ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 3, Summer 1993

# THE MEASURE OF APPROXIMATION IN THE TWO DIMENSIONAL OPERATIONAL FIELD

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ABSTRACT. In this paper we present a measure of approximation for the approximate solution of equations with coefficients in the field $\mathcal{Q}(\mathcal{A})$ of operators of T. Ogata [**4**].

**1. Operational calculus.** Consider the algebraically closed subset $\mathcal{K}$ of the field of Mikusinski operators $\mathcal{F}$ with elements of the form

$$\sum_{i=i_0}^{\infty} h_i l^{\alpha i-\beta},$$

where $h_i$ are complex numbers for $i_0 > -\infty$, $i = i_0, i_0+1, \dots$, $\alpha$ and $\beta$ are rational numbers, and $\alpha > 0$.

Let $\mathcal{A}$ be the set of formal power series of a variable $\lambda$ with coefficients in $\mathcal{K}$. With the usual addition and multiplication given by

$$PQ = \bigg\{ \sum_{\rho=0}^{\infty} \bigg( \sum_{\rho=\mu+\nu} \frac{\nu!\mu!}{(\rho+1)!} p_\nu q_\mu \bigg) \lambda^{\rho+1} \bigg\},$$

where

$$P = \bigg\{ \sum_{\nu=0}^{\infty} p_\nu \lambda^\nu \bigg\}, \qquad Q = \bigg\{ \sum_{\mu=0}^{\infty} g_\mu \lambda^\mu \bigg\},$$

(with $p_\mu, q_\nu \in \mathcal{K}$ for $\mu, \nu = 0, 1, \dots$) the ring $\mathcal{A}$ forms an integral domain without a unit element. The field of operators $\mathcal{Q}(\mathcal{A})$ is the quotient field of ring $\mathcal{A}$.

In the field of Mikusinski operators, $\mathcal{F}$, let us denote, as usual, by $l$ the integral operator, by $s$ the differential operator (recall that $l = s^{-1}$), and in $\mathcal{Q}(\mathcal{A})$ by $L$ the integral operator and by $S$ the differential operator (recall that $L = S^{-1}$).

Received by the editors on June 12, 1990, and in revised form on March 23, 1992.
This research was supported by Science Fund of Serbia, Grant Number 0401 A.

We shall frequently use the following transformations

$$\frac{1}{(s-p)^n} = \left\{ \frac{t^{n-1}}{(n-1)!} e^{pt} \right\}, \qquad n \in \mathbf{N}$$

and

$$\frac{1}{(S-p)^n} = \left\{ \frac{\lambda^{n-1}}{(n-1)!} e^{p\lambda} \right\}, \qquad n \in \mathbf{N}.$$

The rational operator

$$R(S) = \frac{q_m S^m + q_{m-1} S^{m-1} + \cdots + q_0}{p_n S^n + p_{n-1} S^{n-1} + \cdots + p_0}$$

from $\mathcal{Q}(\mathcal{A})$ belongs to $\mathcal{A}$ if $m < n$.

**2. The approximate solution.** Let us consider the linear partial differential equation with constant coefficients

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \alpha_{\mu,\nu} \frac{\partial^{\mu+\nu} x(\lambda, t)}{\partial \lambda^\mu \partial \lambda^\nu} = f_1(\lambda, t), \qquad t > 0,\ \lambda \geq 0, \tag{1}$$

with appropriate conditions

$$\frac{\partial^{\mu+\nu} x(\lambda, 0)}{\partial \lambda^\mu \partial t^\nu} = \phi_{\mu,\nu}(\lambda), \qquad \lambda > 0,\ \mu = 0, \dots, m,\ \nu = 0, \dots, n-1, \tag{2}$$

$$\frac{\partial^\mu x(0, t)}{\partial \lambda^\mu} = \psi_\mu(t), \qquad t > 0,\ \mu = 0, \dots, m-1, \tag{3}$$

where $f_1(\lambda, t)$ is a given continuous function.

