ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 3, Summer 1993

BLOCH TYPE SPACES OF ANALYTIC FUNCTIONS

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1. Introduction. Let $\mathbf{D}$ be the open unit disk in the complex plane $\mathbf{C}$. The Bloch space of $\mathbf{D}$, denoted $\mathcal{B}$, consists of analytic functions f on $\mathbf{D}$ such that

$$\sup\{(1-|z|^2)|f'(z)| : z \in \mathbf{D}\} < +\infty.$$

Functions in the Bloch space have been studied extensively by many authors. See [**1**] for a recent survey of the theory of Bloch functions.

In this paper we study a class of generalized Bloch spaces. Specifically, for each $\alpha > 0$, we let $\mathcal{B}_\alpha$ denote the space of analytic functions f on $\mathbf{D}$ satisfying

$$\sup\{(1-|z|^2)^\alpha|f'(z)| : z \in \mathbf{D}\} < +\infty.$$

These spaces are not new. They are a certain type of Besov space [**12**]. When $\alpha > 1$, the space $\mathcal{B}_\alpha$ can be identified with the space of analytic functions f with

$$\sup\{(1-|z|^2)^{\alpha-1}|f(z)| : z \in \mathbf{D}\} < +\infty;$$

see Proposition 7. Such spaces are studied in [**13**] and [**16**]. When $0 < \alpha < 1$, the space $\mathcal{B}_\alpha$ can be identified with the analytic Lipschitz space $\mathrm{Lip}_{1-\alpha}$ consisting of analytic functions f on $\mathbf{D}$ such that

$$|f(z) - f(w)| \le C|z-w|^{1-\alpha}$$

for some constant $C > 0$ (depending on f) and all $z, w \in \mathbf{D}$; see [**21**] or Theorem B of [**10**]. Thus our results here unify the theory of Bloch functions, Lipschitz functions, and functions studied in [**13**] and [**16**].

Although most results in the paper are new (in the sense that this is the first time they appear in the literature), this paper is expository

Received by the editors on October 18, 1991.
Research supported by the National Science Foundation.

in nature. This is because most arguments used here are standard and well known to experts.

Our approach here is functional analytic. We will make extensive use of the technique of reproducing kernels. Many arguments involving reproducing kernels can also be handled by the Cauchy integral or the so-called fractional integrals introduced by Hardy and Littlewood [**11**].

2. Basic properties of $\mathcal{B}_\alpha$. Recall that for each $\alpha > 0$ the space $\mathcal{B}_\alpha$ consists of analytic functions f on $\mathbf{D}$ with the property that

$$||f||_\alpha = |f(0)| + \sup\{(1-|z|^2)^\alpha |f'(z)| : z \in \mathbf{D}\} < +\infty.$$

Proposition 1. *For each $\alpha > 0$ the space $\mathcal{B}_\alpha$ is a Banach space with the above norm.*

Proof. The proof is standard and elementary. We omit the details. □

We will also be interested in the generalization of the little Bloch space $\mathcal{B}_0$ consisting of functions f in $\mathcal{B}$ such that

$$\lim_{|z|\to 1^-} (1-|z|^2) f'(z) = 0.$$

Thus for each $\alpha > 0$ we let $\mathcal{B}_{\alpha,0}$ denote the subspace of $\mathcal{B}_\alpha$ consisting of functions f with

$$\lim_{|z|\to 1^-} (1-|z|^2)^\alpha f'(z) = 0.$$

We will see later that the above condition for an analytic function to be in $\mathcal{B}_{\alpha,0}$ is equivalent to the condition that $(1-|z|^2)^\alpha f'(z)$ extends continuously to the boundary of the unit disk. This is even true locally: if f is analytic and $(1-|z|^2)^\alpha f'(z)$ has a finite (uniform) limit at a boundary point, then the limit must be 0.

It is clear that each $\mathcal{B}_{\alpha,0}$ contains all functions that are analytic in a neighborhood of the closed disk $\overline{\mathbf{D}}$. In particular, $\mathcal{B}_{\alpha,0}$ contains all polynomials.

Proposition 2. *$\mathcal{B}_{\alpha,0}$ is the closure of the set of polynomials in the norm topology of $\mathcal{B}_\alpha$. In particular, $\mathcal{B}_{\alpha,0}$ is a separable Banach space by itself.*

Proof. Again the proof is standard. It breaks down into two steps. First, one shows that each function in $\mathcal{B}_{\alpha,0}$ can be approximated in norm by functions which are analytic in a neighborhood of $\overline{\mathbf{D}}$. Specifically, one shows that $||f - f_r||_\alpha \to 0$ $(r \to 1^-)$ if f is in $\mathcal{B}_{\alpha,0}$, where $f_r(z) = f(rz)$. Then one shows that each function f_r can be approximated in norm by polynomials. Specifically, one shows that each f_r is approximated in norm by its Taylor polynomials. We omit the actual arguments. $\square$

Proposition 3. *For any $\alpha > -1$ and $z \in \mathbf{D}$ we have*

$$f(z) = (\alpha+1)\int_{\mathbf{D}} \frac{(1-|w|^2)^\alpha f(w)}{(1-z\bar{w})^{2+\alpha}}\, dA(w)$$

if f is an analytic function on $\mathbf{D}$ with

$$\int_{\mathbf{D}} (1-|z|^2)^\alpha |f(z)|\, dA(z) < +\infty,$$

where dA is the normalized area measure on $\mathbf{D}$.

Proof. See 4.2.1 of [**18**].

Corollary 4. *Suppose $\alpha > 0$, $z \in \mathbf{D}$, and $f \in \mathcal{B}_\alpha$. Then*

$$f(z) = f(0) + \int_{\mathbf{D}} \frac{(1-|w|^2)^\alpha f'(w)}{\bar{w}(1-z\bar{w})^{1+\alpha}}\, dA(w).$$

Proof. By Proposition 3,

$$f'(z) = (\alpha+1)\int_{\mathbf{D}} \frac{(1-|w|^2)^\alpha f'(w)}{(1-z\bar{w})^{2+\alpha}}\, dA(w), \qquad z \in \mathbf{D}.$$

Taking the line integral from 0 to z, we get

$$f(z) - f(0) = \int_{\mathbf{D}} \frac{(1-|w|^2)^\alpha f'(w)}{\bar{w}} \left[\frac{1}{(1-z\bar{w})^{1+\alpha}} - 1\right] dA(w).$$

It is easy to see (using Taylor expansion, for example) that

$$\int_{\mathbf{D}} \frac{(1-|w|^2)^\alpha f'(w)}{\bar{w}}\, dA(w) = 0.$$

Thus the desired result follows. $\square$

Proposition 5. *For each nonnegative integer n and each compact set K contained in $\mathbf{D}$, there exists a constant $C > 0$ (depending on α only) such that*

$$\sup\{|f^{(n)}(z)| : z \in K\} \leq C||f||_\alpha$$

for all f in $\mathcal{B}_\alpha$.

Proof. The desired result follows easily from Corollary 4 and the fact that derivatives can be taken inside the integral sign. Note that $|w|^{-1}$ is dA-integrable, so that there is no problem with convergence in the case $n = 0$. $\square$

Proposition 6. *For $s > -1$ and t real, let*

$$I_{s,t}(z) = \int_{\mathbf{D}} \frac{(1-|w|^2)^s}{|1-z\bar{w}|^{2+s+t}}\, dA(w), \qquad z \in \mathbf{D}.$$

We have

1) $I_{s,t}(z)$ *is bounded in z if $t < 0$;*
2) $I_{s,t}(z) \sim -\log(1-|z|^2)$ *as $|z| \to 1^-$ if $t = 0$;*
3) $I_{s,t}(z) \sim (1-|z|^2)^{-t}$ *as $|z| \to 1^-$ if $t > 0$.*

Proof. See 4.2.2 of [**18**]. $\square$

Proposition 7. *Suppose* $\alpha > 1$. *Then* f *is in* $\mathcal{B}_\alpha$ *if and only if* $(1-|z|^2)^{\alpha-1}f(z)$ *is bounded on* $\mathbf{D}$; f *is in* $\mathcal{B}_{\alpha,0}$ *if and only if* $(1-|z|^2)^{\alpha-1}f(z)\to 0$ *as* $|z|\to 1^-$.

Proof. First assume that f is in $\mathcal{B}_\alpha$. By Corollary 4,

$$f(z)=f(0)+\int_{\mathbf{D}}\frac{(1-|w|^2)^\alpha f'(w)}{\bar w(1-z\bar w)^{1+\alpha}}\,dA(w),\qquad z\in\mathbf{D}.$$

It follows that

$$|f(z)-f(0)|\le ||f||_\alpha\int_{\mathbf{D}}\frac{dA(w)}{|w|\,|1-z\bar w|^{1+\alpha}},\qquad z\in\mathbf{D}.$$

The factor $|w|$ in the denominator does not change the growth rate of the integral for z near the boundary. Thus, Proposition 6 implies that there is a constant $C>0$ such that

$$|f(z)-f(0)|\le C||f||_\alpha(1-|z|^2)^{-(\alpha-1)},\qquad z\in\mathbf{D}.$$

This shows that $(1-|z|^2)^{\alpha-1}f(z)$ is bounded on $\mathbf{D}$.

Conversely, if $(1-|z|^2)^{\alpha-1}|f(z)|\le M$ for some constant $M>0$, then

$$f(z)=\alpha\int_{\mathbf{D}}\frac{(1-|w|^2)^{\alpha-1}f(w)}{(1-z\bar w)^{\alpha+1}}\,dA(w),\qquad z\in\mathbf{D},$$

by Proposition 3. Differentiating under the integral sign, we obtain

$$f'(z)=\alpha(\alpha+1)\int_{\mathbf{D}}\frac{\bar w(1-|w|^2)^{\alpha-1}f(w)}{(1-z\bar w)^{\alpha+2}}\,dA(w),\qquad z\in\mathbf{D}.$$

By Proposition 6 there exists a constant $C>0$ such that

$$|f'(z)|\le\alpha(\alpha+1)M\int_{\mathbf{D}}\frac{dA(w)}{|1-z\bar w|^{\alpha+2}}\le CM(1-|z|^2)^{-\alpha}$$

for all $z\in\mathbf{D}$. This clearly shows that f is in $\mathcal{B}_\alpha$.

