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A CHARACTERIZATION OF WEIGHTED COMPOSITION OPERATORS

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ABSTRACT. Let V be a system of weights on a completely regular Hausdorff space X, and let E be a Hausdorff locally convex topological vector space. Then $CV_b(X,E)$ and $CV_0(X,E)$ are weighted spaces of vector-valued continuous functions on X topologized by the family of seminorms $f \to \sup\{v(x)p(f(x)) : x \in X\}$, where $v \in V$ and p is a continuous seminorm on E. In this note, we characterize weighted composition operators wC_T on $CV_b(X,E)$ induced by operator-valued functions w on X and self maps T on X. Some concrete examples are presented to illustrate the theory.

1. Introduction. Let X denote a completely regular Hausdorff space, V a system of weights on X, and let E be a Hausdorff locally convex topological vector space over the field $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$. Then $B(E)$ is the locally convex space of all continuous linear transformations (operators) on E with the topology of uniform convergence on bounded subsets of E, and $CV_b(X,E)$ and $CV_0(X,E)$ are weighted locally convex spaces of E-valued continuous functions on X topologized by the family of semi-norms $f \to \sup\{v(x)p(f(x)) : x \in X\}$, where $v \in V$ and p is a continuous semi-norm on E. If w is a $B(E)$-valued function on X and T is a self map on X such that $w \cdot f \circ T$ belongs to $CV_b(X,E)$ (or $CV_0(X,E)$) whenever $f \in CV_b(X,E)$ (or $CV_0(X,E)$), then the map taking f to $w \cdot f \circ T$ is a linear transformation on $CV_b(X,E)$ (or $CV_0(X,E)$), where $w \cdot f \circ T$ is defined as $(w \cdot f \circ T)(x) = w(x)(f(T(x)))$ for every $x \in X$. If this linear transformation is also continuous, we call it the weighted composition operator on $CV_b(X,E)$ (or $CV_0(X,E)$) induced by the pair (w,T) and denote it by the symbol wC_T. If $w(x) = I$, the identity transformation on E, for every $x \in X$, we write wC_T as C_T and call it the composition operator on $CV_b(X,E)$

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(or $CV_0(X,E)$) induced by T. In case $T(x) = x$ for every $x \in X$, we write wC_T as M_w and call it the multiplication operator on $CV_b(X,E)$ (or $CV_0(X,E)$) induced by w. For examples and details on weighted spaces of continuous functions, we refer to [**10, 13, 15** and **16**]. The class of weighted composition operators has been the subject matter of study of several papers in recent years, for example, see [**3, 5, 6, 11, 12** and **17**].

In this paper, we have characterized weighted composition operators wC_T on weighted spaces $CV_b(X,E)$ induced by the pair (w,T), where w is a continuous $B(E)$-valued function on X and T is a continuous self map on X, and some examples are given to illustrate the results.

2. Preliminaries. Let $\mathbf{R}^+$ denote the set of all nonnegative reals with the usual relative topology. Then a function $v : X \to \mathbf{R}^+$ is called a weight on X if it is upper semi-continuous. A family V of weights on X is directed upward (or a Nachbin family [**16**]) if for every $v_1, v_2 \in V$ and $\alpha > 0$ there exists $v \in V$ such that $\alpha v_1, \alpha v_2 \leq v$ (pointwise on X), and a system of weights on X if it additionally satisfies the condition that for each $x \in X$ there exists $v_x \in V$ such that $v_x(x) \neq 0$. Let $cs(E)$ denote the collection of all continuous semi-norms on E and $C(X,E)$ the vector space of all continuous E-valued functions on X with pointwise linear operations. For a system V of weights on X, we define $CV_b(X,E) = \{f \in C(X,E) : vf(X)$ is bounded in E for each $v \in V\}$ and $CV_0(X,E) = \{f \in C(X,E) : vf$ vanishes at infinity on X for each $v \in V\}$. It is clear that $CV_b(X,E)$ and $CV_0(X,E)$ are vector spaces (over $\mathbf{K}$) with pointwise linear operations, while the upper semi-continuity of the weights implies that $CV_0(X,E) \subseteq CV_b(X,E)$. For $v \in V$, $p \in cs(E)$ and $f \in C(X,E)$, we put

