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SELF-SIMILARITY

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ABSTRACT. A self-similar set is defined as a compact set which is the union of its images under the members of a collection of contractions, the contractions being indexed by a compact set. Self-similarity is characterized by the consideration of points in the self-similar set as limits associated with certain sequences of contractions. Conditions are given for the occurrence of self-similarity. A self-similar set is also treated as a fixed point in hyperspace, and the continuous variation of self-similar sets is shown.

Introduction. Self-similarity has received much attention in recent years in connection with the study of fractals. The small-scale geometry of a self-similar set reproduces after a fashion the large-scale geometry of the set, such as seen in the Cantor space. The self-similar set is made up of arbitrarily small copies, or contracted images, of itself. Historically, Mandelbrot [**6**, p. 18] stated that many of his fractals were self-similar, and, in 1981, Hutchinson [**4**] put on a formal footing the matter of self-similarity in fractals. In 1985, the contribution of Hata [**3**] appeared. Also in 1985, Barnsley and Demko [**2**] formalized an approach to self-similar fractals. Barnsley [**1**] further expanded this approach in a text on fractals in 1988. In the present study the work of Hutchinson and Barnsley related to self-similarity is elaborated upon and expanded. Most studies of self-similarity have dealt with just finitely many contractions acting at once on a set, although Hata [**3**] does consider a sequence of contractions. In the present study the collection of contractions acting on a set may be uncountable.

In Section 1 "self-similar" is defined, and in Section 2 self-similarity is characterized by "addressing," or the consideration of points in the self-similar set as limits associated with certain sequences of contractions. The existence of a self-similar set determined by a certain type of collection of contractions is shown in Section 3. In Section 4 the

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approach to the self-similar set as a fixed point in hyperspace is considered. Finally, in Section 5 the continuous variation of self-similar sets is treated.

1. Self-similarity. In this section *self-similar* is defined. The context for self-similarity in this study is always to include a complete metric space (X, d) on which there is defined the contraction w_λ for every λ in a compact topological space Λ, assumed to have the discrete topology if finite. It is assumed that not all of the contractions indexed by Λ are constant and that all the contractions have a common contractivity factor s. Additionally, $w : \Lambda \times X \to X$ is defined by $w(\lambda, x) = w_\lambda(x)$, Ω is the ordered triple $((X, d), w, \Lambda)$, F_Ω is $\{w_\lambda : \lambda \in \Lambda\}$, and $\mathbf{N}$ is the set of positive integers.

Definition 1.1. i) A *contraction system* is an Ω,

ii) a set A *self-similar under* F_Ω is a nonempty compact subset of X for which $A = \cup\{w_\lambda(A) : \lambda \in \Lambda\}$, and

iii) a *self-similar* set A is a set for which there is an Ω such that A is self-similar under F_Ω.

The essence, then, of self-similarity is representation by a collection of contracted images. To make for uniqueness and determination of distances between sets, the self-similar set is required to be compact. (Other definitions of "self-similar" are used: see [**3**, p. 383] and [**4**, p. 734].)

Example 1.2. The Cantor space $\{\sum_{i=1}^{\infty} n_i/3^i : n_i \in \{0, 2\}$ for $i \in \mathbf{N}\}$ is self-similar under $\{w_1, w_2\}$, w_1 being defined by $w_1(x) = x/3$ and w_2 by $w_2(x) = x/3 + 2/3$ for all real x.

Example 1.3. Define T to be the triangular region in R^2 whose vertices are (0,0), (1,0), and (0,1), but without the interior of the triangular region whose vertices are the midpoints of the sides of the original triangle. Define T_2 to be the region left when the "middle triangular interior" is removed from each of the three triangular regions of T_1. Thus, construct $\{T_n\}$. Define $A = \cap_{n=1}^{\infty} T_n$. Then A is self-similar under $\{w_1, w_2, w_3\}$, w_1 being defined by $w_1(x,y) = (x/2, y/2)$, w_2 by $w_2(x,y) = (x/2, y/2+1/2)$, and w_3 by $w_3(x,y) = (x/2+1/2, y/2)$ for all (x,y) in R^2. (The set A is the classical "Sierpinski Gasket.")

Example 1.4. If (X,d) is a Banach space, let A be a nonempty, compact, nonsingleton convex subset of X. For every $a \in A$, define $w_a : X \to X$ by $w_a(x) = (x+a)/2$. Then A is self-similar under $\{w_a : a \in A\}$.

Example 1.5. If X is the square region in R^2 with vertices (0,0), (0,1), (1,0), and (1,1), then define $\Lambda \equiv \{0,2\} \cup \{1/n : n \in \mathbf{N}\}$ with the topology inherited from R. Define w_0, w_2, and $w_{1/n}$ for $n \in \mathbf{N}$ on X by

$$w_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

$$w_{1/n} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2\ \cos \pi/2^{n+1} & -1/2\ \sin(\pi/2^{n+1} - \pi/2) \\ 1/2\ \sin \pi/2^{n+1} & 1/2\ \cos(\pi/2^{n+1} - \pi/2) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

and

$$w_2 \begin{pmatrix} x \\ y \end{pmatrix} = w_1 \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 - \sqrt{2}/2 \\ 1 - \sqrt{2}/2 \end{pmatrix}.$$

Thus, each w_λ collapses X into a radius of the unit circle, w_2 following this action on X by a translation so that $w_1(X) \cup w_2(X)$ is the diagonal of X from (0,0) to (1,1). The diagonal is at an angle of $\pi/4$ radians from the x-axis, and, for every $n \geq 2$, $w_{1/n}(X)$ is the radius at an angle of $\pi/2^{n+1}$ radians, $w_{1/n+1}(X)$ being at one-half the angle of $w_{1/n}(X)$, while $w_0(X)$ is at an angle of 0 radians. Define A to be the union of all these radii and the diagonal. Then a common contractivity factor for all w_λ is $\sqrt{2}/2$, and A is self-similar under $\{w_\lambda : \lambda \in \Lambda\}$. Further, A is not self-similar under any finite set of contractions, since a connected set self-similar under a finite set of contractions must necessarily be locally connected (see [**3**, p. 391]).