Carefully examining the above proof, we have actually shown that, for $\alpha>1$, the norm $||\quad||_\alpha$ on $\mathcal{B}_\alpha$ is equivalent to the norm

$$||f||=\sup\{(1-|z|^2)^{\alpha-1}|f(z)|:z\in\mathbf{D}\}.$$

Since $\mathcal{B}_{\alpha,0}$ is the closure of the polynomials in $\mathcal{B}_\alpha$ under the norm $\|\ \ \|_\alpha$, and the closure of the polynomials in $\mathcal{B}_\alpha$ under the above norm $\|\ \ \|$ consists of analytic functions f with $(1-|z|^2)^{\alpha-1}f(z)\to 0$ as $|z|\to 1^-$, we conclude that $f\in\mathcal{B}_{\alpha,0}$ if and only if $(1-|z|^2)^{\alpha-1}f(z)\to 0$ as $|z|\to 1^-$. $\square$

Proposition 8. *Suppose $\alpha>0$, and suppose that $n\geq 2$ is an integer. An analytic function f on $\mathbf{D}$ belongs to $\mathcal{B}_\alpha$ if and only if*

$$\sup\{(1-|z|^2)^{\alpha+n-1}|f^{(n)}(z)| : z\in\mathbf{D}\}<+\infty.$$

Similarly, f belongs to $\mathcal{B}_{\alpha,0}$ if and only if $(1-|z|^2)^{\alpha+n-1}f^{(n)}(z)\to 0$ as $|z|\to 1^-$.

Proof. If f is in $\mathcal{B}_\alpha$, then

$$f(z)=f(0)+\int_{\mathbf{D}}\frac{(1-|w|^2)^\alpha f'(w)}{\bar{w}(1-z\bar{w})^{1+\alpha}}\,dA(w),\qquad z\in\mathbf{D},$$

by Corollary 4. Differentiating under the integral sign n times and applying Proposition 6 we easily conclude that

$$\sup\{(1-|z|^2)^{\alpha+n-1}|f^{(n)}(z)| : z\in\mathbf{D}\}<+\infty.$$

To prove the inverse, we may as well assume that the first $n+1$ Taylor coefficients of f all vanish (since subtracting a polynomial from f neither alters the assumption nor does the conclusion). In this case, the function

$$\varphi(z)=\frac{(\alpha+2)(1-|z|^2)^{\alpha+n-1}f^{(n)}(z)}{(\alpha+n)\cdots(\alpha+2)\bar{z}^n}$$

is bounded on $\mathbf{D}$. By Proposition 3,

$$f^{(n)}(z)=(\alpha+n)\int_{\mathbf{D}}\frac{(1-|w|^2)^{\alpha+n-1}f^{(n)}(w)}{(1-z\bar{w})^{\alpha+n+1}}\,dA(w),\qquad z\in\mathbf{D}.$$

Integrating from 0 to z, we get

$$\begin{aligned}f^{(n-1)}(z)&-f^{(n-1)}(0)\\&=\int_{\mathbf{D}}\frac{(1-|w|^2)^{\alpha+n-1}f^{(n)}(w)}{\bar{w}}\left[\frac{1}{(1-z\bar{w})^{\alpha+n}}-1\right]dA(w).\end{aligned}$$

Since $f^{(n-1)}(0) = 0$ and

$$\int_{\mathbf{D}} \frac{(1-|w|^2)^{\alpha+n-1} f^{(n)}(w)}{\bar{w}}\, dA(w) = 0,$$

we have

$$f^{(n-1)}(z) = \int_{\mathbf{D}} \frac{(1-|w|^2)^{\alpha+n-1} f^{(n)}(w)}{\bar{w}(1-z\bar{w})^{\alpha+n}}\, dA(w), \qquad z \in \mathbf{D}.$$

Repeating the above argument $n-1$ times we will get

$$f'(z) = \int_{\mathbf{D}} \frac{\varphi(w)}{(1-z\bar{w})^{\alpha+2}}\, dA(w), \qquad z \in \mathbf{D}.$$

By Proposition 6 there is a constant $C > 0$ such that

$$\begin{aligned} |f'(z)| &\leq ||\varphi||_\infty \int_{\mathbf{D}} \frac{dA(w)}{|1-z\bar{w}|^{\alpha+2}} \\ &\leq C||\varphi||_\infty (1-|z|^2)^{-\alpha} \end{aligned}$$

for all $z \in \mathbf{D}$, and hence f is in $\mathcal{B}_\alpha$.

Note that the above proof actually shows that the norm $|| \quad ||_\alpha$ on $\mathcal{B}_\alpha$ is equivalent to the following norm

$$\begin{aligned} ||f|| = |f(0)| + |f'(0)| + \cdots + |f^{(n-1)}(0)| \\ + \sup\{(1-|z|^2)^{\alpha+n-1}|f^{(n)}(z)| : z \in \mathbf{D}\}. \end{aligned}$$

It is easy to see that the closure of the polynomials in $\mathcal{B}_\alpha$ under the above norm consists of analytic functions f with $(1-|z|^2)^{\alpha+n-1} f^{(n)}(z) \to 0$ as $|z| \to 1^-$. This implies that an analytic function f belongs to $\mathcal{B}_{\alpha,0}$ if and only if $(1-|z|^2)^{\alpha+n-1} f^{(n)}(z) \to 0$ as $|z| \to 1^-$, completing the proof of Proposition 8. $\square$

Proposition 9. *Suppose that $0 < \alpha < 1$. An analytic function f on $\mathbf{D}$ belongs to $\mathcal{B}_\alpha$ if and only if*

$$\sup\left\{\frac{|f(z)-f(w)|}{|z-w|^{1-\alpha}} : z \neq w\right\} < +\infty.$$

In particular, if $0 < \alpha < 1$*, the space* $\mathcal{B}_\alpha$ *is contained in the disk algebra.*

Proof. See Theorem B in [**10**]. □

As a consequence of Propositions 7 and 9 we see that if f is in $\mathcal{B}_\alpha$ then

$$\int_{\mathbf{D}} |f(z)|(1-|z|^2)^{\alpha-1}\, dA(z) < +\infty.$$

In fact, if $\alpha > 1$, Proposition 7 shows that the integrand is a bounded function; if $0 < \alpha < 1$, Proposition 9 shows that f is bounded on $\mathbf{D}$ and hence the integral converges since $(1-|z|^2)^t$ is area integrable for all $t > -1$; if $\alpha = 1$, the desired integrability follows from the well-known fact that each function in $\mathcal{B}$ grows at most as fast as $-\log(1-|z|^2)$.

3. Duality. Let L_a^1 denote the Bergman space of analytic functions f on $\mathbf{D}$ such that

$$||f|| = \int_{\mathbf{D}} |f(z)|\, dA(z) < +\infty.$$

The space L_a^1 is a Banach space with the above norm. We prove in this section that if $f \in L_a^1$ and $g \in \mathcal{B}_\alpha$ then

$$\lim_{r\to 1^-} \int_{\mathbf{D}} f(rz)\overline{g(rz)}(1-|z|^2)^{\alpha-1}\, dA(z)$$

exists. (If $\alpha > 1$ this is clear in virtue of Proposition 7.) Moreover, we will show that the above pairing induces the following dualities:

$$\mathcal{B}_{\alpha,0}^* = L_a^1, \qquad L_a^{1*} = \mathcal{B}_\alpha.$$

The case $\alpha > 1$ follows from Proposition 7 and a result in [**16**]. The case $\alpha = 1$ is proved in [**2**]. Thus the main interest here is the case $0 < \alpha < 1$; even this case can be deduced from the case $1 < \alpha < 2$ by considering f' instead of f. Nevertheless, this section is justified by at least two considerations: it gives a unified approach to some known duality results and it develops several results along the way which are of independent interest.

To prove the above dualities we need to introduce an auxiliary operator. For any $\alpha > 0$ let Q_α denote the operator defined by

$$Q_\alpha f(z) = \alpha \int_{\mathbf{D}} \frac{f(w)}{(1 - z\bar{w})^{1+\alpha}} \, dA(w), \qquad z \in \mathbf{D}.$$

Note that Q_α is not a projection; it does not reproduce analytic functions unless $\alpha = 1$.

We will also need to use several subspaces of $L^\infty(\mathbf{D})$. $\mathbf{C}_0(\mathbf{D})$ is the space of complex-valued continuous functions on $\mathbf{D}$ which vanish on the boundary. $\mathbf{C}(\overline{\mathbf{D}})$ is the space of complex-valued continuous functions on the closed unit disk $\overline{\mathbf{D}}$. $B\mathbf{C}(\mathbf{D})$ will be the space of bounded complex-valued continuous functions on $\mathbf{D}$.

Proposition 10. *For each $\alpha > 0$ the operator Q_α maps $L^\infty(\mathbf{D})$ boundedly onto $\mathcal{B}_\alpha$. Q_α also maps $B\mathbf{C}(\mathbf{D})$ onto $\mathcal{B}_\alpha$.*

Proof. Let $g \in L^\infty(\mathbf{D})$, and let $f = Q_\alpha g$. Thus

$$f(z) = \alpha \int_{\mathbf{D}} \frac{g(w)}{(1 - z\bar{w})^{1+\alpha}} \, dA(w)$$

and

$$f'(z) = \alpha(\alpha + 1) \int_{\mathbf{D}} \frac{\bar{w} g(w)}{(1 - z\bar{w})^{2+\alpha}} \, dA(w), \qquad z \in \mathbf{D}.$$

By Proposition 6 there exists a constant $C > 0$ such that

$$|f'(z)| \le \alpha(\alpha + 1)||g||_\infty \int_{\mathbf{D}} \frac{dA(w)}{|1 - z\bar{w}|^{2+\alpha}} \le C||g||_\infty (1 - |z|^2)^{-\alpha}$$

for all $z \in \mathbf{D}$. It is also clear that $|f(0)| \le \alpha||g||_\infty$. Thus $||f||_\alpha \le (C + \alpha)||g||_\infty$ and hence Q_α maps $L^\infty(\mathbf{D})$ boundedly into $\mathcal{B}_\alpha$.