$$||f||_{v,p} = \sup\{v(x)p(f(x)) : x \in X\}.$$

Then $||\cdot||_{v,p}$ is a semi-norm on $CV_b(X,E)$ (and hence on $CV_0(X,E)$) and the family $\{||\cdot||_{v,p} : v \in V, p \in cs(E)\}$ of semi-norms generates a locally convex Hausdorff topology on $CV_b(X,E)$ as well as on $CV_0(X,E)$. The spaces $CV_b(X,E)$ and $CV_0(X,E)$ with the corresponding topology are known as the weighted spaces of vector-valued continuous functions on X. In case $E = \mathbf{K}$, we shall omit E from our notation and write $CV_0(X)$ in place of $CV_0(X,E)$. If $(E, ||\cdot||)$ is a normed linear space, we shall write $||\cdot||_v = ||\cdot||_{v,p}$ for each $v \in V$, where $p(t) = ||t||$, $t \in E$.

The spaces $CV_b(X)$ and $CV_0(X)$ were first introduced by Nachbin [**8**] whereas the corresponding vector-valued spaces $CV_b(X,E)$ and $CV_0(X,E)$ were studied in detail by Bierstedt [**1, 2**] and Prolla [**9**].

The object $B(E)$ stands for the vector space of all continuous vector space endomorphisms (operators) on E while the symbol $\mathcal{B}$ denotes the collection of all bounded subsets of E. For each $p \in cs(E)$ and $M \in \mathcal{B}$, we define the semi-norm $||\cdot||_{p,M}$ on $B(E)$ as

$$||A||_{p,M} = \sup\{p(A(t)) : t \in M\}.$$

The family $\{||\cdot||_{p,M} : p \in cs(E), M \in \mathcal{B}\}$ of semi-norms defines a locally convex Hausdorff topology on $B(E)$, and the vector space $B(E)$ endowed with this topology becomes a locally convex space of operators on E. The convergence in this topology is the uniform convergence on bounded subsets of E. For details about topologies on spaces of operators, we refer to Grothendieck [**4**] and Köthe [**7**].

3. Functions inducing weighted composition operators. As assumed by Singh and Summers [**15**] and Singh and Manhas [**14**], we will work under the following requirements:

(a) X is a completely regular Hausdorff space.

(b) V is a system of weights on X.

(c) E is a nonzero locally convex Hausdorff topological vector space.

(d) Corresponding to each $x \in X$, there exists an $f_x \in CV_0(X)$ such that $f_x(x) \neq 0$.

In case X is locally compact, the condition (d) is automatically satisfied.

Before characterizing weighted composition operators on $CV_b(X,E)$, we first present the following proposition.

Proposition 3.1. *Let $w \in C(X, B(E))$ and $T \in C(X,X)$. If $wC_T : CV_0(X,E) \to CV_b(X,E)$ is continuous, then for every $v \in V$ and $p \in cs(E)$, there exists $u \in V$ and $q \in cs(E)$ such that*

$$v(x)p(w(x)t) \leq u(T(x))q(t)$$

for each $x \in X$ and $t \in E$.

Proof. This follows from Theorem 2.1 of [**14**] by replacing u by $u \circ T$. □

Theorem 3.2. *Let $T \in C(X,X)$ and $w \in C(X, B(E))$ such that $w(X)$ is equicontinuous. Then wC_T is a weighted composition operator on $CV_b(X,E)$ if and only if for every $v \in V$ and $p \in cs(E)$, there exists $u \in V$ and $q \in cs(E)$ such that $v(x)p(w(x)t) \leq u(T(x))q(t)$ for each $x \in X$ and $t \in E$.*

Proof. The necessary part follows from the Proposition 3.1. For sufficient part, we suppose that for every $v \in V$ and $p \in cs(E)$, there exists $u \in V$ and $q \in cs(E)$ such that $v(x)p(w(x)t) \leq u(T(x))q(t)$ for each $x \in X$ and $t \in E$. We will show that wC_T is continuous. First, we check that it maps $CV_b(X,E)$ into itself. To do this, let $f \in CV_b(X,E)$ and let $\{x_\alpha : \alpha \in A\}$ be a net in X such that x_α converges to some x in X. Then we show that for every $p \in cs(E)$ and $\varepsilon > 0$, there exists and index $\alpha_0 \in A$ such that $p[w(x_\alpha)h(x_\alpha) - w(x)h(x)] < \varepsilon$ for each $\alpha \geq \alpha_0$, where $h = f \circ T$. Now

(a) $$p[w(x_\alpha)h(x_\alpha) - w(x)h(x)] \leq p[\{w(x_\alpha) - w(x)\}h(x)] + p[w(x_a)\{h(x_\alpha) - h(x)\}]$$