In the field of Mikusinski operators, $\mathcal{F}$, equation (1) with conditions (2) and (3) corresponds to the problem

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \alpha_{\mu,\nu} s^\nu x^{(\mu)}(\lambda) = f(\lambda) \tag{4}$$

$$x^{(\mu)}(0) = \psi_\mu, \qquad \mu = 0, \dots, m-1, \tag{5}$$

where

$$f(\lambda) = \{f_1(\lambda, t)\} + \sum_{\mu=0}^{m}\sum_{\nu=0}^{n}\sum_{k=0}^{\nu-1} \alpha_{\mu,\nu} s^{\nu-\mu-1} \phi_{\mu,k}(\lambda) \tag{6}$$

(see [**3**]).

In the field $\mathcal{Q}(\mathcal{A})$ problem (4) and (5) corresponds to the equation

$$\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} \alpha_{\mu,\nu} s^{\nu} S^{\mu} X = \sum_{\mu=0}^{m-1} b_\mu S^\mu + F, \tag{7}$$

where $F = \{f(\lambda)\}$. The solution of the previous equation can be written as

$$X = \frac{\sum_{\mu=0}^{m-1} b_\mu S^\mu + F}{P(S)} \tag{8}$$

where

$$P(S) = \sum_{\mu=0}^{m}\sum_{\nu=0}^{n} \alpha_{\mu,\nu} s^{\nu} S^{\mu}$$

and the coefficients $b_\mu$ are of the form

$$b_\mu = a_{\mu+1}\psi_0 + \cdots + a_m \psi_{m-\mu-1}, \qquad \mu = 0, \ldots, m-1,$$

where $a_{\mu+1}, a_{\mu+2}, \ldots, a_m$ are given by (12).

It is known [**5**] that the solutions $w_j$, $j = 1, \ldots, m$, of equation $P(S) = 0$, have the form

$$w_j = \sum_{i=0}^{\infty} c_{i,j} l^{i\alpha_j - \beta_j}, \tag{9}$$

(in fact, they are the solutions of the characteristic equation of the equation (4)). In (9), $l \in \mathcal{F}$ and $c_{i,j}$ are complex numbers, $j = 1, \ldots, m$. The approximate solution of equation $P(S) = 0$ can be treated in the form

$$w_{j,n} = \sum_{i=0}^{n} c_{i,j} l^{i\alpha_j - \beta_j}, \qquad j = 1, \ldots, m. \tag{10}$$

Since $\mathcal{K}$ is an algebraically closed set, we can decompose

$$P(S) = a_m \prod_{j=1}^{m} (S - w_j), \tag{11}$$

where $w_j$, $j = 1, \dots, m$, has the form (9) and

$$a_\mu = \sum_{\nu=0}^{n} \alpha_{\mu,\nu} s^\nu, \qquad j = 1, \dots, m. \tag{12}$$

In this paper we shall consider only such equations of type (1) with conditions (2) and (3) where $w_j$, $j = 1, \dots, m$, are simple solutions of equation $P(S) = 0$. Also, we shall suppose that $\beta_j \le -1$ in (9).

If $F$ has the form

$$F = \sum_{i=1}^{r} \frac{d_i}{S - \alpha_i}, \tag{13}$$

where $d_i \in \mathcal{K}$ for $i = 1, \dots, r$, then the approximate solution in the space $\mathcal{Q}(\mathcal{A})$ is constructed in [**6**] as

$$\begin{aligned} X_n = \sum_{j=1}^{m} \frac{1}{S - w_{j,n}} \cdot \frac{1}{P'(w_{j,n})} \left( \sum_{\mu=1}^{m-1} b_\mu w_{j,n}^\mu + \sum_{i=1}^{r} \frac{d_i}{w_{j,n} - \alpha_i} \right) \\ + \sum_{i=1}^{r} \frac{d_i}{P(\alpha_i)(S - \alpha_i)}. \end{aligned} \tag{14}$$

If

$$F = \sum_{i=1}^{p} \frac{d_i}{S^i}, \tag{15}$$

where $d_i \in \mathcal{K}$, and $p \in \mathbf{N}$, then the approximate solution constructed in [**7**] has the form

$$\begin{aligned} X_n = \sum_{j=1}^{m} \frac{1}{P'(w_{j,n})} \Bigg( \frac{1}{S - w_{j,n}} \left( \sum_{\mu=0}^{m-1} b_\mu w_{j,n}^\mu + \sum_{i=1}^{p} d_i \frac{1}{w_{j,n}^i} \right) \\ + \sum_{i=1}^{p} (-d_i) \sum_{\rho=1}^{i} \frac{1}{S^r w_{j,n}^{i-\rho+1}} \Bigg). \end{aligned} \tag{16}$$

**3. The error of approximation.** By using the results of T. Boehme [**1**] and J. Burzyk [**2**] connected with topological convergences, the error of approximation in papers [**5, 6, 7**] was estimated in the field of Mikusinski operators.