2. Addressing. In this section the relationship of *addressing* is used to characterize a self-similar set and is shown to be continuous. A salient feature of self-similarity is that if $A = \cup\{w_\lambda(A) : \lambda \in \Lambda\}$, then $A = \cup\{w_\lambda(\cup\{w_\tau(A) : \tau \in \Lambda\}) : \lambda \in \Lambda\}$, and so forth. This dissection of the set A raises the question of the location of each particular element of

A, a question answered by addressing: each element of A is associated with a certain sequence of indices of contractions in a manner as just suggested.

The product space $\prod_{i=1}^{\infty} \Lambda_i$, Λ_i being Λ, is denoted by Σ and has the standard product topology, and σ_i is the ith coordinate of σ in Σ. If $\sigma \in \Sigma$ and $\lambda \in \Lambda$, then $\lambda\sigma$ is α in Σ for which $\alpha_1 = \lambda$ and $\alpha_i = \sigma_{i-1}$ for $i > 1$. The ith projection function for Σ, $i \in \mathbf{N}$, is denoted by PROJ_i; $N(q)$ is an open neighborhood of q in the topological space indicated by context, and $N(q, \delta)$ is $\{p \in Q : d_Q(q,p) < \delta\}$, (Q, d_Q) being a metric space indicated by context and $\delta > 0$.

In this section it is assumed that A is self-similar under F_Ω.

The following is immediate.

Proposition 2.1. i) *For every $n \in \mathbf{N}$,*

$$A = \cup\{w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(A) : \sigma \in \Sigma\},$$

and

ii) *if B is a bounded subset of X and if, for every $n \in \mathbf{N}$, $g_n : \Sigma \to R$ is defined by $g_n(\sigma) = \mathrm{diam}\,(w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(B))$, then $\{g_n\}$ converges uniformly to 0.*

So Proposition 2.1 states that A is made up of arbitrarily small copies, or contracted images, of itself. If $x \in A$, then $x \in w_{\lambda_1}(A)$ for some $\lambda_1 \in \Lambda$, $x \in w_{\lambda_1}(w_{\lambda_2}(A))$ for some $\lambda_2 \in \Lambda$, and so forth to ever finer levels. Such trailing of individual elements of A gives rise to "addressing," a term originated by Barnsley [**1**, p. 118]. The "address" of x is the σ in Σ specified by $(\lambda_1, \lambda_2, \dots)$. Since x is in all the ever shrinking pieces $w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(A)$ as n increases, then x is the limit as n increases of the $w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(y)$ for every y in A.

Lemma 2.2. *If B is a nonempty bounded subset of X, and $\cup\{w_\lambda(B) : \lambda \in \Lambda\} \subset B$, then*

i) *$\lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(b)$ exists for every $b \in B$ and every $\sigma \in \Sigma$ and is independent of b, and*

ii) *if $g_n : \Sigma \times B \to X$ is defined by $g_n(\sigma, b) = w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(b)$*

for every $n \in \mathbf{N}$*, and* $g : \Sigma \times B \to X$ *is defined by* $g(\sigma, b) = \lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(b)$*, then* $\{g_n\}$ *converges uniformly to* g*.*

Proof. i) Since $w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_{n+1}}(B) \subset w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(B)$ for every $\sigma \in \Sigma$, and since X is complete, the result follows from Proposition 2.1 (ii).

ii) The result follows from Proposition 2.1 (ii). □

Theorem 2.3. i) $\lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(a)$ *exists in* A *for every* $a \in A$ *and every* $\sigma \in \Sigma$ *and is independent of* a*,*

ii) *if* $g_n : \Sigma \times A \to A$ *is defined by* $g_n(\sigma, a) = w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(a)$ *for every* $n \in \mathbf{N}$*, and* $g : \Sigma \times A \to A$ *is defined by* $g(\sigma, a) = \lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(a)$*, then* $\{g_n\}$ *converges uniformly to* g*, and*

iii) $A = \{\lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(a) : \sigma \in \Sigma\}$ *for every* $a \in A$*.*

Proof. i) The result follows from Lemma 2.2 (i), since A is compact.

ii) The result follows from Lemma 2.2 (ii).

iii) Let $a \in A$. By Proposition 2.1 (i), a sequence $\{\lambda_n\}$ can be chosen inductively so that $a \in w_{\lambda_1} \circ w_{\lambda_2} \circ \cdots \circ w_{\lambda_n}(A)$ for every $n \in \mathbf{N}$. The result then follows from (i) and Proposition 2.1 (ii). □

In a different manner, Barnsley [**1**, p. 127] arrives at Theorem 2.3 (i) and (iii) in the case of finite Λ.

The following definition is well-formed according to Theorem 2.3.

Definition 2.4. i) *The* Ω-address function *is* $\Phi_\Omega : \Sigma \to A$ *defined by* $\Phi_\Omega(\sigma) = \lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(a)$ *for every* $a \in A$*, and*

ii) *an* Ω-address *of* a *in* A *is any* $\sigma \in \Sigma$ *for which* $\Phi_\Omega(\sigma) = a$*.*

The address function is introduced by Williams [**8**, p. 56] and in a form different from that here; Hutchinson [**4**, p. 725] makes use of the function in showing that its image of Σ for finite Λ is compact. Hata [**3**, p. 384] draws more attention to the address function, but it is left to Barnsley [**1**, p. 118], who introduces the term "address," to define the address function as it is here and to give addressing a significant role in his pursuit of fractals. In the present study addressing has even more of a role.

It is now seen that there is some freedom in finding the values of the address function.