Next we show that Q_α maps $B\mathbf{C}(\mathbf{D})$ onto $\mathcal{B}_\alpha$. First observe that if p is a polynomial then there exists $g \in \mathbf{C}_0(\mathbf{D})$ such that $p = Q_\alpha g$. In fact, for any nonnegative integer n,

$$\alpha \int_{\mathbf{D}} \frac{(1 - |w|^2)^2 w^n}{(1 - z\bar{w})^{1+\alpha}} \, dA(w) = \frac{2\alpha(\alpha + 1) \cdots (\alpha + n)}{(n + 3)!} z^n, \qquad z \in \mathbf{D}.$$

Given f in $\mathcal{B}_\alpha$, we can write

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2}z^2 + f_1(z)$$

with f_1 still in $\mathcal{B}_\alpha$. By the above observation, we can find a function g in $\mathbf{C}_0(\mathbf{D})$ with

$$f(0) + f'(0)z + \frac{f''(0)}{2}z^2 = Q_\alpha g(z).$$

By Corollary 4, we also have $f_1 = Q_\alpha g_1$ with

$$g_1(z) = \frac{(1-|z|^2)^\alpha f_1'(z)}{\alpha \bar{z}}$$

belonging to $B\mathbf{C}(\mathbf{D})$. Thus $f = Q_\alpha(g+g_1)$ and hence Q_α maps $B\mathbf{C}(\mathbf{D})$ onto $\mathcal{B}_\alpha$. □

Proposition 11. *For each $\alpha > 0$ the operator Q_α maps $\mathbf{C}(\overline{\mathbf{D}})$ boundedly onto $\mathcal{B}_{\alpha,0}$. Q_α also maps $\mathbf{C}_0(\mathbf{D})$ onto $\mathcal{B}_{\alpha,0}$.*

Proof. The "onto" parts follow from the proof of Proposition 10. It remains to show that Q_α maps $\mathbf{C}(\overline{\mathbf{D}})$ into $\mathcal{B}_{\alpha,0}$. By the Stone-Weierstrass approximation theorem, each function in $\mathbf{C}(\overline{\mathbf{D}})$ can be uniformly approximated by finite linear combinations of functions of the form $z^n\bar{z}^m$ $(n, m \geq 0)$. Since Q_α maps $L^\infty(\mathbf{D})$ boundedly into $\mathcal{B}_\alpha$ and $\mathcal{B}_{\alpha,0}$ is closed in $\mathcal{B}_\alpha$, it suffices to show that Q_α maps each function of the form $z^n\bar{z}^m$ $(n, m \geq 0)$ into $\mathcal{B}_{\alpha,0}$. But this is clear since an easy calculation using polar coordinates shows that Q_α maps each function of the form $z^n\bar{z}^m$ to a monomial. □

Corollary 12. *For each $\alpha > 0$ there exists a constant $C > 0$ such that*

$$C^{-1}||f||_\alpha \leq inf\{||g||_\infty : f = Q_\alpha g, g \in L^\infty(\mathbf{D})\} \leq C||f||_\alpha$$

for all f in $\mathcal{B}_\alpha$ and

$$C^{-1}||f||_\alpha \leq inf\{||g||_\infty : f = Q_\alpha g, g \in \mathbf{C}_0(\mathbf{D})\} \leq C||f||_\alpha$$

for all f in $\mathcal{B}_{\alpha,0}$.

Proof. This follows from consideration of the quotient norm and the open mapping theorem. $\square$

We need to introduce another operator before we prove the duality results. For each $\alpha > 0$ we let T_α denote the operator defined by

$$T_\alpha f(z) = 3(1-|z|^2)^2 \int_{\mathbf{D}} \frac{f(w)}{(1-z\bar{w})^4}(1-|w|^2)^{\alpha-1}\, dA(w).$$

Proposition 13. *For each $\alpha > 0$ the operator T_α maps $\mathcal{B}_\alpha$ boundedly into $L^\infty(\mathbf{D})$; T_α maps $\mathcal{B}_{\alpha,0}$ boundedly into $\mathbf{C}_0(\mathbf{D})$. Moreover, there exists a constant $C > 0$ (depending on α only) such that*

$$C^{-1}||f||_\alpha \le ||T_\alpha f||_\infty \le C||f||_\alpha$$

for all f in $\mathcal{B}_\alpha$. In particular, for an analytic function f on $\mathbf{D}$ we have $f \in \mathcal{B}_\alpha$ if and only if $T_\alpha f \in L^\infty(\mathbf{D})$ and $f \in \mathcal{B}_{\alpha,0}$ if and only if $T_\alpha f \in \mathbf{C}_0(\mathbf{D})$.

Proof. Given f in $\mathcal{B}_\alpha$, there exists g in $L^\infty(\mathbf{D})$ such that $f = Q_\alpha g$. By Fubini's theorem and Proposition 3,

$$\begin{aligned} T_\alpha f(z) &= 3\alpha \int_{\mathbf{D}} \frac{(1-|z|^2)^2}{(1-z\bar{w})^4}(1-|w|^2)^{\alpha-1}\, dA(w) \int_{\mathbf{D}} \frac{g(u)\, dA(u)}{(1-w\bar{u})^{1+\alpha}} \\ &= 3(1-|z|^2)^2 \int_{\mathbf{D}} g(u)\, dA(u) \alpha \int_{\mathbf{D}} \frac{(1-|w|^2)^{\alpha-1}\, dA(w)}{(1-z\bar{w})^4(1-w\bar{u})^{1+\alpha}} \\ &= 3(1-|z|^2)^2 \int_{\mathbf{D}} \frac{g(u)\, dA(u)}{(1-z\bar{u})^4}. \end{aligned}$$

Thus,

$$|T_\alpha f(z)| \le 3||g||_\infty (1-|z|^2)^2 \int_{\mathbf{D}} \frac{dA(u)}{|1-z\bar{u}|^4} = 3||g||_\infty$$

for all z in $\mathbf{D}$ and hence $||T_\alpha f||_\infty \le 3||g||_\infty$ for all g in $L^\infty(\mathbf{D})$ with $Q_\alpha g = f$. It follows from Corollary 12 that there is a constant $C > 0$ such that $||T_\alpha f||_\infty \le 3C||f||_\alpha$ for all f in $\mathcal{B}_\alpha$. Therefore, T_α maps $\mathcal{B}_\alpha$ boundedly into $L^\infty(\mathbf{D})$.

On the other hand, Proposition 3 and Fubini's theorem easily imply that $f = Q_\alpha T_\alpha f$ for all f in $\mathcal{B}_\alpha$. In fact,

$$\begin{aligned} Q_\alpha T_\alpha f(z) &= 3\alpha \int_{\mathbf{D}} \frac{dA(w)}{(1-z\bar{w})^{1+\alpha}} \int_{\mathbf{D}} \frac{(1-|w|^2)^2}{(1-w\bar{u})^4} (1-|u|^2)^{\alpha-1} f(u)\, dA(u) \\ &= \alpha \int_{\mathbf{D}} f(u)(1-|u|^2)^{\alpha-1}\, dA(u) 3 \int_{\mathbf{D}} \frac{(1-|w|^2)^2\, dA(w)}{(1-w\bar{u})^4(1-z\bar{w})^{1+\alpha}} \\ &= \alpha \int_{\mathbf{D}} \frac{(1-|u|^2)^{\alpha-1}}{(1-z\bar{u})^{1+\alpha}} f(u)\, dA(u) \\ &= f(z). \end{aligned}$$

Thus $T_\alpha f \in L^\infty(\mathbf{D})$ implies that $f \in \mathcal{B}_\alpha$ by Proposition 10, and there is a constant $C > 0$ such that

$$||f||_\alpha = ||Q_\alpha T_\alpha f||_\alpha \le C||T_\alpha f||_\infty$$

for all f in $\mathcal{B}_\alpha$. Also Proposition 11 shows that $T_\alpha f \in \mathbf{C}_0(\mathbf{D})$ implies that $f \in \mathcal{B}_{\alpha,0}$.

It remains to show that T_α maps $\mathcal{B}_{\alpha,0}$ into $\mathbf{C}_0(\mathbf{D})$. By the symmetry of the disk, the operator T_α maps each polynomial to a polynomial times the function $(1-|z|^2)^2$. In particular, T_α maps each polynomial to a function in $\mathbf{C}_0(\mathbf{D})$. Since $T_\alpha : \mathcal{B}_\alpha \to L^\infty(\mathbf{D})$ is bounded, $\mathcal{B}_{\alpha,0}$ is the closure of the set of polynomials in $\mathcal{B}_\alpha$, and $\mathbf{C}_0(\mathbf{D})$ is closed in $L^\infty(\mathbf{D})$, we conclude that T_α maps $\mathcal{B}_{\alpha,0}$ into $\mathbf{C}_0(\mathbf{D})$. □

We can now prove the main result of this section.

Theorem 14. *For each $\alpha > 0$ the Banach dual of L_a^1 can be identified with $\mathcal{B}_\alpha$ (with equivalent norms) under the pairing*

$$\langle f, g\rangle = \lim_{r\to 1^-} \int_{\mathbf{D}} f(rz)\overline{g(rz)}(1-|z|^2)^{\alpha-1}\, dA(z), \qquad f \in L_a^1,\ g \in \mathcal{B}_\alpha.$$

Proof. Recall that $g \in \mathcal{B}_\alpha$ implies that

$$\int_{\mathbf{D}} |g(z)|(1-|z|^2)^{\alpha-1}\, dA(z) < +\infty.$$

We break the proof of the theorem down into several simple steps.

Step 1. Suppose that $f \in L^1_a$ is bounded and $g \in \mathcal{B}_\alpha$. We show that

$$\left| \int_{\mathbf{D}} f(z)\overline{g(z)}(1-|z|^2)^{\alpha-1}\, dA(z) \right| \le C||f||_{L^1_a}||g||_\alpha$$

for some constant $C > 0$ independent of f and g. Writing $g = Q_\alpha \varphi$ for some $\varphi \in L^\infty(\mathbf{D})$ and applying Fubini's theorem, we have

$$\begin{aligned}
&\int_{\mathbf{D}} f(z)\overline{g(z)}(1-|z|^2)^{\alpha-1}\, dA(z) \\
&\qquad = \alpha \int_{\mathbf{D}} f(z)(1-|z|^2)^{\alpha-1}\, dA(z) \int_{\mathbf{D}} \frac{\overline{\varphi(w)}\, dA(w)}{(1-w\bar{z})^{1+\alpha}} \\
&\qquad = \alpha \int_{\mathbf{D}} \overline{\varphi(w)}\, dA(w) \int_{\mathbf{D}} \frac{(1-|z|^2)^{\alpha-1} f(z)}{(1-w\bar{z})^{1+\alpha}}\, dA(z).
\end{aligned}$$

Using Proposition 3, we see that

$$\int_{\mathbf{D}} f(z)\overline{g(z)}(1-|z|^2)^{\alpha-1}\, dA(z) = \int_{\mathbf{D}} f(w)\overline{\varphi(w)}\, dA(w)$$

and hence

$$\left| \int_{\mathbf{D}} f(z)\overline{g(z)}(1-|z|^2)^{\alpha-1}\, dA(z) \right| \le ||f||_{L^1_a}||\varphi||_\infty.$$

Taking the infimum over φ and applying Corollary 12 we get a constant $C > 0$ such that

$$\left| \int_{\mathbf{D}} f(z)\overline{g(z)}(1-|z|^2)^{\alpha-1}\, dA(z) \right| \le C||f||_{L^1_a}||g||_\alpha.$$

Step 2. We show that if f is in L^1_a and g is in $\mathcal{B}_\alpha$ then

$$\lim_{r\to 1^-} \int_{\mathbf{D}} f(rz)\overline{g(z)}(1-|z|^2)^{\alpha-1}\, dA(z)$$

exists and the absolute value of the above limit is less than or equal to $C||f||_{L^1_a}||g||_\alpha$, where C is a constant independent of f and g.

