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ON GENERALIZED ABSOLUTE CESÀRO SUMMABILITY FACTORS

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ABSTRACT. In this paper using δ-quasi-monotone sequences a theorem on $|C,\alpha|_k$, $0 < \alpha \leq 1$, summability factors of infinite series, which generalizes a theorem of Mazhar [**6**] on $|C,1|_k$ summability factors, has been proved.

1. Introduction. A sequence (b_n) of positive numbers is said to be quasi-monotone if $n\Delta b_n \geq -\beta b_n$ for some β and is said to be δ-quasi-monotone, if $b_n \to 0$, $b_n > 0$ ultimately and $\Delta b_n \geq -\delta_n$, where (δ_n) is a sequence of positive numbers (see [**1**]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n^α and t_n^α the n-th Cesàro means of order α, $\alpha > -1$, of the sequences (s_n) and (na_n), respectively, i.e.,

$$u_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \tag{1.1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1.2}$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = \mathcal{O}(n^\alpha), \qquad \alpha > -1, \quad A_0^\alpha = 1 \quad \text{and} \quad A_{-n}^\alpha = 0 \quad \text{for } n > 0. \tag{1.3}$$

The series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \geq 1$, if (see [**3**])

$$\sum_{n=1}^{\infty} n^{k-1}|u_n^\alpha - u_{n-1}^\alpha|^k < \infty. \tag{1.4}$$

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But since $t_n^\alpha = n(u_n^\alpha - u_{n-1}^\alpha)$ (see [**4**]) condition (1.4) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty. \tag{1.5}$$

Mazhar [**5**] has proved the following theorem for $|C,1|_k$ summability factors.

Theorem A. *Let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $\sum n\delta_n \log n < \infty$, $\sum B_n \log n$ is convergent and $|\Delta\lambda_n| \leq |B_n|$ for all n.*

If

$$\sum_{n=1}^{m} \frac{1}{n} |t_n^1|^k = \mathcal{O}(\log m) \qquad \text{as } m \to \infty, \tag{1.6}$$

where (t_n^1) is the n-th $(C,1)$ mean of the sequence (na_n), then the series $\sum a_n \lambda_n$ is summable $|C,1|_k$, $k \geq 1$.

Section 2. The aim of this paper is to generalize Theorem A for $|C,\alpha|_k$ summability. Now, we shall prove the following theorem.

Theorem. *Let $0 < \alpha \leq 1$ and $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $\sum n\delta_n \log n < \infty$, $\sum B_n \log n$ is convergent and $|\Delta\lambda_n| \leq |B_n|$ for all n. If the sequence (w_n^α), defined by*

$$w_n^\alpha = \begin{cases} |t_n^\alpha n|, & \alpha = 1 \\ \max_{1 \leq v \leq n} |t_v^\alpha|, & 0 < \alpha < 1, \end{cases} \tag{2.1}$$

satisfies the condition

$$\sum_{n=1}^{m} \frac{1}{n} (w_n^\alpha)^k = \mathcal{O}(\log m) \qquad \textit{as } m \to \infty, \tag{2.2}$$

then the series $\sum a_n \lambda_n$ *is summable* $|C, \alpha|_k$, $k \geq 1$.

It should be noted that, if we take $\alpha = 1$ in this theorem, then we get Theorem A. In fact, in this case the condition (2.1) is reduced to the condition (1.6).

Section 3. We need the following lemmas for the proof of our theorem.

Lemma 1 ([**2**]). *If* $0 < \alpha \leq 1$ *and* $1 \leq v \leq n$, *then*

$$\left| \sum_{p=1}^{v} A_{n-p}^{\alpha-1} a_p \right| \leq \max_{1 \leq m \leq v} \left| \sum_{p=1}^{m} A_{m-p}^{\alpha-1} a_p \right|, \tag{3.1}$$

where A_n^α *is as in* (1.3).

Lemma 2 ([**5**]). *If* (B_n) *is* δ*-quasi-monotone with* $\sum n\delta_n \log n < \infty$ *and* $\sum B_n \log n$ *is convergent, then*

$$mB_m \log m = \mathcal{O}(1) \qquad \text{as } m \to \infty \tag{3.2}$$

$$\sum_{n=1}^{\infty} n \log n |\Delta B_n| < \infty. \tag{3.3}$$

Proof of the Theorem. Let (T_n^α) be the n-th (C, α) mean of the sequence $(na_n\lambda_n)$, where $0 < \lambda \leq 1$. Then, by (1.2), we have

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v.$$

By Abel's transformation, we have

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda n}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v,$$

so that, making use of Lemma 1, we get

$$\begin{aligned}|T_n^\alpha| &\le \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}|\Delta\lambda_v|\left|\sum_{p=1}^{v}A_{n-p}^{\alpha-1}pa_p\right| \\ &\quad + \frac{|\lambda_n|}{A_n^\alpha}\left|\sum_{v=1}^{n}A_{n-v}^{\alpha-1}va_v\right| \\ &\le \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}A_v^\alpha w_v^\alpha|\Delta\lambda_v| + |\lambda_n|w_n^\alpha \\ &= T_{n,1}^\alpha + T_{n,2}^\alpha, \text{ say.}\end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality for $k>1$, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{1}{n}|T_{n,r}^\alpha|^k < \infty, \qquad \text{for } r=1,2, \text{ by (1.5).}$$