Lemma 2.5. *If $B \subset X$, and $B \cup \{w_\lambda(b) : \lambda \in \Lambda, b \in B\}$ is bounded, then $\{w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(b) : \sigma \in \Sigma, n \in \mathbf{N}, b \in B\}$ is bounded.*

Proof. It may be assumed that B is nonempty. Let $\alpha \in \Sigma$, $m \in \mathbf{N}$, $c \in B$ and $x \in X$. Define $D \equiv \operatorname{diam}(\{x\} \cup B \cup \{w_\lambda(b) : \lambda \in \Lambda, b \in B\})$. Then

$$
\begin{aligned}
d(w_{\alpha_1} & \circ w_{\alpha_2} \circ \cdots \circ w_{\alpha_m}(c), x) \\
&\leq d(w_{\alpha_1} \circ w_{\alpha_2} \circ \cdots \circ w_{\alpha_m}(c), w_{\alpha_1}(c)) + d(w_{\alpha_1}(c), x) \\
&\leq sd(w_{\alpha_2} \circ w_{\alpha_3} \circ \cdots \circ w_{\alpha_m}(c), c) + D \\
&\leq s[d(w_{\alpha_2} \circ w_{\alpha_3} \circ \cdots \circ w_{\alpha_m}(c), w_{\alpha_2}(c)) + d(w_{\alpha_2}(c), c)] + D \\
&\leq s^2 d(w_{\alpha_3} \circ w_{\alpha_4} \circ \cdots \circ w_{\alpha_m}(c), c) + sD + D \\
&\ \ . \\
&\ \ . \\
&\ \ . \\
&\leq s^{m-1}D + s^{m-2}D + \cdots + sD + D \\
&\leq D\bigg(\sum_{i=0}^{\infty} s^i\bigg) = D/(1-s).
\end{aligned}
$$

Hence, $\{w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(b) : \sigma \in \Sigma, n \in \mathbf{N}, b \in B\}$ is bounded. □

Proposition 2.6. *If $y \in X$ and $\{w_\lambda(y) : \lambda \in \Lambda\}$ is bounded, then*

i) *$\Phi_\Omega(\sigma) = \lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(y)$ for every $\sigma \in \Sigma$, and*

ii) *if $g_n : \Sigma \to X$ is defined by $g_n(\sigma) = w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(y)$ for every $n \in \mathbf{N}$, then $\{g_n\}$ converges uniformly to Φ_Ω.*

Proof. i) Define $B \equiv \{w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(y) : \sigma \in \Sigma, n \in \mathbf{N}\} \cup \{y\}$. Since $\{y\} \cup \{w_\lambda(y) : \lambda \in \Lambda\}$ is bounded, $\{w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(y) : \sigma \in \Sigma, n \in \mathbf{N}\}$ is bounded by Lemma 2.5, and so $A \cup B$ is bounded. Also, $\cup\{w_\lambda(A \cup B) : \lambda \in \Lambda\} \subset A \cup B$. Let $\alpha \in \Sigma$, $a \in A$. By Lemma 2.2 (i), then, $\lim_{n\to\infty} w_{\alpha_1} \circ w_{\alpha_2} \circ \cdots \circ w_{\alpha_n}(y)$ exists and equals $\lim_{n\to\infty} w_{\alpha_1} \circ w_{\alpha_2} \circ \cdots \circ w_{\alpha_n}(a)$. Thus $\Phi_\Omega(\alpha) = \lim_{n\to\infty} w_{\alpha_1} \circ w_{\alpha_2} \circ \cdots \circ w_{\alpha_n}(y)$.

ii) Define B as in (i). Define $f_n : \Sigma \times (A \cup B) \to X$ by $f_n(\sigma, x) = w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(x)$ for every $n \in \mathbf{N}$. Define $f : \Sigma \times (A \cup B) \to X$ by $f(\sigma, x) = \lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(x)$, which exists by Lemma 2.2 (i). By Lemma 2.2 (ii), $\{f_n\}$ converges uniformly to f. Therefore, $\{f_n \mid (\Sigma \times \{y\})\}$ converges uniformly to $f \mid (\Sigma \times \{y\})$, and $\{g_n\}$ converges uniformly to Φ_Ω by (i). □

Proposition 2.7. *If w is continuous, then*

i) $\Phi_\Omega(\sigma) = \lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(x)$ *for every* $\sigma \in \Sigma$ *and every* $x \in X$, *and*

ii) *if K is a compact subset of X, $g_n : \Sigma \times K \to X$ is defined by $g_n(\sigma, x) = w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(x)$ for every $n \in \mathbf{N}$, and $g : \Sigma \times K \to X$ is defined by $g(\sigma, x) = \lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(x)$, then $\{g_n\}$ converges uniformly to g.*

Proof. i) Let $y \in X$. Since $w(\Lambda \times \{y\})$ is compact, the result follows from Proposition 2.6 (i).

ii) Since $w(\Lambda \times K)$ is compact, $K \cup \{w_\lambda(x) : \lambda \in \Lambda, x \in K\}$ is bounded, and by Lemma 2.5, if

$$B \equiv \{w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(x) : \sigma \in \Sigma, n \in \mathbf{N}, x \in K\},$$

then B is bounded. Further, $B \cup K$ is bounded, and $\cup\{w_\lambda(B \cup K) : \lambda \in \Lambda\} \subset B \cup K$. Define $f_n : \Sigma \times (B \cup K) \to X$ by $f_n(\sigma, x) = w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(x)$ for every $n \in \mathbf{N}$. Define $f : \Sigma \times (B \cup K) \to X$ by $f(\sigma, x) = \lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(x)$, which exists by Lemma 2.2 (i). By Lemma 2.2 (ii), $\{f_n\}$ converges uniformly to f. Hence, $\{f_n \mid (\Sigma \times K)\}$ converges uniformly to $f \mid (\Sigma \times K)$, or $\{g_n\}$ converges uniformly to g. □

Corollary 2.8. *If w is continuous, K is a compact subset of X, and $\varepsilon > 0$, then there is an $m \in \mathbf{N}$ such that for every $a \in A$ there is a*

$\sigma \in \Sigma$ such that $d(w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(x), a) < \varepsilon$ for every $x \in K$ and for every $n \geq m$, and for every $\alpha \in \Sigma$ there is a $b \in A$ such that $d(w_{\alpha_1} \circ w_{\alpha_2} \circ \cdots \circ w_{\alpha_n}(x), b) < \varepsilon$ for every $x \in K$ and for every $n \geq m$.