Given g in $\mathcal{B}_\alpha$, Step 1 shows that

$$F_g(f) = \int_{\mathbf{D}} f(z)\overline{g(z)}(1-|z|^2)^{\alpha-1}\,dA(z), \qquad f \in H^\infty(\mathbf{D})$$

extends to a bounded linear functional on L_a^1 with $||F_g|| \le C||g||_\alpha$. So we may assume that F_g is defined on the whole space L_a^1 (but not via the above formula). Fix f in L_a^1 and g in $\mathcal{B}_\alpha$. Let $f_r(z) = f(rz)$, $0 < r < 1$. Each f_r is in $H^\infty(\mathbf{D})$ and $||f_r - f||_{L_a^1} \to 0$ as $r \to 1^-$. It follows that

$$\lim_{r\to 1^-} \int_{\mathbf{D}} f(rz)\overline{g(z)}(1-|z|^2)^{\alpha-1}\,dA(z) = \lim_{r\to 1^-} F_g(f_r) = F_g(f)$$

exists with $|F_g(f)| \le ||F_g||\,||f||_{L_a^1} \le C||g||_\alpha ||f||_{L_a^1}$.

Step 3. We show that if F is a bounded linear functional on L_a^1 then there exists a function g in $\mathcal{B}_\alpha$ such that

$$F(f) = \lim_{r\to 1^-} \int_{\mathbf{D}} f(rz)\overline{g(z)}(1-|z|^2)^{\alpha-1}\,dA(z), \qquad f \in L_a^1.$$

By the Hahn-Banach extension theorem, F extends to a bounded linear functional on $L^1(\mathbf{D}, dA)$ without increasing the norm. Since $(L^1)^* = L^\infty$, there is a function $\varphi \in L^\infty(\mathbf{D})$ such that

$$F(f) = \int_{\mathbf{D}} f(z)\overline{\varphi(z)}\,dA(z), \qquad f \in L_a^1.$$

Using an equality proved in Step 1 we see that for each f in L_a^1,

$$\begin{aligned} F(f) &= \lim_{r\to 1^-} \int_{\mathbf{D}} f_r(z)\overline{\varphi(z)}\,dA(z) \\ &= \lim_{r\to 1^-} \int_{\mathbf{D}} f(rz)\overline{Q_\alpha\varphi(z)}(1-|z|^2)^{\alpha-1}\,dA(z). \end{aligned}$$

Let $g = Q_\alpha\varphi$. Then g is in $\mathcal{B}_\alpha$ by Proposition 10 and

$$F(f) = \lim_{r\to 1^-} \int_{\mathbf{D}} f(rz)\overline{g(z)}(1-|z|^2)^{\alpha-1}\,dA(z)$$

for all f in L_a^1.

Step 4. By the rotational invariance of the measure $(1-|z|^2)^{\alpha-1}\,dA(z)$, we have

$$\int_{\mathbf{D}} f(rz)\overline{g(z)}(1-|z|^2)^{\alpha-1}\,dA(z) = \int_{\mathbf{D}} f(sz)\overline{g(sz)}(1-|z|^2)^{\alpha-1}\,dA(z),$$

where $s=\sqrt{r}$. This clearly implies that

$$\lim_{r\to 1^-}\int_{\mathbf{D}} f(rz)\overline{g(z)}(1-|z|^2)^{\alpha-1}\,dA(z) = \lim_{r\to 1^-}\int_{\mathbf{D}} f(rz)\overline{g(rz)}(1-|z|^2)^{\alpha-1}\,dA(z),$$

completing the proof of Theorem 14. $\square$

Theorem 15. *For each $\alpha>0$ the Banach dual of $\mathcal{B}_{\alpha,0}$ can be identified with L^1_a (with equivalent norms) under the pairing*

$$\langle f,g\rangle = \lim_{r\to 1^-}\int_{\mathbf{D}} f(rz)\overline{g(rz)}(1-|z|^2)^{\alpha-1}\,dA(z), \qquad f\in\mathcal{B}_{\alpha,0}, g\in L^1_a.$$

Proof. In view of Theorem 14, we only need to show that each bounded linear functional F on $\mathcal{B}_{\alpha,0}$ arises from a function g in L^1_a in the following fashion:

$$F(f) = \lim_{r\to 1^-}\int_{\mathbf{D}} f(rz)\overline{g(rz)}(1-|z|^2)^{\alpha-1}\,dA(z), \qquad f\in\mathcal{B}_{\alpha,0}.$$

By Proposition 13, the operator $T_\alpha:\mathcal{B}_{\alpha,0}\to \mathbf{C}_0(\mathbf{D})$ satisfies

$$C^{-1}||f||_\alpha \le ||T_\alpha f||_\infty \le C||f||_\alpha$$

for some constant C independent of f. Let X be the image of $\mathcal{B}_{\alpha,0}$ under the mapping T_α. Then X is a closed subspace of $\mathbf{C}_0(\mathbf{D})$ and $F\circ T_\alpha^{-1}: X\to\mathbf{C}$ is a bounded linear functional. Extending the above bounded linear functional to the whole space $\mathbf{C}_0(\mathbf{D})$, and using the well-known Riesz representation theorem, we obtain a finite Borel measure μ on $\mathbf{D}$ such that

$$F\circ T_\alpha^{-1}(\varphi) = \int_{\mathbf{D}} \varphi(z)\,d\bar{\mu}(z), \qquad \varphi\in X.$$

This clearly implies that, for each f in $\mathcal{B}_{\alpha,0}$,

$$F(f) = \int_{\mathbf{D}} T_\alpha f(z)\, d\bar{\mu}(z).$$

Applying Fubini's theorem, we get

$$\begin{aligned} F(f) &= 3\int_{\mathbf{D}} (1-|z|^2)^2\, d\bar{\mu}(z) \int_{\mathbf{D}} \frac{(1-|w|^2)^{\alpha-1} f(w)}{(1-z\bar{w})^4}\, dA(w) \\ &= 3\lim_{r\to 1^-} \int_{\mathbf{D}} (1-|z|^2)^2\, d\bar{\mu}(z) \int_{\mathbf{D}} \frac{(1-|w|^2)^{\alpha-1} f(w)}{(1-rz\bar{w})^4}\, dA(w) \\ &= \lim_{r\to 1^-} \int_{\mathbf{D}} f(z)\overline{g(rz)}(1-|z|^2)^{\alpha-1}\, dA(z) \\ &= \lim_{r\to 1^-} \int_{\mathbf{D}} f(rz)\overline{g(rz)}(1-|z|^2)^{\alpha-1}\, dA(z), \end{aligned}$$

where the analytic function g defined by

$$g(w) = 3\int_{\mathbf{D}} \frac{(1-|z|^2)^2\, d\mu(z)}{(1-w\bar{z})^4}$$

belongs to L_a^1. In fact,

$$\begin{aligned} \int_{\mathbf{D}} |g(w)|\, dA(w) &\le 3\int_{\mathbf{D}} (1-|z|^2)^2\, d|\mu|(z) \int_{\mathbf{D}} \frac{d(w)}{|1-z\bar{w}|^4} \\ &= 3\int_{\mathbf{D}} d|\mu|(z) = 3||\mu||. \end{aligned}$$

This completes the proof of Theorem 15. □

Remark 1. In terms of Taylor coefficients the pairing that induces the duality between L_a^1 and $\mathcal{B}_\alpha$ can be written as follows:

$$\lim_{r\to 1^-} \int_{\mathbf{D}} f(rz)\overline{g(rz)}(1-|z|^2)^{\alpha-1}\, dA(z) = \lim_{r\to 1^-} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha)n!}{\Gamma(\alpha+n+1)} a_n \bar{b}_n r^{2n},$$

where a_n and b_n are the Taylor coefficients of f and g, respectively. It is easy to see that

$$\frac{\Gamma(\alpha)n!}{\Gamma(\alpha+n+1)} \sim \frac{1}{n^\alpha}$$

as $n \to +\infty$. Taylor coefficients of functions in $\mathcal{B}_\alpha$ will be further studied in Section 5.

Remark 2. By Proposition 2, each function in $\mathcal{B}_{\alpha,0}$ can be approximated in norm by a sequence of polynomials. It is then natural to ask whether each function in $\mathcal{B}_{\alpha,0}$ can be approximated in norm by its Taylor polynomials. As an application of Theorem 15, we can show that the answer to the above question is negative. In fact, it is fairly easy to see [**19**] that if X is a separable Banach space of analytic functions on $\mathbf{D}$, then every function in X can be approximated in norm by its Taylor polynomials if and only if the operators S_n are uniformly bounded on X, where S_n is the operator that sends each analytic function on $\mathbf{D}$ to its nth Taylor polynomial. It is shown in [**19**] that the operators S_n are not uniformly bounded on L_a^1. Since the adjoint of S_n on $\mathcal{B}_{\alpha,0}$ is just S_n on L_a^1 under the pairing given in Theorem 15, we see that the operators S_n are not uniformly bounded on $\mathcal{B}_{\alpha,0}$ for each $\alpha > 0$. It follows that for each $\alpha > 0$ there exist functions in $\mathcal{B}_{\alpha,0}$ whose Taylor polynomials do not converge in norm.