Since $\{h(x)\} \in \mathcal{B}$, for every $p \in cs(E)$ and $\varepsilon > 0$, there exists an $\alpha_1 \in A$ such that

(b) $$p[\{w(x_\alpha) - w(x)\}h(x)] < \varepsilon/2 \qquad \text{for each } \alpha \geq \alpha_1.$$

Since $w(X)$ is equicontinuous, for every $p \in cs(E)$ and $\varepsilon > 0$, there exists a neighborhood G of origin in E such that $p(w(y)t) < \varepsilon/2$ for each $t \in G$ and $y \in X$. By continuity of h, we have an $\alpha_2 \in A$ such that $h(x_\alpha) - h(x) \in G$ for $\alpha \geq \alpha_2$, and consequently

(c) $$p[w(x_\alpha)\{h(x_\alpha) - h(x)\}] < \varepsilon/2$$

for each $\alpha \geq \alpha_2$. Pick $\alpha_0 \in A$ with $\alpha_1 \leq \alpha_0$ and $\alpha_2 \leq \alpha_0$. Then it follows from (a), (b) and (c) that $p[w(x_\alpha)h(x_\alpha) - w(x)h(x)] < \varepsilon$ for each $\alpha \geq \alpha_0$, and so $wC_T(f) \in C(X,E)$.

Also let $v \in V$, $p \in cs(E)$. Then

$$||wC_T(f)||_{v,p} \leq ||f \circ T||_{u \circ T,q} \leq ||f||_{u,q} < \infty,$$

which shows that $wC_T(f) \in CV_b(X, E)$ and is also enough to conclude the continuity of the linear transformation wC_T on $CV_b(X, E)$. □

Remark 3.3. (i) The condition of the above theorem is not sufficient for wC_T to be a weighted composition operator on $CV_0(X, E)$ as can be seen from the example preceding Theorem 2.3 of [**15**].

(ii) If E is a Banach space and $w(X)$ is not equicontinuous, even then wC_T is a weighted composition operator on $CV_b(X, E)$ as soon as the inequality of Theorem 3.2 is satisfied.

4. Example. Before giving some examples of weighted composition operators wC_T on the weighted spaces, we first give the following propositions.

Proposition 4.1. *If $w \in C(X, B(E))$ induces a multiplication operator M_w on $CV_b(X, E)$ and $T \in C(X, X)$ induces a composition operator C_T on $CV_b(X, E)$, then the pair (w, T) induces a weighted composition operator wC_T on $CV_b(X, E)$.*

The converse of the above proposition may not be true. For example, take $w(x) = 0$ for each $x \in X$ and T to be any self on X which does not induce a composition operator on $CV_b(X, E)$. But obviously wC_T is a weighted composition operator on $CV_b(X, E)$.

Proposition 4.2. *Let X be a completely regular Hausdorff space, let V be a system of weights on X, and let E be a Banach space. If $w \in C(X, B(E))$ is a bounded map and $T : X \to X$ induces a composition operator C_T on $CV_b(X, E)$, then (w, T) induces a weighted composition operator wC_T on $CV_b(X, E)$.*

Proof. Let $w \in C(X, B(E))$ be a bounded operator-valued mapping. Then there exists $m > 0$ such that $||w(x)|| \leq m$ for every $x \in X$. Let $v \in V$. Take $u = mv$. Then $u \in V$. Now

$$\begin{aligned} v(x)||w(x)t|| &\leq v(x)||w(x)||\,||t|| \\ &\leq mv(x)||t|| \\ &= u(x)||t||, \end{aligned}$$

for every $x \in X$ and $t \in E$. It follows from Remark 3.3 (ii) that M_w is a multiplication operator on $CV_b(X,E)$ and, hence, from Proposition 4.1 that wC_T is a weighted composition operator on $CV_b(X,E)$. $\square$

Proposition 4.3. *Let X be a completely regular Hausdorff space, E a Banach space, and take $V = \{\alpha\chi_K : \alpha \geq 0, K \subseteq X, K \text{ compact}\}$. Let $w \in C(X, B(E))$ and $T \in C(X,X)$. Then (w,T) induces a weighted composition operator wC_T on $CV_b(X,E)$.*