Proof. The result follows from Theorem 2.3 (iii) and Proposition 2.7. □

The previous result indicates that travel along the many address trails is uniform, so that in some sense convergence is to the set A itself rather than just its points individually. This collective approach to A as a limit in hyperspace will be explored in Section 4 as an alternate development of self-similarity.

The conclusions of Proposition 2.7 and Corollary 2.8 certainly hold for a contraction system with finite Λ, and Barnsley [**1**, p. 127] arrives at the conclusion of Proposition 2.7 (i) for such a system.

Next, the continuity of the address function is seen to be equivalent to a restricted continuity of w.

Lemma 2.9. *If B is a nonempty bounded subset of X, $\{w_\lambda(B) : \lambda \in \Lambda\} \subset B$, and $\Gamma : \Sigma \to X$ is defined by $\Gamma(\sigma) = \lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(b)$ for every $b \in B$, then Γ is continuous if and only if $w \mid (\Lambda \times \{x\})$ is continuous for every $x \in \Gamma(\Sigma)$.*

Proof. By Lemma 2.2 (i), Γ is well defined. If $w \mid (\Lambda \times \{x\})$ is continuous for every $x \in \Gamma(\Sigma)$, then let $\alpha \in \Sigma$, $b \in B$. Let $\varepsilon > 0$. There is a $k \in \mathbf{N}$ such that $s^k(\operatorname{diam} B) < \varepsilon/4$. Define $\delta \equiv \varepsilon(1-s)/4$. If $i \in \{1, 2, \ldots, k\}$, then $x_i \equiv \lim_{n\to\infty} w_{\alpha_{i+1}} \circ w_{\alpha_{i+2}} \circ \cdots \circ w_{\alpha_n}(b)$ exists by Lemma 2.2 (i), and so, by continuity of $w \mid (\Lambda \times \{x_i\})$, there is an open neighborhood $N(\alpha_i)$ such that if $\lambda \in N(\alpha_i)$, then $d(w_{\alpha_i}(x_i), w_\lambda(x_i)) < \delta/2$.

Now if $\sigma \in \cap_{i=1}^k \mathrm{PROJ}_i^{-1}(N(\alpha_i))$, a neighborhood of α, then $d(w_{\alpha_i}(x_i), w_{\sigma_i}(x_i)) < \delta/2$ for every $i \in \{1, 2, \ldots, k\}$. Also for every $i \in \{1, 2, \ldots, k\}$, continuity of w_{α_i} and w_{σ_i} ensures that there is an $m_i \in \mathbf{N}$ such that, for every $m \geq m_i$,

$$d(w_{\alpha_i}(x_i), w_{\alpha_i} \circ w_{\alpha_{i+1}} \circ \cdots \circ w_{\alpha_m}(b)) < \delta/4$$

and

$$d(w_{\sigma_i}(x_i), w_{\sigma_i} \circ w_{\alpha_{i+1}} \circ w_{\alpha_{i+2}} \circ \cdots \circ w_{\alpha_m}(b)) < \delta/4.$$

So for every $i \in \{1, 2, \dots, k\}$ and for every $m \geq m_i$,

$$\begin{aligned}
d(w_{\alpha_i} \circ w_{\alpha_{i+1}} \circ \cdots \circ w_{\alpha_m}(b), w_{\sigma_i} \circ w_{\alpha_{i+1}} \circ w_{\alpha_{i+2}} \circ \cdots \circ w_{\alpha_m}(b)) \\
\leq d(w_{\alpha_i} \circ w_{\alpha_{i+1}} \circ \cdots \circ w_{\alpha_m}(b), w_{\alpha_i}(x_i)) \\
+ d(w_{\alpha_i}(x_i), w_{\sigma_i}(x_i)) \\
+ d(w_{\sigma_i}(x_i), w_{\sigma_i} \circ w_{\alpha_{i+1}} \circ w_{\alpha_{i+2}} \circ \cdots \circ w_{\alpha_m}(b)) \\
< \delta/4 + \delta/2 + \delta/4 = \delta.
\end{aligned}$$

Then, for every $m > \max\{k, m_1, m_2, \dots, m_k\}$,

$$\begin{aligned}
& d(w_{\alpha_1} \circ w_{\alpha_2} \circ \cdots \circ w_{\alpha_m}(b), w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_m}(b)) \\
& \leq d(w_{\alpha_1} \circ w_{\alpha_2} \circ \cdots \circ w_{\alpha_m}(b), w_{\sigma_1} \circ w_{\alpha_2} \circ w_{\alpha_3} \circ \cdots \circ w_{\alpha_m}(b)) \\
& \quad + d(w_{\sigma_1} \circ w_{\alpha_2} \circ w_{\alpha_3} \circ \cdots \circ w_{\alpha_m}(b), w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_m}(b)) \\
& < \delta + sd(w_{\alpha_2} \circ w_{\alpha_3} \circ \cdots \circ w_{\alpha_m}(b), w_{\sigma_2} \circ w_{\sigma_3} \circ \cdots \circ w_{\sigma_m}(b)) \\
& < \delta + s[\delta + sd(w_{\alpha_3} \circ w_{\alpha_4} \circ \cdots \circ w_{\alpha_m}(b), w_{\sigma_3} \circ w_{\sigma_4} \circ \cdots \circ w_{\sigma_m}(b))] \\
& = \delta + s\delta + s^2 d(w_{\alpha_3} \circ w_{\alpha_4} \circ \cdots \circ w_{\alpha_m}(b), w_{\sigma_3} \circ w_{\sigma_4} \circ \cdots \circ w_{\sigma_m}(b)) \\
& \quad . \\
& \quad . \\
& \quad . \\
& < \delta(1 + s + \cdots + s^{k-1}) \\
& \quad + s^k d(w_{\alpha_{k+1}} \circ w_{\alpha_{k+2}} \circ \cdots \circ w_{\alpha_m}(b), w_{\sigma_{k+1}} \circ w_{\sigma_{k+2}} \circ \cdots \circ w_{\sigma_m}(b)) \\
& < \delta/(1-s) + s^k(\operatorname{diam} B) < \varepsilon/4 + \varepsilon/4 = \varepsilon/2.
\end{aligned}$$

