has a normal 2-complement, it follows that L has a normal Sylow t-subgroup T. Certainly, then $N_G(J(T))$ contains L. Now let $e = e(q,t)$, $k = e/(e,2)$, $m = [n/k]$, and $d = n - km$. The Sylow t-subgroups of G are described in [**20**]. If T_0 is a Sylow t-subgroup of $Sp_{2k}(q)$ and T_1 is a Sylow t-subgroup of S_m, then T is isomorphic to $T_0 \operatorname{wr}(T_1, m)$. It now follows from Lemmas 2.6 and 2.4 that $J(T)$ is the base subgroup of $T_0 \operatorname{wr}(T_1, m)$. The obvious imbedding of $T_0 \operatorname{wr}(T_1, m)$ into $Sp_{2n}(q)$ shows that V is the orthogonal direct sum

$$V = V_1 \oplus \cdots \oplus V_m \oplus V_0$$

where V_i is a nonsingular symplectic subspace of dimension $2k$ for $1 \leq i \leq m$, V_0 is a nonsingular symplectic subspace of dimension $2d$, T

acts trivially on V_0, T permutes $\{V_1, \dots, V_m\}$ and $J(T)$ fixes each V_i. Then it is not difficult to show that $N_G(J(T))$ fixes V_0 and permutes $\{V_1, \dots, V_m\}$, Therefore, L is isomorphic to a Hall $2,t$-subgroup of $(Sp_{2k}(q)\mathrm{wr}\,(S_m, m)) \times Sp_{2d}(q)$. (Here if $d = 0$, we interpret $Sp_{2d}(q)$ to be the identity.) This implies that $Sp_{2k}(q)$ contains a t-closed Hall $2,t$-subgroup and S_m satisfies $E_{2,t}$.

Using Lemma 2.6, we see that $e(q, 2) = e$ (in order for $Sp_{2k}(q)$ to have a t-closed Hall $2,t$-subgroup). Then $e = 1$ or 2. This implies that $k = 1$ and $m = n$. Now S_n never satisfies $E_{2,t}$ if 2 and t both divide $|S_n|$ [**12**, Theorem A4]. Hence, t does not divide $|S_n|$; thus, $t > n$.

We now see that $e(q, t) = e(q, 2)$ and $t > n$ for all $t \in \tau$. It now follows that $Sp_2(q)$ contains an Abelian Hall τ-subgroup. Since G contains the direct product of n copies of $Sp_2(q)$, G contains an Abelian subgroup of order $(q^2 - 1)^n_\tau$. This is the right order and so G contains an Abelian Hall τ-subgroup. Then G satisfies D_τ [**21**]. This implies that every τ-subgroup of G is contained in a Hall τ-subgroup; hence, every τ-subgroup of G is Abelian. Then H has an Abelian normal 2-complement. It now follows that every pair of Hall π-subgroups of G have Sylow towers of the same complexion. Hence, by Lemma 2.3, G satisfies C_π. □

Theorem 3.7 *Let $S = C_n(q)$. Assume S satisfies E_π with $3 \notin \pi$. Then S satisfies C_π and a Hall π-subgroup of S has a Sylow tower. Further, if $S \leq G \leq \mathrm{Aut}\,(S)$, then G satisfies C_π and a Hall π-subgroup of G is solvable.*

Proof. If $2 \notin \pi$, this follows from [**9**], and if π contains the characteristic of $GF(q)$, we obtain the conclusion from Theorem 3.1. Finally, if π contains 2 but not the characteristic of $GF(q)$, then the result is a consequence of the previous theorem. □

Note that Theorems 3.3 and 3.7 together yield the proposition stated in the introduction.

Theorem 3.8 *Let $G = Sp_{2n}(q)$ with $(q, 2) = 1$. Let $\pi \subseteq \pi(G)$ with $2 \in \pi$ but $3 \notin \pi$. Let $\tau = \pi - \{2\}$. Then G satisfies E_π if and only if all of the following are true*:

(i) *π does not contain the characteristic of $GF(q)$.*

(ii) *$e(q,t) = e(q,2)$ for all $t \in \tau$.*

(iii) *$t > n$ for all $t \in \tau$.*

Proof. If G satisfies E_π, then the result follows from Theorem 3.6. Assume therefore that (i), (ii) and (iii) all hold. Let $e = e(q,2)$; then $e = 1$ or 2. It follows from Lemma 2.6 that $Sp_2(q)$ contains a Hall π-subgroup T. If P is a Sylow 2-subgroup of S_n, then P is a Hall π-subgroup of S_n. Then it is straightforward to verify that $T \operatorname{wr} (P,n)$ is a Hall π-subgroup of G. Hence, G satisfies E_π. □

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