4. Lipschitz type theorems. It is well known [**18**] that an analytic function f on $\mathbf{D}$ belongs to the Bloch space $\mathcal{B}$ if and only if

$$|f(z) - f(w)| \le C\beta(z,w)$$

for some constant $C > 0$ and all z, w in $\mathbf{D}$, where β is the Bergman distance on $\mathbf{D}$. The purpose of this section is to extend the above result to all the spaces $\mathcal{B}_\alpha$. Specifically, we will prove that for each $\alpha > 0$ there exists a distance d_α on $\mathbf{D}$ such that an analytic function f on $\mathbf{D}$ belongs to $\mathcal{B}_\alpha$ if and only if

$$|f(z) - f(w)| \le C\, d_\alpha(z,w)$$

for some constant $C > 0$ and all z, w in $\mathbf{D}$.

Proposition 16. *For $\alpha > 0$ and z, w in $\mathbf{D}$, let*

$$d_\alpha(z,w) = \sup\{|f(z) - f(w)| : (1-|u|^2)^\alpha |f'(u)| \le 1, u \in \mathbf{D}\}.$$

Then d_α is a distance on $\mathbf{D}$.

Proof. That d_α is well defined (i.e., $d_\alpha(z,w) \neq +\infty$) follows from Proposition 5. The triangle inequality is obvious. That $d_\alpha(z,w) = 0$ if and only if $z = w$ follows from the fact that the function $f(z) = z$ belongs to each $\mathcal{B}_\alpha$. $\square$

Note that the distance d_α can also be written as

$$d_\alpha(z,w) = \sup\{|f(z) - f(w)| : ||f||_\alpha \leq 1\}$$

since adding a constant to f does not alter the difference $f(z) - f(w)$.

Theorem 17. *For any $\alpha > 0$ and $z \in \mathbf{D}$, we have*

$$\lim_{w \to z} \frac{d_\alpha(z,w)}{|z-w|} = (1 - |z|^2)^{-\alpha}.$$

Proof. By the definition of d_α,

$$\frac{d_\alpha(z,w)}{|z-w|} \geq \frac{|f(z) - f(w)|}{|z-w|}$$

for all $||f||_\alpha \leq 1$ and $z, w \in \mathbf{D}$. Let $w \to z$. We obtain

$$\liminf_{w \to z} \frac{d_\alpha(z,w)}{|z-w|} \geq |f'(z)|$$

for all $z \in \mathbf{D}$ and $||f||_\alpha \leq 1$. For $z \in \mathbf{D} - \{0\}$ let f_z be the anti-derivative of

$$g(w) = \left(1 - \frac{\bar{z}^2}{|z|^2} w^2\right)^{-\alpha}, \qquad w \in \mathbf{D},$$

with $f_z(0) = 0$. For $z = 0$, let $f_z(w) = w$. It is elementary to check that $||f_z||_\alpha = 1$ and $f_z(z) = (1 - |z|^2)^{-\alpha}$. Thus,

$$\liminf_{w \to z} \frac{d_\alpha(z,w)}{|z-w|} \geq (1 - |z|^2)^{-\alpha}$$

for all $z \in \mathbf{D}$. It remains to show that

$$\limsup_{w \to z} \frac{d_\alpha(z,w)}{|z-w|} \leq (1 - |z|^2)^{-\alpha}$$

for all $z \in \mathbf{D}$.

Fix $z \in \mathbf{D}$ and let $r = (1-|z|)/2$. It is clear that the closed disk with center z and radius r is contained in $\mathbf{D}$. For $|w-z| < r$ we have

$$f(w) = f(z) + f'(z)(w-z) + f_2(w)(w-z)^2,$$

where

$$f_2(w) = \frac{1}{2\pi i} \int_{|\zeta - z| = r} \frac{f(\zeta)\, d\zeta}{(\zeta - z)^2(\zeta - w)}.$$

By Proposition 5, there exists a constant $C > 0$ (depending on z) such that $|f_2(w)| \le C||f||_\alpha$ for all $|w-z| < r$. It follows that

$$|f(z) - f(w)| \le |f'(z)|\,|w-z| + C||f||_\alpha |w-z|^2$$

for all $f \in \mathcal{B}_\alpha$ and $|w-z| < r$. This implies that

$$|f(z) - f(w)| \le |z-w|(1-|z|^2)^{-\alpha} + C|z-w|^2$$

for all $||f||_\alpha \le 1$ and $|w-z| < r$. Taking the supremum over all such f, we get

$$d_\alpha(z,w) \le |z-w|(1-|z|^2)^{-\alpha} + C|z-w|^2$$

for all $|w-z| < r$. Letting $w \to z$ we obtain

$$\limsup_{w \to z} \frac{d_\alpha(z,w)}{|z-w|} \le (1-|z|^2)^{-\alpha}. \qquad \square$$

Theorem 18. *Suppose $\alpha > 0$ and f is analytic on $\mathbf{D}$. Then f is in $\mathcal{B}_\alpha$ if and only if there exists a constant $C > 0$ such that*

$$|f(z) - f(w)| \le C d_\alpha(z,w), \qquad z, w \in \mathbf{D}.$$

Moreover, we have

$$\sup\{(1-|z|^2)^\alpha |f'(z)| : z \in \mathbf{D}\} = \sup\left\{\frac{|f(z) - f(w)|}{d_\alpha(z,w)} : z \neq w\right\}$$

for all $f \in \mathcal{B}_\alpha$.

Proof. It follows from the definition of d_α that

$$M = \sup\left\{\frac{|f(z)-f(w)|}{d_\alpha(z,w)} : z \neq w\right\} \leq \sup\{(1-|z|^2)^\alpha |f'(z)| : z \in \mathbf{D}\}.$$

On the other hand, for any $z \in \mathbf{D}$, we clearly have

$$M \geq \lim_{w\to z} \frac{|f(z)-f(w)|}{d_\alpha(z,w)} = \lim_{w\to z} \frac{|f(z)-f(w)|}{|z-w|}\frac{|z-w|}{d_\alpha(z,w)}.$$

Applying Theorem 17 we obtain $M \geq (1-|z|^2)^\alpha |f'(z)|$ for all $z \in \mathbf{D}$. It follows that

$$\sup\left\{\frac{|f(z)-f(w)|}{d_\alpha(z,w)} : z \neq w\right\} \geq \sup\{(1-|z|^2)^\alpha |f'(z)| : z \in \mathbf{D}\},$$

which completes the proof of Theorem 18. □

Recall that if f is in $\mathcal{B}_\alpha$ then $f(z)(1-|z|^2)^{\alpha-1}$ is in $L^1(\mathbf{D}, dA)$. Using Proposition 3, we can write

$$f(z)-f(w) = \alpha \int_{\mathbf{D}} \left[\frac{1}{(1-z\bar{u})^{1+\alpha}} - \frac{1}{(1-w\bar{u})^{1+\alpha}}\right] f(u)(1-|u|^2)^{\alpha-1}\, dA(u).$$

It follows from the duality between L^1_a and $\mathcal{B}_\alpha$ (Theorem 14) that there is a constant $C > 0$ such that

$$C^{-1}\, d_\alpha(z,w) \leq \int_{\mathbf{D}} \left|\frac{1}{(1-z\bar{u})^{1+\alpha}} - \frac{1}{(1-w\bar{u})^{1+\alpha}}\right| dA(u) \leq C d_\alpha(z,w)$$

for all z and w in $\mathbf{D}$. This gives an asymptotic formula for the distance function $d_\alpha(z,w)$. In particular, we can use Proposition 6 to obtain the growth rate of $d_\alpha(0,z)$. Specifically, $d_\alpha(0,z)$ is bounded on $\mathbf{D}$ if $0 < \alpha < 1$; $d_\alpha(0,z) \sim -\log(1-|z|^2)$ if $\alpha = 1$; and $d_\alpha(0,z) \sim (1-|z|^2)^{1-\alpha}$ if $\alpha > 1$. $d_\alpha(0,z)$ is the maximal boundary growth rate a function in $\mathcal{B}_\alpha$ can achieve.

When $\alpha = 1$, the distance d_1 is precisely the Bergman distance on $\mathbf{D}$, which is given by

$$d_1(z,w) = \frac{1}{2}\log\frac{|1-z\bar{w}|+|z-w|}{|1-z\bar{w}|-|z-w|}, \qquad z, w \in \mathbf{D}.$$

We are unable to find a precise formula for any of the other distances d_α.

When $0 < \alpha < 1$, an analytic function f on $\mathbf{D}$ belongs to $\mathcal{B}_\alpha$ if and only if there exists a constant $C > 0$ (depending on f) such that

$$|f(z) - f(w)| \leq C|z - w|^{1-\alpha}$$

for all z and w in $\mathbf{D}$. It is further true that there exists a constant $C > 0$ (independent of f) such that

$$C^{-1}||f||_\alpha \leq |f(0)| + \sup\left\{\frac{|f(z) - f(w)|}{|z - w|^{1-\alpha}} : z \neq w\right\} \leq C||f||_\alpha$$

for all f in $\mathcal{B}_\alpha$. This naturally suggests that the distance d_α is comparable to the distance d'_α defined by $d'_\alpha(z, w) = |z - w|^{1-\alpha}$. However, this is not true by Theorem 17. Thus we have an interesting example of two distances on the disk that are not mutually equivalent but induce the same Lipschitz space of analytic functions.

In the rest of this section we show that the generalized Bloch spaces $\mathcal{B}_\alpha$ can also be described using Riemannian distances. Fix $\alpha > 0$ and suppose that $\gamma(t)$, $a \leq t \leq b$, is a continuous and piecewise smooth curve in $\mathbf{D}$. Let

$$L_\alpha(\gamma) = \int_a^b \frac{|\gamma'(t)|}{(1 - |\gamma(t)|^2)^\alpha}\, dt.$$

For z and w in $\mathbf{D}$ we define

$$\lambda_\alpha(z, w) = \inf\{L_\alpha(\gamma) : \gamma(0) = z,\\ \gamma(1) = w, \gamma \text{ is continuous and piecewise smooth}\}.$$

It is clear that λ_α is a (Riemannian) distance for each $\alpha > 0$. λ_1 $(= d_1)$ is the Bergman (or hyperbolic) distance on $\mathbf{D}$.

Theorem 19. *Suppose $\alpha > 0$ and f is analytic on $\mathbf{D}$. Then f is in $\mathcal{B}_\alpha$ if and only if there exists a constant $C > 0$ such that*

$$|f(z) - f(w)| \leq C\lambda_\alpha(z, w), \qquad z, w \in \mathbf{D}.$$

Furthermore, we have

$$\sup\{(1 - |z|^2)^\alpha |f'(z)| : z \in \mathbf{D}\} = \sup\left\{\frac{|f(z) - f(w)|}{\lambda_\alpha(z, w)} : z \neq w\right\}$$

for all $f \in \mathcal{B}_\alpha$.