Furthermore, there is a $p > \max\{k, m_1, m_2, \dots, m_k\}$ such that

$$d(\lim_{n\to\infty} w_{\alpha_1} \circ w_{\alpha_2} \circ \cdots \circ w_{\alpha_n}(b), w_{\alpha_1} \circ w_{\alpha_2} \circ \cdots \circ w_{\alpha_p}(b)) < \varepsilon/4$$

and

$$d(\lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(b), w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_p}(b)) < \varepsilon/4.$$

Thus

$$\begin{aligned} d(\Gamma(\alpha),\Gamma(\sigma)) &= d(\lim_{n\to\infty} w_{\alpha_1}\circ w_{\alpha_2}\circ\cdots\circ w_{\alpha_n}(b), \\ &\qquad\qquad \lim_{n\to\infty} w_{\sigma_1}\circ w_{\sigma_2}\circ\cdots\circ w_{\sigma_n}(b)) \\ &\le d(\lim_{n\to\infty} w_{\alpha_1}\circ w_{\alpha_2}\circ\cdots\circ w_{\alpha_n}(b), \\ &\qquad\qquad w_{\alpha_1}\circ w_{\alpha_2}\circ\cdots\circ w_{\alpha_p}(b)) \\ &\quad + d(w_{\alpha_1}\circ w_{\alpha_2}\circ\cdots\circ w_{\alpha_p}(b), w_{\sigma_1}\circ w_{\sigma_2}\circ\cdots\circ w_{\sigma_p}(b)) \\ &\quad + d(w_{\sigma_1}\circ w_{\sigma_2}\circ\cdots\circ w_{\sigma_p}(b), \\ &\qquad\qquad \lim_{n\to\infty} w_{\sigma_1}\circ w_{\sigma_2}\circ\cdots\circ w_{\sigma_n}(b)) \\ &< \varepsilon/4+\varepsilon/2+\varepsilon/4=\varepsilon, \end{aligned}$$

and Γ is continuous at α.

Conversely, if Γ is continuous, then let $\alpha\in\Sigma$, $b\in B$. Let $\tau\in\Lambda$; let $\varepsilon>0$. Since Γ is continuous, there is an open neighborhood $N(\tau\alpha)$ such that if $\sigma\in N(\tau\alpha)$, then $d(\Gamma(\tau\alpha),\Gamma(\sigma))<\varepsilon$. Now, if $\lambda\in \mathrm{PROJ}_1(N(\tau\alpha))$, a neighborhood of τ, then $\lambda\alpha\in N(\tau\alpha)$, and

$$\begin{aligned} d(w_\tau(\Gamma(\alpha)), w_\lambda(\Gamma(\alpha))) &= d(w_\tau(\lim_{n\to\infty} w_{\alpha_1}\circ w_{\alpha_2}\circ\cdots\circ w_{\alpha_n}(b)), \\ &\qquad w_\lambda(\lim_{n\to\infty} w_{\alpha_1}\circ w_{\alpha_2}\circ\cdots\circ w_{\alpha_n}(b))) \\ &= d(\lim_{n\to\infty} w_\tau\circ w_{\alpha_1}\circ w_{\alpha_2}\circ\cdots\circ w_{\alpha_n}(b), \\ &\qquad \lim_{n\to\infty} w_\lambda\circ w_{\alpha_1}\circ w_{\alpha_1}\circ\cdots\circ w_{\alpha_n}(b)) \\ &= d(\Gamma(\tau\alpha),\Gamma(\lambda\alpha)<\varepsilon, \end{aligned}$$

because w_τ and w_λ are continuous. Hence $w \mid (\Lambda\times\{\Gamma(\alpha)\})$ is continuous. □

Theorem 2.10. *The address function Φ_Ω is continuous if and only if $w\mid(\Lambda\times\{a\})$ is continuous for every $a\in A$.*

Proof. By Theorem 2.3 (iii), $\Phi_\Omega(\Sigma)=A$. Now the result follows from Lemma 2.9. □

It can be shown that the following two are equivalent: (i) $w\mid(\Lambda\times\{x\})$ is continuous for every $x\in X$, and (ii) w is continuous. If X is compact, then it can be shown that each of (i) and (ii) is equivalent to: (iii) $\{w \mid (\Lambda \times \{x\}) : x \in X\}$ is equicontinuous. So, with respect to A, these different forms of continuity for the collection of contractions w_λ are identical, and the continuity of Φ_Ω is equivalent to the continuity of $w \mid (\Lambda \times A)$.

According to Theorem 2.10, Φ_Ω is certainly continuous if Λ is finite. Each of Williams [**8**, p. 56], Hutchinson [**4**, p. 725], Hata [**3**, p. 384], and Barnsley [**1**, p. 127] concludes the continuity of his version of the address function in the case of finite Λ.

If Φ_Ω is continuous and one-to-one, then A is homeomorphic to Σ, since Σ is compact and A Is Hausdorff. If, in addition, Λ is finite, then A is homeomorphic to the Cantor space. One-to-oneness, however, is not readily achieved, as the following shows.

Proposition 2.11. *The address function Φ_Ω is one-to-one if and only if for every λ, $\tau \in \Lambda$, with $\lambda \neq \tau$, $w_\lambda \mid A$ is one-to-one and $w_\lambda(A) \cap w_\tau(A) = \phi$.*

Proof. If Φ_Ω is one-to-one, and $w_\tau(a) = w_\delta(b)$ for some $\tau, \delta \in \Lambda$ and some $a, b \in A$, then, since Φ_Ω is onto A, there are $\alpha, \beta \in \Sigma$ such that $\Phi_\Omega(\alpha) = a$ and $\Phi_\Omega(\beta) = b$. It follows from the continuity of w_τ and w_δ that $\Phi_\Omega(\tau\alpha) = w_\tau(a)$ and $\Phi_\Omega(\delta\beta) = w_\delta(b)$. Since Φ_Ω is one-to-one, $\tau\alpha = \delta\beta$, and thus $\tau = \delta$ and $\alpha = \beta$. Then $a = b$, and hence $w_\lambda \mid A$ is one-to-one for every $\lambda \in \Lambda$, and any two different elements of $\{w_\lambda(A) : \lambda \in \Lambda\}$ do not meet.