Now, when $k>1$, applying Hölder's inequality with indices k and k', where $1/k+1/k'=1$, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\frac{1}{n}|T_{n,1}^\alpha|^k &\le \sum_{n=2}^{m+1}\frac{1}{n(A_n^\alpha)^k}\left\{\sum_{v=1}^{n-1}A_v^\alpha w_v^\alpha|B_v|\right\}^k \\ &= \mathcal{O}(1)\sum_{n=2}^{m+1}n^{-1}n^{-\alpha_k}\left\{\sum_{v=1}^{n-1}A_v^\alpha(w_v^\alpha)^k|B_v|\right\} \\ &\qquad \cdot\left\{\sum_{v=1}^{n-1}A_v^\alpha|B_v|\right\}^{k-1} \\ &= \mathcal{O}(1)\sum_{n=2}^{m+1}n^{-1-\alpha k}(n^\alpha)^{k-1}\left\{\sum_{v=1}^{n-1}v^\alpha(w_v^\alpha)^k|B_v|\right\} \\ &\qquad \cdot\left\{\sum_{v=1}^{n-1}|B_v|\right\}^{k-1} \\ &= \mathcal{O}(1)\sum_{n=2}^{m+1}n^{-1-\alpha}\left\{\sum_{v=1}^{n-1}v^\alpha(w_v^\alpha)^k|B_v|\right\}\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{O}(1) \sum_{v=1}^{m} v^{\alpha} (w_v^{\alpha})^k |B_v| \sum_{n=v+1}^{m+1} \frac{1}{n^{1+\alpha}} \\
&= \mathcal{O}(1) \sum_{v=1}^{m} v^{\alpha} (w_v^{\alpha})^k |B_v| \int_v^{\infty} \frac{dx}{x^{1+\alpha}} \\
&= \mathcal{O}(1) \sum_{v=1}^{m} (w_v^{\alpha})^k |B_v| = \mathcal{O}(1) \sum_{v=1}^{m} v |B_v| \frac{(w_v^{\alpha})^k}{v} \\
&= \mathcal{O}(1) \sum_{v=1}^{m-1} \Delta(v|B_v|) \sum_{r=1}^{v} \frac{1}{r} (w_r^{\alpha})^k + \mathcal{O}(1) \sum_{v=1}^{m} \frac{1}{v} (w_v^{\alpha})^k \\
&= \mathcal{O}(1) \sum_{v=1}^{m-1} |\Delta(v|B_v|) \log v + \mathcal{O}(1) m |B_m| \log m \\
&= \mathcal{O}(1) \sum_{v=1}^{m-1} v |\Delta B_v| \log v \\
&\quad + \mathcal{O}(1) \sum_{v=1}^{m-1} |B_{v+1}| \log v + \mathcal{O}(1) m |B_m| \log m \\
&= \mathcal{O}(1) \qquad \text{as } m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses and Lemma 2. Again, we have that

$$
\begin{aligned}
\sum_{n=1}^{m} \frac{1}{n} |T_{n,2}^{\alpha}|^k &= \sum_{n=1}^{m} |\lambda_n| \, |\lambda_n|^{k-1} \frac{1}{n} (w_n^{\alpha})^k \\
&= \mathcal{O}(1) \sum_{n=1}^{m} |\lambda_n| \frac{1}{n} (w_n^{\alpha})^k \\
&= \mathcal{O}(1) \sum_{n=1}^{m} \frac{1}{n} (w_n^{\alpha})^k \sum_{v=n}^{\infty} |\Delta \lambda_v| \\
&= \mathcal{O}(1) \sum_{v=1}^{\infty} |\Delta \lambda_v| \sum_{n=1}^{v} \frac{1}{n} (w_n^{\alpha})^k \\
&= \mathcal{O}(1) \sum_{v=1}^{\infty} |B_v| \log v < \infty
\end{aligned}
$$

by virtue of the hypotheses of the theorem. Therefore, we get that

$$\sum_{n=1}^{m} \frac{1}{n} |T_{n,r}^{\alpha}|^k = \mathcal{O}(1) \qquad \text{as } m \to \infty, \quad \text{for } r = 1, 2.$$

This completes the proof of the theorem. □

Remark. It is natural to ask whether our theorem is true with $\alpha > 1$. All that we can say with certainty is that our proof fails if $\alpha > 1$, for our estimate of $T_{n,1}^{\alpha}$ depends upon Lemma 1, and Lemma 1 is known to be false when $\alpha > 1$ (see [**2**] for details).

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