Proof. It follows from Proposition 2.3 of [**14**] that w induces a multiplication operator M_w on $CV_b(X,E)$. To see that $T \in C(X,X)$ induces a composition operator C_T on $CV_b(X,E)$, it is enough to show that for every $v \in V$, there exists $u \in V$ such that $v \leq u \circ T$. Let $v = \alpha\chi_K$, where K is a compact subset of X. Then there exists a compact subset F of X such that $K \subseteq T^{-1}(F)$. Let $u = \alpha\chi_F$. Then $u \in V$ and

$$v = \alpha\chi_K \leq \alpha\chi_{T^{-1}(F)} = \alpha\chi_F \circ T = u \circ T.$$

Hence, by Proposition 4.1, wC_T is a weighted composition operator on $CV_b(X,E)$. $\square$

Corollary 4.4. *Let X have the discrete topology, let E be a Banach space, and take*

$$V = \{\alpha\chi_K : \alpha \geq 0, K \subseteq X, K \text{ finite}\}.$$

Suppose that $w : X \to B(E)$ and $T : X \to X$ are any functions. Then wC_T is a weighted composition operator on $CV_b(X,E)$.

Example 4.5. Let $X = \mathbf{N}$ with the discrete topology, let V be the system of constant weights on X, and let $E = l^2$, the Hilbert space of all square summable sequences of complex numbers. If $T : \mathbf{N} \to \mathbf{N}$ is any function, then T induces a composition operator on $CV_b(X,E)$. Define $w : \mathbf{N} \to B(l^2)$ as $w(n) = A^n$ for every $n \in \mathbf{N}$, where A is the unilateral shift (or projection) operator on l^2. Then w is a bounded operator-valued function and so, in view of Proposition 4.2, it follows that wC_T is a weighted composition operator on $CV_b(\mathbf{N},E)$.

Example 4.6. Let $X = \mathbf{N}$ with the discrete topology, let V be the system of constant weights on $\mathbf{N}$, and let $E = l^\infty$, the Banach algebra of all bounded sequences of complex numbers. Let T be a self map on $\mathbf{N}$. Then T induces a composition operator C_T on $CV_b(X, E)$. Now define $w : \mathbf{N} \to B(l^\infty)$ as $w(n) = C_{A^n}$ for every $n \in \mathbf{N}$ where C_A is the composition operator on l^∞ induced by $A : \mathbf{N} \to \mathbf{N}$. Since w is a bounded operator-valued function, it follows, in view of Proposition 4.2, that wC_T is a weighted composition operator on $CV_b(\mathbf{N}, l^\infty)$.

As in Proposition 4.1, it is obvious that wC_T is a weighted composition operator on $CV_b(X, E)$ whenever w induces a multiplication operator M_w on $CV_b(X, E)$ and T induces a composition operator C_T on $CV_b(X, E)$. It is remarkable to observe that even if one of w or T does not induce the corresponding operator, the pair (w, T) may still induce a weighted composition operator. This we shall see in the following examples:

Example 4.7. Let $X = \mathbf{N}$ with the discrete topology, $V = \{\alpha v : \alpha \geq 0\}$ where $v(n) = n$ for every $n \in \mathbf{N}$, and let $E = l^2$ be the Hilbert space of all square summable sequences of complex numbers. Let us define $w : \mathbf{N} \to B(l^2)$ as $w(n) = (1/n)U^n$ for every $n \in \mathbf{N}$, where U denotes the unilateral shift operator on l^2. Then w is a bounded operator-valued function and so it induces a multiplication operator M_w on $CV_b(X, E)$. Now we define $T : \mathbf{N} \to \mathbf{N}$ as

$$T(n) = \begin{cases} \sqrt{n}, & \text{if } n \text{ is a perfect square} \\ n, & \text{otherwise.} \end{cases}$$

Then it is easy to check that this does not define a composition operator on $CV_b(X, E)$. However, for every $v \in V$ there is a $u \in V$ such that $v(n)||w(n)t|| \leq u(T(n))||t||$ for every $n \in \mathbf{N}$ and $t \in l^2$. And so, in view of Remark 3.3 (ii), it follows that wC_T is a weighted composition operator on $CV_b(X, E)$.

Example 4.8. With same X, V and E as in Example 4.7, if we define $w : \mathbf{N} \to B(l^2)$ as $w(n) = nU^n$ for every $n \in \mathbf{N}$ and $T : \mathbf{N} \to \mathbf{N}$ as $T(n) = n^2$ for every $n \in \mathbf{N}$, then one can easily check that w does not induce a multiplication operator M_w on $CV_b(X, E)$ and T defines a composition operator C_T on $CV_b(X, E)$, but the pair (w, T) induces a weighted composition operator wC_T on $CV_b(X, E)$.

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