Conversely, if $w_\lambda \mid A$ is one-to-one for every $\lambda \in \Lambda$, and $w_\lambda(A) \cap w_\tau(A) = \phi$ for every $\lambda, \tau \in \Lambda$, with $\lambda \neq \tau$, then let $\alpha, \beta \in \Sigma$, $a \in A$. If $\Phi_\Omega(\alpha) = \Phi_\Omega(\beta)$, then

$$\lim_{n\to\infty} w_{\alpha_1} \circ w_{\alpha_2} \circ \cdots \circ w_{\alpha_n}(a) = \lim_{n\to\infty} w_{\beta_1} \circ w_{\beta_2} \circ \cdots \circ w_{\beta_n}(a),$$

and

$$w_{\alpha_1}(\lim_{n\to\infty} w_{\alpha_2} \circ w_{\alpha_3} \circ \cdots \circ w_{\alpha_n}(a)) = w_{\beta_1}(\lim_{n\to\infty} w_{\beta_2} \circ w_{\beta_3} \circ \cdots \circ w_{\beta_n}(a)),$$

since w_{α_1} and w_{β_1} are continuous. Thus, $\alpha_1 = \beta_1$ and

$$\lim_{n\to\infty} w_{\alpha_2} \circ w_{\alpha_3} \circ \cdots \circ w_{\alpha_n}(a) = \lim_{n\to\infty} w_{\beta_2} \circ w_{\beta_3} \circ \cdots \circ w_{\beta_n}(a).$$

Induction being used, if $k \in \mathbf{N}$ and $\alpha_k = \beta_k$, and

$$\lim_{n\to\infty} w_{\alpha_{k+1}} \circ w_{\alpha_{k+2}} \circ \cdots \circ w_{\alpha_n}(a) = \lim_{n\to\infty} w_{\beta_{k+1}} \circ w_{\beta_{k+2}} \circ \cdots \circ w_{\beta_n}(a),$$

then by an argument similar to that just used, $\alpha_{k+1} = \beta_{k+1}$ and the proper equation for limits follows. So $\alpha_i = \beta_i$ for every $i \in \mathbf{N}$. Thus $\alpha = \beta$ and Φ_Ω is one-to-one. □

3. Existence of self-similar sets. It has been seen that a self-similar set is composed of the limits of certain sequences. In this section it is shown that if w is continuous, then, conversely, the limits of these sequences exist and comprise a self-similar set. The following development is suggested by Lemma 2.2 (i).

For every $\lambda \in \Lambda$, the fixed point of w_λ is denoted by z_λ.

Theorem 3.1. *If A is self-similar under F_Ω, then A is the unique such set.*

Proof. If B is self-similar under F_Ω, then let $a \in A$, $b \in B$. By Theorem 2.3 (iii), $B = \{\lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(b) : \sigma \in \Sigma\}$ and $A = \{\lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(a) : \sigma \in \Sigma\}$. By Proposition 2.6 (i), $\lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(a) = \lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(b)$ for every $\sigma \in \Sigma$. Hence, $A = B$. □

Theorem 3.2. *If w is continuous, then there exists a unique set self-similar under F_Ω.*

Proof. Define $z : \Lambda \to X$ by $z(\lambda) = z_\lambda$. Let $\tau \in \Lambda$. Let $\varepsilon > 0$. Since w is continuous, there is an open neighborhood $N(\tau)$ such that if $\lambda \in N(\tau)$, then $d(w_\tau(z_\tau), w_\lambda(z_\tau)) < \varepsilon(1-s)$. So if $\lambda \in N(\tau)$, then

$$\begin{aligned} d(z_\tau, z_\lambda) &= d(w_\tau(z_\tau), w_\lambda(z_\lambda)) \\ &\le d(w_\tau(z_\tau), w_\lambda(z_\tau)) + d(w_\lambda(z_\tau), w_\lambda(z_\lambda)) \\ &< \varepsilon(1-s) + s d(z_\tau, z_\lambda), \end{aligned}$$

$(1-s)d(z_\tau, z_\lambda) < \varepsilon(1-s)$, and $d(z_\tau, z_\lambda) < \varepsilon$. Hence z is continuous, and so $\{z_\lambda : \lambda \in \Lambda\}$ is bounded, being a continuous image of the compact Λ.

Define $D \equiv \text{diam}\,\{z_\lambda : \lambda \in \Lambda\}$. Define $B \equiv \cap\{N(z_\lambda, D(1+s)/(1-s)) : \lambda \in \Lambda\}$. Because $\{z_\lambda : \lambda \in \Lambda\} \subset B$, B is nonempty, and certainly B is bounded. Let $\tau, \delta \in \Lambda$, $b \in B$. Then

$$\begin{aligned} d(w_\tau(b), z_\delta) &\le d(w_\tau(b), w_\tau(z_\delta)) + d(w_\tau(z_\delta), w_\tau(z_\tau)) \\ &\quad + d(w_\tau(z_\tau), z_\delta) \\ &\le sd(b, z_\delta) + sd(z_\delta, z_\tau) + d(z_\tau, z_\delta) \\ &< sD(1+s)/(1-s) + sD + D \\ &= D(1+s)/(1-s), \end{aligned}$$

and thus $w_\tau(B) \subset B$.

Now, according to Lemma 2.2 (i), $\lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(b)$ exists for every $\sigma \in \Sigma$. Define $A \equiv \{\lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(b) : \sigma \in \Sigma\}$. Define $\Gamma : \Sigma \to A$ by $\Gamma(\sigma) = \lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(b)$. By Lemma 2.9, Γ is continuous. Since Σ is compact due to the compactness of Λ, it follows that A, being $\Gamma(\Sigma)$, is compact.