Proof. First assume that

$$|f(z) - f(w)| \le C\lambda_\alpha(z, w), \qquad z, w \in \mathbf{D}.$$

We may assume that C is the smallest constant satisfying the above condition. Fix $z \in \mathbf{D}$ and let $\gamma(s)$ be a geodesic (parametrized by arc-length) in the underlying Riemannian metric that starts at z. Since $\lambda_\alpha(\gamma(0), \gamma(s)) = s$, we have

$$|f(\gamma(0)) - f(\gamma(s))| \le Cs, \qquad 0 < s < \varepsilon.$$

Dividing both sides by s and then letting $s \to 0$ in the above inequality, we obtain $|f'(z)|\,|\gamma'(0)| \le C$. By the minimal length property of geodesics,

$$s = \lambda_\alpha(\gamma(0), \gamma(s)) = \int_0^s \frac{|\gamma'(t)|}{(1 - |\gamma(t)|^2)^\alpha}\, dt, \qquad 0 < s < \varepsilon.$$

Dividing by s and then letting $s \to 0$ we see that $|\gamma'(0)| = (1 - |z|^2)^\alpha$. It follows that $(1 - |z|^2)^\alpha |f'(z)| \le C$ and hence $f \in \mathcal{B}_\alpha$ with

$$\sup\{(1 - |z|^2)^\alpha |f'(z)| : z \in \mathbf{D}\} \le \sup\left\{\frac{|f(z) - f(w)|}{\lambda_\alpha(z, w)} : z \ne w\right\}.$$

On the other hand, if f is in $\mathcal{B}_\alpha$, then

$$C = \sup\{(1 - |z|^2)^\alpha |f'(z)| : z \in \mathbf{D}\} < +\infty,$$

and hence $|f'(z)| \le C(1 - |z|^2)^{-\alpha}$ for all $z \in \mathbf{D}$. If $\gamma(t)$, $0 \le t \le 1$, is a smooth curve from z to w, the fundamental theorem of calculus shows that

$$\begin{aligned} |f(z) - f(w)| &= \left| \int_0^1 \frac{d}{dt} f(\gamma(t))\, dt \right| \\ &\le \int_0^1 |f'(\gamma(t))|\, |\gamma'(t)|\, dt \\ &\le C \int_0^1 \frac{|\gamma'(t)|}{(1 - |\gamma(t)|^2)^\alpha}\, dt \\ &= C L_\alpha(\gamma). \end{aligned}$$

It is easy to see that this also holds if γ is continuous but only piecewise smooth. Taking the infimum over all piecewise smooth curves connecting z to w, we conclude that

$$|f(z)-f(w)| \le C\lambda_\alpha(z,w), \qquad z,w\in \mathbf{D}.$$

This completes the proof of Theorem 19. $\square$

Remark . It follows from the above theorem and the definition of d_α that $d_\alpha(z,w)\le \lambda_\alpha(z,w)$ for all z and w in $\mathbf{D}$. We conjecture that $d_\alpha=\lambda_\alpha$ for all $\alpha>0$. This is of course known when $\alpha=1$. We also know that d_α and λ_α are "locally" the same. This can be seen from Theorem 17 and the following equality

$$\lim_{w\to z}\frac{\lambda_\alpha(z,w)}{|z-w|}=(1-|z|^2)^{-\alpha}, \qquad z\in\mathbf{D},$$

which can easily be proved using geodesics (as used in the first paragraph of the proof of Theorem 19). More information on local behavior of distances and Lipschitz spaces induced by Riemannian distances can be found in [**20**].

5. Taylor coefficients of functions in $\mathcal{B}_\alpha$. In this section we study Taylor coefficients of functions in $\mathcal{B}_\alpha$.

Proposition 20. *If $f(z)=\sum_{n=0}^\infty a_n z^n$ belongs to $\mathcal{B}_\alpha$, then $a_n = O(n^{\alpha-1})$ as $n\to+\infty$; if f is further in $\mathcal{B}_{\alpha,0}$, then $a_n=o(n^{\alpha-1})$ as $n\to+\infty$.*

Proof. Given $f\in\mathcal{B}_\alpha$, there exists $g\in L^\infty(\mathbf{D})$ such that $f=Q_\alpha g$, that is,

$$f(z)=\alpha\int_{\mathbf{D}}\frac{g(w)\,dA(w)}{(1-z\bar w)^{1+\alpha}}, \qquad z\in\mathbf{D}.$$

Recall that

$$\frac{\alpha}{(1-z\bar w)^{1+\alpha}}=\sum_{n=0}^\infty \frac{\alpha(\alpha+1)\cdots(\alpha+n)}{n!}z^n\bar w^n.$$

This clearly implies that

$$\begin{aligned} a_n &= \frac{\alpha(\alpha+1)\cdots(\alpha+n)}{n!}\int_{\mathbf{D}} g(w)\bar{w}^n\,dA(w) \\ &= \frac{\alpha(\alpha+1)\cdots(\alpha+n)}{(n+1)!}(n+1)\int_{\mathbf{D}} g(w)\bar{w}^n\,dA(w). \end{aligned}$$

The desired results now follow from the following easily checked facts:

1) $(\alpha(\alpha+1)\cdots(\alpha+n))/(n+1)! \sim n^{\alpha-1}$ as $n \to +\infty$;
2) $(n+1)\int_{\mathbf{D}} g(w)\bar{w}^n\,dA(w)$ is bounded in n if $g \in L^\infty(\mathbf{D})$;
3) $(n+1)\int_{\mathbf{D}} g(w)\bar{w}^n\,dA(w) \to 0$ as $n \to +\infty$ if $g \in \mathbf{C}_0(\mathbf{D})$. $\square$

The next (probably well-known) result will show that the above estimates are best possible, and it will also show that $\mathcal{B}_\alpha$ contains lots of interesting functions.

Theorem 21. *Let $\{\lambda_n\}$ be a sequence of positive integers satisfying*

$$1 < \lambda \le \frac{\lambda_{n+1}}{\lambda_n} \le M < +\infty, \qquad n \ge 1.$$

Suppose $\alpha > 0$ and $f(z) = \sum_{n=1}^\infty a_n z^{\lambda_n}$. Then $f \in \mathcal{B}_\alpha$ if and only if $a_n = O(\lambda_n^{\alpha-1})$ as $n \to +\infty$; $f \in \mathcal{B}_{\alpha,0}$ if and only if $a_n = o(\lambda_n^{\alpha-1})$ as $n \to +\infty$.

Proof. The "only if" parts follow from Proposition 20.

Suppose $|a_n| \le C\lambda_n^{\alpha-1}$ for all $n \ge 1$. It is easy to see that

$$|zf'(z)| \le C\sum_{n=1}^\infty \lambda_n^\alpha |z|^{\lambda_n}, \qquad z \in \mathbf{D}.$$

Since $\lambda_{n+1}/\lambda_n \ge \lambda$, we have $\lambda_{n+1} - \lambda_n \ge (\lambda-1)\lambda_n$. Thus

$$\begin{aligned} (\lambda_n+1)^{\alpha-1}|z|^{\lambda_n+1} + (\lambda_n+2)^{\alpha-1}|z|^{\lambda_n+2} + \cdots + \lambda_{n+1}^{\alpha-1}|z|^{\lambda_{n+1}} \\ \ge \lambda_n^{\alpha-1}[|z|^{\lambda_n+1} + |z|^{\lambda_n+2} + \cdots + |z|^{\lambda_{n+1}}] \\ \ge \lambda_n^{\alpha-1}(\lambda_{n+1}-\lambda_n)|z|^{\lambda_{n+1}} \\ \ge \lambda\lambda_n^\alpha |z|^{\lambda_{n+1}} \\ \ge \frac{\lambda}{M^\alpha}\lambda_{n+1}^\alpha |z|^{\lambda_{n+1}}. \end{aligned}$$

It follows that
$$\sum_{n=2}^{\infty} \lambda_n^{\alpha} |z|^{\lambda_n} \le \frac{M^{\alpha}}{\lambda} \sum_{k=1}^{\infty} k^{\alpha-1} |z|^k.$$
Since
$$\sum_{k=1}^{\infty} k^{\alpha-1} |z|^k \le \frac{C_1}{(1-|z|^2)^{\alpha}}, \qquad z \in \mathbf{D},$$
for some constant $C_1 > 0$, we can find another constant $C_2 > 0$ such that $|zf'(z)|(1-|z|^2)^{\alpha} \le C_2$ for all z in $\mathbf{D}$. This shows that f belongs to $\mathcal{B}_{\alpha}$.

That $a_n = o(\lambda_n^{\alpha-1})$ implies $f \in \mathcal{B}_{\alpha,0}$ can be proven in a similar fashion; we omit the details. □

In the rest of this section we will obtain partial results about the following problem: Characterize those functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{B}_{\alpha}$ with the property that $a_n = o(n^{\alpha-1})$ as $n \to +\infty$. See [**1**] and [**2**] for the question and results in the special case $\alpha = 1$.

Proposition 22. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function in $\mathcal{B}_{\alpha}$. Then $a_n = o(n^{\alpha-1})$ as $n \to +\infty$ if and only if there exists a function φ in $B\mathbf{C}(\mathbf{D})$ such that $f = Q_{\alpha}\varphi$ and*
$$\lim_{n\to+\infty} \frac{1}{2\pi} \int_0^{2\pi} \varphi(re^{it}) e^{-\mathrm{int}}\, dt = 0$$
uniformly for $r \in (0,1)$.

Proof. First assume that $f = Q_{\alpha}\varphi$ with Taylor coefficients a_n. Then
$$f(z) = \alpha \int_{\mathbf{D}} \frac{\varphi(w)}{(1-z\bar{w})^{1+\alpha}}\, dA(w), \qquad z \in \mathbf{D}.$$
Developing $(1-z\bar{w})^{-(1+\alpha)}$ into its Taylor series, we get
$$\begin{aligned} a_n &= \frac{\alpha(\alpha+1)\cdots(\alpha+n)}{n!} \int_{\mathbf{D}} \varphi(w)\bar{w}^n\, dA(w) \\ &= \frac{\alpha(\alpha+1)\cdots(\alpha+n)}{(n+1)!} \frac{n+1}{\pi} \int_0^1 r^{n+1}\, dr \int_0^{2\pi} \varphi(re^{it}) e^{-\mathrm{int}}\, dt. \end{aligned}$$

Since

$$\frac{\alpha(\alpha+1)\cdots(\alpha+n)}{(n+1)!} \sim n^{\alpha-1},$$

we see that $a_n = o(n^{\alpha-1})$ as $n \to +\infty$ if

$$\lim_{n\to+\infty} \frac{1}{2\pi}\int_0^{2\pi} \varphi(re^{it})e^{-\mathrm{int}}\,dt = 0$$

uniformly for $r \in (0,1)$.