In this paper we shall estimate the error of approximation in the space $\mathcal{Q}(\mathcal{A})$. Therefore, let us consider the functional

$$A_Q(X) = \sum_{i=0}^{\infty} \frac{B_{i*i,1/i}(X)}{1 + B_{i*i,1/i}(X)} e^{-i^2 e^{i^2}}, \qquad X \in \mathcal{Q}(\mathcal{A}), \tag{17}$$

where

$$B_{T*T,\varepsilon} = \inf\{||H||_{T*T};\ X = H/G;\ ||G||_{T*T} < 1;\ ||Ll - lLG||_{T*T} < \varepsilon\} \tag{18}$$

and

$$H_{T*T} = \int_0^T \int_0^T |h(\lambda, t)|\, d\lambda\, dt.$$

One easily shows

**Lemma 1.** *The operators*

$$g_k = \{F_k(\lambda, t)\}, \qquad \lambda \geq 0, t \geq 0$$

*in the space $\mathcal{Q}(\mathcal{A})$, which correspond to the function*

$$F_k(\lambda, t) = k^4 \exp(-k^2 t - k^2 \lambda), \qquad \lambda \geq 0, t \geq 0,$$

*satisfy the conditions*

$$||g_k||_{k*k} \leq 1; \qquad ||lG - lG_k||_{k*k} < 1/k.$$

**Proposition 1.** *If $X$ and $X_n$ are given by relations* (8) *and* (14), *respectively, then it holds that*

$$B_{T*T,1/T}(X - X_n) \leq \sum_{j=1}^{m} \Gamma^{-1}\left(\frac{(n+1)\alpha_j - \beta_j}{2} + 1\right) k_{1,j}^1 e^{k_{2,j}^2 e^{k_{3,j}^3 T}} T,$$

*where $k^1_{1,j}, k^2_{2,j}$ and $k^3_{3,j}$ are constants not depending on $T$ nor $\lambda$.*

*Proof.* Let us suppose that the first part of (14) can be written as

$$X_{1,n} = \sum_{j=1}^{m} \frac{1}{S - w_{j,n}} \cdot \frac{1}{P'(w_{j,n})} \sum_{\mu=0}^{m-1} b_\mu w_{j,n}^\mu$$

and the corresponding part of the exact solution as

$$X_1 = \sum_{j=1}^{m} \frac{1}{S - w_j} \cdot \frac{1}{P'(w_j)} \sum_{\mu=0}^{m-1} b_\mu w_j^\mu.$$

Then, by using relation (18), we have

$$B_{T*T,1/T}(X_1 - X_{1,n}) \leq \sum_{j=1}^{m} \Gamma^{-1}\left(\frac{(n+1)\alpha_j - \beta_j}{2} + 1\right) r^1_{1,j} e^{r^2_{2,j} e^{r^3_{3,j} T}} T,$$

where $r^1_{1,j}, r^2_{2,j}$ and $r^3_{3,j}$ are constants and $\alpha_j$ and $\beta_j$ appear in (9).