Also,

$$\begin{aligned} \cup\{w_\lambda(A) :&\lambda \in \Lambda\} \\ &= \cup\{w_\lambda(\{\lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(b) : \sigma \in \Sigma\}) : \lambda \in \Lambda\} \\ &= \cup\{\{w_\lambda(\lim_{n\to\infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(b)) : \sigma \in \Sigma\} : \lambda \in \Lambda\} \\ &= \cup\{\{\lim_{n\to\infty} w_\lambda \circ w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(b) : \sigma \in \Sigma\} : \lambda \in \Lambda\} \\ &= \cup\{\Gamma(\{\lambda\sigma : \sigma \in \Sigma\}) : \lambda \in \Lambda\} \\ &= \Gamma(\cup\{\{\lambda\sigma : \sigma \in \Sigma\} : \lambda \in \Lambda\}) = \Gamma(\Sigma) = A. \end{aligned}$$

Thus, A is self-similar under F_Ω, and, by Theorem 3.1, A is unique. □

So, in some sense, in the presence of the continuity of w, addressability is equivalent to self-similarity.

If Λ is finite, then w is trivially continuous, and so Ω is a stage for self-similarity by Theorem 3.2. Hutchinson [**4**, p. 724], Hata [**3**, p. 384], and Barnsley [**1**, p. 82] all have this finite case result.

4. Self-similar sets and hyperspace. In this section an approach to self-similarity other than addressing is taken. The self-similar set is viewed as a point in the hyperspace of compact subsets of X and is achieved as the fixed point of a suitable contraction on hyperspace. For finite Λ, this approach is indicated by Hutchinson [**4**, p. 728] and used extensively by Barnsley [**1**, p. 82]. Here the hyperspace approach is suggested by Corollary 2.8.

The set of all nonempty compact subsets of X is denoted by $H(X)$, and h is the Hausdorff metric on $H(X)$. Also, for $K \in H(X)$ and $\varepsilon > 0$, $M(K,\varepsilon)$ is $\cup\{N(x,\varepsilon) : x \in K\}$, and the function W is defined on $H(X)$ by $W(K) = \cup\{w_\lambda(K) : \lambda \in \Lambda\}$.

Theorem 4.1. *If w is continuous, then there exists a unique set A self-similar under F_Ω, and $W^n(K) \to A$ in $(H(X),h)$ for every $K \in H(X)$.*

Proof. By Theorem 3.2, there exists a unique set A self-similar under F_Ω. Let $K \in H(X)$. Since w is continuous, $W(K)$, which is $w(\Lambda \times K)$, is compact, and by induction it follows that $W^n(K) \in H(X)$ for every $n \in \mathbf{N}$ and $W^n(K) = \cup\{w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(K) : \sigma \in \Sigma\}$. Now let $\varepsilon > 0$. By Corollary 2.8, there is an $m \in \mathbf{N}$ such that, for every $k \geq m$, $W^k(K) \subset M(A,\varepsilon)$ and $A \subset M(W^k(K),\varepsilon)$. It follows that $h(A, W^k(K)) < \varepsilon$ and hence that $W^n(K) \to A$ in $(H(X),h)$. □

The above result may be achieved differently. If A is self-similar under F_Ω, then $W(A) = A$ and A is a fixed point of W; also, in the previous result the iterates of W converge to A. Thus, the Banach fixed point theorem is brought to mind. The details follow.

Proposition 4.2. *If $W(\{x\}) \in H(x)$ for every $x \in X$, then W is a contraction on $H(X)$ with contractivity factor s.*

Proof. Let $K \in H(X)$. Let $\{x_n\}$ be a sequence in $W(K)$; so there is a sequence $\{y_n\}$ in K and a sequence $\{\lambda_n\}$ in Λ such that $w_{\lambda_n}(y_n) = x_n$. Since K is compact, there is a subsequence $\{y_{n_k}\}$ and there is a $y \in K$ such that $y_{n_k} \to y$. Since $\{w(\lambda_{n_k}, y)\}$ is a sequence in the compact

$\{w_\lambda(y) : \lambda \in \Lambda\}$, it has a subsequence $\{w(\lambda_{t_n}, y)\}$ for which there is a $\tau \in \Lambda$ such that $w(\lambda_{t_n}, y) \to w_\tau(y)$. Also, $y_{t_n} \to y$.

Let $\varepsilon > 0$. There is a $p \in \mathbf{N}$ such that, for every $m \geq p$, $d(y_{t_m}, y) < \varepsilon/2s$. There is a $q \in \mathbf{N}$ such that, for every $m \geq q$, $d(w(\lambda_{t_m}, y), w_\tau(y)) < \varepsilon/2$. So, for every $m \geq \max\{p, q\}$,

$$\begin{aligned} d(w(\lambda_{t_m}, y_{t_m}), w_\tau(y)) &\leq d(w(\lambda_{t_m}, y_{t_m}), w(\lambda_{t_m}, y)) + d(w(\lambda_{t_m}, y), w_\tau(y)) \\ &< sd(y_{t_m}, y) + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence, $w(\lambda_{t_n}, y_{t_n}) \to w_\tau(y)$, and $W(K)$ is compact. Thus, W is a self-map on $H(X)$.

Let $K, L \in H(X)$. Let $\varepsilon > 0$. Let $\lambda \in \Lambda$, $x \in K$. There is a $y \in L$ such that $d(x, y) < h(K, L) + \varepsilon$. Then

$$d(w_\lambda(x), w_\lambda(y)) \leq sd(x, y) < sh(K, L) + s\varepsilon,$$

and $W(K) \subset M(W(L), sh(K, L) + s\varepsilon)$. Similarly, $W(L) \subset M(W(K), sh(K, L) + s\varepsilon)$. So $h(W(K), W(L)) < sh(K, L) + s\varepsilon$, and thus $h(W(K), W(L)) \leq sh(K, L)$. Therefore, W is a contraction on $H(X)$ with contractivity factor s. □

Theorem 4.3. *If $W(\{x\}) \in H(X)$ for every $x \in X$, then there exists a unique set A self-similar under F_Ω, and $W^n(K) \to A$ in $(H(X), h)$ for every $K \in H(X)$.*

Proof. The nonempty compact set A is self-similar under F_Ω if and only if $W(A) = A$. Since $(H(X), h)$ is complete (see [**5**, p. 45]) and W is a contraction on $H(X)$ by Proposition 4.2, the Banach fixed point theorem now applies. □

Although Theorem 4.3 has a weaker hypothesis than that of Theorem 4.1, it is to be seen that the two theorems are equivalent.