Conversely, if $a_n = o(n^{\alpha-1})$, we show that $f = Q_\alpha\varphi$ for some function φ in $B\mathbf{C}(\mathbf{D})$ with the desired property. Without loss of generality, we may assume that $f(0) = f'(0) = f''(0) = 0$. In this case, we can let

$$\varphi(z) = \frac{\alpha(1-|z|^2)^\alpha f'(z)}{\bar{z}}$$

and use Corollary 4 to obtain $f = Q_\alpha\varphi$. It remains to show that φ has the desired property.

Let

$$A_n(r) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(re^{it})e^{-\mathrm{int}}\,dt, \qquad n \ge 0.$$

When $n \ge 1$,

$$\begin{aligned} A_n(r) &= \frac{\alpha}{2\pi}\int_0^{2\pi} \frac{(1-r^2)^\alpha}{r} f'(re^{it})e^{-i(n-1)t}\,dt \\ &= \alpha(1+r)^\alpha a_n n r^{n-2}(1-r)^\alpha. \end{aligned}$$

By elementary calculus

$$nr^{n-2}(1-r)^\alpha = O(n^{1-\alpha}), \qquad n \to +\infty$$

uniformly for $r \in (0,1)$. Since $a_n = o(n^{\alpha-1})$, we conclude that $A_n(r) \to 0$ as $n \to +\infty$ uniformly for $r \in (0,1)$. $\square$

Corollary 23. *Suppose that $\varphi \in B\mathbf{C}(\mathbf{D})$ has radial boundary values almost everywhere and $f(z) = Q_\alpha\varphi(z) = \sum_{n=0}^\infty a_n z^n$. Then $a_n = o(n^{\alpha-1})$ as $n \to +\infty$.*

Proof. It is easy to show that if $\varphi \in B\mathbf{C}(\mathbf{D})$ has radial boundary values almost everywhere then

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(re^{it})e^{-\mathrm{int}}\,dt \to 0, \qquad n \to +\infty$$

uniformly for $r \in (0,1)$. $\square$

Corollary 24. *Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is in $\mathcal{B}_\alpha$ and $m \geq 1$ is an integer. If the function $(1-|z|^2)^{\alpha+m-1} f^{(m)}(z)$ has radial limit almost everywhere, then $a_n = o(n^{\alpha-1})$ as $n \to +\infty$.*

Proof. If we apply the operator Q_α to the function $(1-|z|^2)^{\alpha+m} f^{(m)}(z)$, the resulting function is "more or less" f; see Proposition 3 and Corollary 4. We leave the easy details to the interested reader. $\square$

Recall that $H^\infty(\mathbf{D})$ is a Banach algebra. Let $\mathcal{M}$ be the maximal ideal space of $H^\infty(\mathbf{D})$, and let $\mathbf{C}(\mathcal{M})$ be the space of complex-valued continuous functions on $\mathcal{M}$. It is well known that $\mathbf{C}(\mathcal{M})$ can be identified with the closed self-adjoint subalgebra of $L^\infty(\mathbf{D})$ generated by $H^\infty(\mathbf{D})$. The following question was asked in [**6**] in the case $\alpha = 1$: Is it true that $Q_\alpha \mathbf{C}(\mathcal{M}) = \mathcal{B}_\alpha$? This question was answered negatively in [**7**], where the function $f(z) = \sum_{n=0}^{\infty} z^{n!}$ was shown to be in $\mathcal{B} - P\mathbf{C}(\mathcal{M})$ (note that $\mathcal{B} = \mathcal{B}_1$ is the Bloch space and $P = Q_1$ is the Bergman projection). We will see a little later that every Lacunary series $f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$ belongs to $\mathcal{B} - P\mathbf{C}(\mathcal{M})$ provided that $\{a_n\}$ is a bounded sequence which does not approach 0. We will actually establish the corresponding result for all $\alpha > 0$.

Theorem 25. *Suppose that $\alpha > 0$ and f is in $\mathcal{B}_\alpha$. The function $(1-|z|^2)^\alpha f'(z)$ belongs to $\mathbf{C}(\mathcal{M})$ if and only if there is a function φ in $\mathbf{C}(\mathcal{M})$ such that $f = Q_\alpha \varphi$.*

Proof. The proof in the general case is similar to that in the $\alpha = 1$ case given in [**7**]. We omit the details. $\square$

Corollary 26. *Suppose $\{\lambda_n\}$ is a sequence of positive integers* *satisfying*

$$1 < \lambda \le \frac{\lambda_{n+1}}{\lambda_n} \le M < +\infty, \qquad n \ge 1.$$

If $\{a_n\}$ is a sequence with $a_n = O(\lambda_n^{\alpha-1})$ but $a_n \ne o(\lambda_n^{\alpha-1})$ as $n \to +\infty$, then the function $f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$ is in $\mathcal{B}_\alpha$, but $(1-|z|^2)^\alpha f'(z)$ is not in $\mathbf{C}(\mathcal{M})$.

Proof. If the function $(1-|z|^2)^\alpha f'(z)$ is in $\mathbf{C}(\mathcal{M})$, it must have radial limit almost everywhere. By Corollary 24, $a_n = o(\lambda_n^{\alpha-1})$, which is a contradiction. □

The above corollary shows that $Q_\alpha \mathbf{C}(\mathcal{M})$ is a proper subspace of $\mathcal{B}_\alpha$ for each $\alpha > 0$.

6. Multipliers. In this section we characterize the pointwise and coefficient multipliers of the generalized Bloch spaces $\mathcal{B}_\alpha$.

Recall that if X is a Banach space of analytic functions on $\mathbf{D}$, we say that an analytic function f is a pointwise multiplier of X if $fg \in X$ for all $g \in X$. We let $M(X)$ denote the space of all pointwise multipliers of X. See [**8**] for basic properties of pointwise multipliers. Here we will need to use two general properties of multipliers: Any pointwise multiplier of X actually induces a bounded multiplication operator on X and the space $M(X)$ is always contained in $X \cap H^\infty(\mathbf{D})$.

Theorem 27.

1) $M(\mathcal{B}_\alpha) = M(\mathcal{B}_{\alpha,0}) = H^\infty(\mathbf{D})$ *if* $\alpha > 1$;

2) $M(\mathcal{B}_\alpha) = \mathcal{B}_\alpha$, $M(\mathcal{B}_{\alpha,0}) = \mathcal{B}_{\alpha,0}$ *if* $0 < \alpha < 1$;

3) $M(\mathcal{B}_\alpha) = M(\mathcal{B}_{\alpha,0}) = \{f \in H^\infty(\mathbf{D}) : (1-|z|^2)f'(z)\log(1-|z|^2)$ *is bounded on* $\mathbf{D}\}$ *if* $\alpha = 1$.

Proof. Part 1) follows from Proposition 7 and the general fact that $M(X) \subset H^\infty(\mathbf{D})$. Part 2) follows from Proposition 9 and the general fact that $M(X) \subset X$. Part 3) is proved in [**3**] and [**17**]. □

Corollary 28. *For each $0 < \alpha < 1$, the spaces $\mathcal{B}_\alpha$ and $\mathcal{B}_{\alpha,0}$ are Banach algebras.*

Note that $\mathcal{B}_0 \cap H^\infty(\mathbf{D})$ is also a Banach algebra.

Suppose that X is a Banach space of analytic functions and that $\{c_n\}$ is a sequence of complex numbers. Let T be the operator on the space of analytic functions defined by

$$T\bigg(\sum_{n=0}^{\infty} a_n z^n\bigg) = \sum_{n=0}^{\infty} c_n a_n z^n.$$

We say that $\{c_n\}$ multiplies X if the above operator T maps X into X. In this case T is actually a bounded linear operator on X by the closed graph theorem.

Theorem 29. *Suppose that $\{c_n\}$ is a sequence of complex numbers and that T is the corresponding operator defined above. For any $\alpha > 0$ the following conditions are equivalent*:

1) *T maps $\mathcal{B}_\alpha$ boundedly into $\mathcal{B}_\alpha$;*

2) *T maps $\mathcal{B}_{\alpha,0}$ boundedly into $\mathcal{B}_{\alpha,0}$;*

3) *T maps L^1_a boundedly into L^1_a;*

4) *The series $h(z) = \sum_{n=1}^{\infty} c_n z^n$ converges to the analytic function h on $\mathbf{D}$ with the property that*

$$\sup_{r\in(0,1)} \frac{1-r}{2\pi} \int_0^{2\pi} |h'(re^{it})|\, dt < +\infty.$$

Proof. The equivalence of 3) and 4) follows from Theorem 3 in [**16**]. The equivalences of 1), 2) and 3) now follow from Theorems 14 and 15 and the following easily checked facts: The adjoint of T on L^1_a is simply T on $\mathcal{B}_\alpha$ and the adjoint of T on $\mathcal{B}_{\alpha,0}$ is simply T on L^1_a under the pairing

$$\langle f, g\rangle = \lim_{r\to 1^-} \int_{\mathbf{D}} f(rz)\overline{g(rz)}(1-|z|^2)^{\alpha-1}\, dA(z). \qquad \square$$

7. Hankel operators on the Bergman space. In this section we show how the generalized Bloch spaces $\mathcal{B}_\alpha$ can be used to study Hankel operators on the Bergman space. The results and methods here are motivated by those in the special case $\alpha = 1$; see [**5, 12**] and [**18**].

Let L_a^2 denote the Bergman space of analytic functions f on $\mathbf{D}$ with

$$||f|| = \left[\int_{\mathbf{D}} |f(z)|^2 \, dA(z) \right]^{1/2} < +\infty.$$

The space L_a^2 is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{\mathbf{D}} f(z)\overline{g(z)} \, dA(z).$$

All unspecified norms and inner products in this section will denote those in the space $L^2 = L^2(\mathbf{D}, dA)$.