Similarly, we can estimate the other two parts of relation (14) (of course, compared with its exact representation). Let us remark that $r^1_{1,j} \leq k^1_{1,j}$, $r^2_{2,j} \leq k^2_{2,j}$, $r^3_{3,j} \leq k^3_{3,j}$ and that these constants do not depend on variable $\lambda$. □

Also, one can prove

**Proposition 2.** *If $X$ and $X_n$ are given by* (8) *and* (16), *respectively, then it holds that*

$$B_{T*T,1/k}(X - X_n) \leq \sum_{j=1}^{n} \Gamma^{-1}\left(\frac{(n+1)\alpha_j - \beta_j}{2} + 1\right) T \gamma_{1,j} e^{\gamma_{2,j} e^{\gamma_{3,j} T}}$$

*where $\gamma_{1,j}, \gamma_{2,j}, \gamma_{3,j}$ are constants.*

**Proposition 3.** *If $X$ and $X_n$ are given by* (8) *and* (16), *respectively, then it holds that*

$$A_Q(X - X_n) \leq \sum_{j=1}^{m} \Gamma\left(\frac{(n+1)\alpha_j - \beta_j}{2} + 1\right) R_j,$$

*where*

$$R_j = \sum_{i=0}^{\infty} \frac{k_{1,j} i e^{k_{2,j} e^{k_{3,j} i}}}{e^{-i^2 e^{i^2}}},$$

*where* $k_{1,j}, k_{2,j}$ *and* $k_{3,j}$ *are equal to* $k_{1,j}^1, k_{2,j}^2$ *and* $k_{3,j}^3$ *respectively, or to* $\gamma_{1,j}, \gamma_{2,j}$ *and* $\gamma_{3,j}$*, respectively.*

Let us remark that, as $n$ increases, $A_Q(X - X_n)$ approaches zero so $A_Q(X - X_n)$ can be taken as the error of approximation in the space $\mathcal{Q}(\mathcal{A})$ if the approximate solution is given by (14) or (16).

**4. An example.** Consider the differential equation [**4**]

$$\frac{\partial^4 x(\lambda, t)}{\partial \lambda^2 \partial t^2} - 2\frac{\partial^3 x(\lambda, t)}{\partial \lambda^2 \partial t} + \frac{\partial^2 x(\lambda, t)}{\partial \lambda^2} = x(\lambda, t) - 4e^{\lambda};$$
$$\lambda > 0, t \geq 0;$$

with conditions

$$x_{\lambda\lambda}(\lambda, 0) = e^{\lambda}, \qquad x_{\lambda\lambda t}(\lambda, 0) - 2x_{\lambda\lambda}(\lambda, 0) = \lambda,$$
$$x(0, t) = 1 + 2t, \qquad x_{\lambda}(0, t) = 2t.$$

In the field $\mathcal{Q}(\mathcal{A})$ the corresponding equation has the form

$$(s-1)^2 S^2 X - X = (s-1)^2((l + 2l^2)S + 2l^2) + \frac{s - 4l}{S - 1} + \frac{1}{S^2},$$

and the exact solution has the form

$$X = \frac{l(s-1)^2}{2(S - 1/(s-1))} - \frac{l(s-1)^2}{2(S + 1/(s-1))} + \frac{l + 2l^2}{S - 1} - \frac{1}{S^2}.$$

In this case we have $P(S) = (s-1)^2 S^2 - 1$, and the solutions of equation $P(S) = 0$ are

$$w_1 = \frac{1}{s-1} = \sum_{i=0}^{\infty} l^{i+1} = \{e^t\}, \qquad w_2 = -\frac{1}{s-1} = -\{e^t\}.$$

If we consider the approximate solution of equation $P(S) = 0$ with

$$w_{1,n} = \sum_{i=0}^{n} l^{i+1}; \qquad w_{2,n} = -\sum_{i=0}^{n} l^{i+1}$$

$(\alpha_1 = \alpha_2 = 1;\ \beta_1 = \beta_2 = -1)$, then the approximate solution is

$$X_n = \frac{l(s-1)^2}{2} \cdot \left( S - \sum_{i=0}^{n} l^{i+1} \right)^{-1} - \frac{l(s-1)^2}{2} \cdot \left( S + \sum_{i=0}^{n} l^{i+1} \right)^{-1} + \frac{l+2l^2}{S-1} - \frac{1}{S^2}.$$

And the error of approximation is

$$A_Q(X - X_n) = \frac{1}{\Gamma((n+2)/2+1)} R,$$

where

$$R = 4(e^{e^4} + 1/(e^4 - 1)).$$

**Acknowledgment.** The author wishes to thank the referee for useful remarks.

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