Proposition 4.4. *The following are all equivalent*:

i) *Theorem* 4.1,

ii) *Theorem* 4.3, *and*

iii) *the Banach fixed point theorem.*

Proof. The proof of Theorem 4.3 shows that (iii) implies (ii). If w is continuous, then $W(\{x\}) \in H(X)$, and so (ii) implies (i).

Now assume (i). If (Q, d_Q) is a complete metric space, and f is a nonconstant contradiction on Q, then $\Lambda \equiv \{1\}$, and $w : \Lambda \times Q \to Q$ is defined by $w(1, q) = f(q)$. Then $((Q, d_Q), w, \Lambda)$ is a contraction system, w is continuous, and, by (i), there exists a unique set A self-similar under $\{w_1\}$. Let $q \in Q$. Again, by (i), $W^n(\{q\}) \to A$ and so $\{f^n(q)\} \to A$. It must be then that A consists of a single point p and $f^n(q) \to p$.

Since $W(A) = A$, $f(p) = p$. This fixed point is unique, because if r is a fixed point of f, then $f(r) = r$, $f(\{p, r\}) = \{p, r\}$, $A = \{p, r\}$, and thus $p = r$. Therefore, (i) implies (iii). □

5. Continuous variation of self-similar sets. In this section it is shown that self-similar sets determined by compact subsets of Λ vary continuously as the index space varies. It is also shown that a self-similar set is the limit of self-similar sets with finite index spaces.

It is assumed in this section that Λ is metrizable by the metric d_Λ, in which case Σ is metrizable by the metric d_Σ defined by $d_\Sigma(\sigma, \alpha) = \sum_{i=1}^{\infty}(1/2^i)(d_\Lambda(\sigma_i, \alpha_i))$. If $\Psi \in H(\Lambda)$, then $\sum_\Psi$ is $\prod_{i=1}^{\infty} \Psi_i$ with the standard product topology, Ψ_i being Ψ for every $i \in \mathbf{N}$, and A_Ψ is the set self-similar under F_{Ω_Ψ}, Ω_Ψ being $((X, d), w \mid (\Psi \times X), \Psi)$. If Q is a metric space, then the Hausdorff metric for $H(Q)$ is denoted by h_Q.

In this section it is assumed that Λ is metrizable, w is continuous, and the set A is self-similar under F_Ω.

Theorem 5.1. *If $T : (H(\Lambda), h_\Lambda) \to (H(A), h)$ is defined by $T(\Psi) = A_\Psi$, then T is uniformly continuous.*

Proof. By Theorem 3.2, A_Ψ exists for every $\Psi \in H(\Lambda)$; by Theorem 2.3 (iii), since $\Phi_{\Omega_\Psi} = \Phi_\Omega \mid \Sigma_\Psi$, it follows that $A_\Psi \in H(A)$; so T is well-defined.

Let $\varepsilon > 0$. According to Theorem 2.10, Φ_Ω is uniformly continuous; so there is a $\delta > 0$ such that, for every $\sigma, \alpha \in \Sigma$, if $d_\Sigma(\sigma, \alpha) < \delta$, then $d(\Phi_\Omega(\sigma), \Phi_\Omega(\alpha)) < \varepsilon$. Now, if $\Upsilon, \Psi \in H(\Lambda)$ and $h_\Lambda(\Upsilon, \Psi) < \delta$, then $\Upsilon \subset M(\Psi, \delta)$. Let $\alpha \in \Sigma_\Upsilon$. So $\alpha_i \in \Upsilon$ for every $i \in \mathbf{N}$, and hence there is a $\beta_i \subset \Psi$ for which $d_\Lambda(\alpha_i, \beta_i) < \delta$. Then $\beta \in \Sigma_\Psi$, $d_\Sigma(\alpha, \beta) = \sum_{i=1}^{\infty}(1/2^i)(d_\Lambda(\alpha_i, \beta_i)) < \delta$, and thus $d(\Phi_\Omega(\alpha), \Phi_\Omega(\beta)) < \varepsilon$ and $\Phi_\Omega(\Sigma_\Upsilon) \subset M(\Phi_\Omega(\Sigma_\Psi), \varepsilon)$. Similarly, $\Phi_\Omega(\Sigma_\Psi) \subset M(\Phi_\Omega(\Sigma_\Upsilon), \varepsilon)$. Consequently, $h(\Phi_\Omega(\Sigma_\Upsilon), \Phi_\Omega(\Sigma_\Psi)) < \varepsilon$, $h(A_\Upsilon, A_\Psi) < \varepsilon$ by Theorem 2.3 (iii), and T is uniformly continuous. □

The next result shows that, for any self-similar set, there is an arbitrarily close set in hyperspace which is self-similar under a finite set of contractions.

Theorem 5.2. *There exists an increasing sequence $\{\Lambda_n\}$ of finite sets in $H(\Lambda)$ for which $\{A_{\Lambda_n}\}$ is increasing and $A_{\Lambda_n} \to A$ in $(H(A), h)$.*

Proof. Since Λ is a compact metric space, for every $n \in \mathbf{N}$ there is a $1/n$-net Ψ_n in Λ. Define $\Lambda_n \equiv \cup_{i=1}^{n} \Psi_i$ for every $n \in \mathbf{N}$. Then, for every $n \in \mathbf{N}$, Λ_n is a $1/n$-net in Λ and $\Lambda_n \subset \Lambda_{n+1}$. By Theorem 3.2 and Theorem 2.3 (iii), the sequence $\{A_{\Lambda_n}\}$ exists and is increasing. Because for every $n \in \mathbf{N}$, $\Lambda_n \subset \Lambda$ and $\Lambda \subset M(\Lambda_n, 1/n)$, it follows that $h_\Lambda(\Lambda, \Lambda_n) < 1/n$. Then $\Lambda_n \to \Lambda$ in $(H(\Lambda), h_\Lambda)$, and by Theorem 5.1, $A_{\Lambda_n} \to A$ in $(H(A), h)$.

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