Given $\alpha > 0$ and a function φ on $\mathbf{D}$ we define a linear operator $h_\varphi^{(\alpha)}$ from L_a^2 into L^2 as follows:

$$h_\varphi^{(\alpha)} f(z) = \int_{\mathbf{D}} \frac{\varphi(w) f(w)}{(1 - \bar{z}w)^2} (1 - |w|^2)^{\alpha - 1} \, dA(w), \qquad f \in L_a^2.$$

The operator $h_\varphi^{(\alpha)}$ will be called the Hankel operator on L_a^2 with symbol φ. Clearly $h_\varphi^{(\alpha)}$ is densely defined (its domain contains H^∞ for example) provided that

$$\int_{\mathbf{D}} |\varphi(w)|(1 - |w|^2)^{\alpha - 1} \, dA(w) < +\infty.$$

Throughout this section we assume that the symbol of any Hankel operator under consideration satisfies the above condition.

Let $\overline{L_a^2}$ denote the space of all conjugate analytic functions in L^2. It is clear that if $h_\varphi^{(\alpha)}$ is a bounded operator from L_a^2 into L^2 then the range of $h_\varphi^{(\alpha)}$ is contained in $\overline{L_a^2}$. Conversely, if there exists a constant $C > 0$ such that

$$|\langle h_\varphi^{(\alpha)} f, \bar{g} \rangle| \le C ||f|| \, ||g||$$

for all f and g in H^∞, then the operator $h_\varphi^{(\alpha)}$ is bounded on L_a^2.

Theorem 30. *Suppose that* $\alpha > 0$ *and* φ *is analytic on* $\mathbf{D}$. *Then* $h_\varphi^{(\alpha)}$ *is bounded on* L_a^2 *if and only if* φ *is in* $\mathcal{B}_\alpha$. *Moreover, there exists a constant* $C > 0$ *such that* $C^{-1}||\varphi||_\alpha \le ||h_{\bar\varphi}^{(\alpha)}|| \le C||\varphi||_\alpha$ *for all* $\varphi \in \mathcal{B}_\alpha$.

Proof. First assume that $\varphi \in \mathcal{B}_\alpha$. We show that $h_{\bar\varphi}^{(\alpha)}$ is bounded on L_a^2. Given f and g in H^∞ we can apply Fubini's theorem to obtain

$$\begin{aligned}\langle h_{\bar\varphi}^{(\alpha)} f, \bar g\rangle &= \int_{\mathbf{D}} g(z)\, dA(z) \int_{\mathbf{D}} \frac{\overline{\varphi(w)} f(w)}{(1-\bar z w)^2}(1-|w|^2)^{\alpha-1}\, dA(w)\\ &= \int_{\mathbf{D}} \overline{\varphi(w)} f(w)(1-|w|^2)^{\alpha-1}\, dA(w) \int_{\mathbf{D}} \frac{g(z)\, dA(z)}{(1-\bar w z)^2}\\ &= \int_{\mathbf{D}} f(w) g(w)\overline{\varphi(w)}(1-|w|^2)^{\alpha-1}\, dA(w).\end{aligned}$$

By Theorem 14, there exists a constant $C > 0$ such that

$$|\langle h_{\bar\varphi}^{(\alpha)} f, \bar g\rangle| \le C||\varphi||_\alpha ||fg||_{L^1} \le C||\varphi||_\alpha ||f||\, ||g||$$

for all f and g in H^∞. This shows that $h_{\bar\varphi}^{(\alpha)}$ is bounded on L_a^2 with $||h_{\bar\varphi}^{(\alpha)}|| \le C||\varphi||_\alpha$.

Next we assume that $h_{\bar\varphi}^{(\alpha)}$ is bounded on L_a^2 and show that φ is in $\mathcal{B}_\alpha$. Recall that for f and g in H^∞, we have

$$\langle h_{\bar\varphi}^{(\alpha)} f, \bar g\rangle = \int_{\mathbf{D}} f(w) g(w)\overline{\varphi(w)}(1-|w|^2)^{\alpha-1}\, dA(w).$$

Let $f = g = k_z$, $z \in \mathbf{D}$, in the above equality, where

$$k_z(w) = \frac{1-|z|^2}{(1-\bar z w)^2}, \qquad w \in \mathbf{D}$$

are the normalized reproducing kernels for L_a^2; we have

$$\langle h_{\bar\varphi}^{(\alpha)} k_z, \bar k_z\rangle = \int_{\mathbf{D}} k_z(w)^2 \overline{\varphi(w)}(1-|w|^2)^{\alpha-1}\, dA(w).$$

Since each k_z is a unit vector in L_a^2, the funtion h on $\mathbf{D}$ defined by

$$h(z) = 3\langle \bar k_z, h_{\bar\varphi}^{(\alpha)} k_z\rangle = 3\int_{\mathbf{D}} \varphi(w)\bar k_z(w)^2(1-|w|^2)^{\alpha-1}\, dA(w)$$

is in $L^\infty(\mathbf{D})$ with $||h||_\infty \le 3||h_{\bar\varphi}^{(\alpha)}||$. Using the operator T_α defined in Section 3 we can write $h = T_\alpha \varphi$ and deduce $||T_\alpha \varphi||_\infty \le 3||h_{\bar\varphi}^{(\alpha)}||$. By Proposition 13 we have $\varphi \in \mathcal{B}_\alpha$ with $||\varphi||_\alpha \le C||h_{\bar\varphi}^{(\alpha)}||$ for some constant $C > 0$ independent of φ. This completes the proof of Theorem 30. □

Theorem 31. *Suppose that $\alpha > 0$ and φ is analytic on $\mathbf{D}$. Then the Hankel operator $h_{\bar\varphi}^{(\alpha)}$ is compact on L_a^2 if and only if φ is in $\mathcal{B}_{\alpha,0}$.*

Proof. If φ is a polynomial, it is easy to see that $h_{\bar\varphi}^{(\alpha)}$ is a finite rank operator. In particular, $h_{\bar\varphi}^{(\alpha)}$ is compact when φ is a polynomial. According to Proposition 2, for each φ in $\mathcal{B}_{\alpha,0}$ there exists a sequence of polynomials $\{p_n\}$ such that $||\varphi - p_n||_\alpha \to 0$ as $n \to +\infty$. By Theorem 30 we then have $||h_{\bar\varphi}^{(\alpha)} - h_{\bar p_n}^{(\alpha)}|| \le C||\varphi - p_n||_\alpha \to 0$ as $n \to +\infty$, and hence $h_{\bar\varphi}^{(\alpha)}$ is compact.

On the other hand, if $h_{\bar\varphi}^{(\alpha)}$ is compact on L_a^2, then

$$T_a\varphi(z) = 3\langle \bar k_z, h_{\bar\varphi}^{(\alpha)} k_z\rangle \to 0, \qquad |z| \to 1^-$$

since $k_z \to 0$ weakly in L_a^2 as $|z| \to 1^-$. By Proposition 13, we have $\varphi \in \mathcal{B}_{\alpha,0}$, completing the proof of Theorem 31. □

Remark . Using the techniques developed in [**12**] we can also characterize those Hankel operators $h_{\bar\varphi}^{(\alpha)}$ with φ analytic which belong to the Schatten classes S_p: For $\alpha > 0$, $p \ge 1$, and φ analytic on $\mathbf{D}$ the following conditions are equivalent:

1) $h_{\bar\varphi}^{(\alpha)}$ belongs to the Schatten class S_p;

2) $(1-|z|^2)^{m+\alpha-1}\varphi^{(m)}(z)$ belongs to $L^p(\mathbf{D}, d\lambda)$, where $d\lambda(z) = (1-|z|^2)^{-2}\, dA(z)$ and m is any positive integer satisfying $p(m+\alpha-1) > 1$;

3) $\varphi \in Q_\alpha L^p(\mathbf{D}, d\lambda)$.

We omit the details here.

8. Further remarks and questions. In this final section we make some further remarks and pose some natural questions.

First we note that almost all the results in the paper can be generalized to the open unit ball. The methods are the same but one needs to use the corresponding versions of Propositions 3 and 6 in the context of the open unit ball, which can be found in [**15**] for example.

It is interesting to note that all spaces $\mathcal{B}_\alpha$ have the same space L_a^1 as a predual (independent of α) and L_a^1 has all the spaces $\mathcal{B}_{\alpha,0}$ as preduals; the dependence on α is reflected in the duality pairing. It is possible that other types of duality results can be proven for the spaces $\mathcal{B}_\alpha$ and $\mathcal{B}_{\alpha,0}$. For example, it was proved in [**4**] that under the Möbius invariant pairing the dual of $\mathcal{B}_0$ is the Besov space B_1, and the dual of B_1 is $\mathcal{B}$ under the same pairing.

It is well known that the Bloch space is the area version of BMO on $\mathbf{D}$. It is also known that the Bloch space is the space of analytic functions on $\mathbf{D}$ with bounded mean oscillation in the Bergman metric. It would be interesting to realize each $\mathcal{B}_\alpha$ as some type of BMO (in terms of the metric d_α or λ_α introduced in Section 4 for example).

Most questions asked in [**2**] also make sense for the spaces $\mathcal{B}_\alpha$. For example, one can consider the problem of characterizing the cyclic vectors for $\mathcal{B}_{\alpha,0}$ or the weak-star cyclic vectors for $\mathcal{B}_\alpha$.

Zeros of functions in $\mathcal{B}_\alpha$ (for $\alpha > 1$) are studied in [**13**]. In particular, certain types of infinite products with prescribed zeros are constructed in [**13**]. It would be interesting to know when such a product in $\mathcal{B}_\alpha$ is actually in $\mathcal{B}_{\alpha,0}$.

In Section 4 we conjectured that $d_\alpha = \lambda_\alpha$ for all $\alpha > 0$. It seems difficult to find a precise formula for either d_α or λ_α when $\alpha \neq 1$. It is easy to show that the distance d_α can be described using only functions in $\mathcal{B}_{\alpha,0}$:

$$d_\alpha(z,w) = \sup\{|f(z) - f(w)| : ||f||_\alpha \leq 1, f \in \mathcal{B}_{\alpha,0}\}.$$

Finally, we mention that the Bloch space is maximal among "decent" Möbius invariant Banach spaces [**14**]. We believe that the spaces $\mathcal{B}_\alpha$ also have a certain maximal property, but we are unable to formulate and prove such a result.

Acknowledgment. The author wishes to thank Boris Korenblum for many useful conversations.

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