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ON THE SECOND EIGENVALUE AND THE SECOND EIGENFUNCTION OF A CLASS OF SELFADJOINT SECOND ORDER LINEAR SYSTEMS

CHAO-LIANG SHEN

ABSTRACT. The multiplicity of the second eigenvalue and the sign properties of the components of the second eigenfunction of the second order linear system $y''+\lambda P(X)y = 0$, $y(-1/2) = y(1/2) = 0$, where $P(x)$ is a 2×2 positive definite matrix-valued function with positive entries, are investigated. Sufficient conditions for the nondegeneracy of the second eigenvalue and the constant sign property for the components of the second eigenfunction are obtained.

1. Introduction. Let $P(x)$ be a continuous real $n \times n$ positive definite matrix-valued function defined on the interval $a \le x \le b$. Suppose $P(x) = (p_{ij}(x))$, $p_{ij}(x) > 0$ for $x \in [-1/2, 1/2]$. It is interesting to consider the following eigenvalue problem

$$(1.1)\qquad \begin{aligned} y''(x) + \lambda P(x)y(x) = 0,\\ y(-1/2) = y(1/2) = 0, \end{aligned}$$

where $y(x)$ is an $\mathbf{R}^n$-valued function. The eigenvalue problem (1.1) is much more complicated than the classical scalar Sturm-Liouville eigenvalue problem. For the scalar Sturm-Liouville problem, all eigenvalues are nondegenerate, and the number of nodal points of the n^{th} eigenfunction is related to n in a clear way. But these scalar results are no longer true for (1.1) except for the first eigenvalue λ_1. In [**1,2,3**] S. Ahmad and A.C. Lazer proved some interesting results which tell us that under the assumption on the coefficient matrix $P(x)$ the first eigenvalue λ_1 of (1.1) is nondegenerate, and the components of the first

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eigenfunction have the same sign and are nonvanishing in the interval $a < x < b$.

In this paper we study the second eigenvalue and the second eigenfunctions of (1.1). For technical reasons we only consider the case $n = 2$. We find conditions which guarantee the nondegeneracy of the second eigenvalue λ_2, and we also find that, under some further conditions, we can describe the sign of the components of the second eigenfunctions of (1.1) explicitly. To be more precise, we consider (1.1) with

$$P(x) = \begin{bmatrix} a(x) & b(x) \\ b(x) & c(x) \end{bmatrix}, \tag{1.2}$$

where $a(x), b(x)$ and $c(x)$ are strictly positive C^1-functions defined on the interval $-1/2 \le x \le 1/2$. For each x, we use $\rho_1(x)$ and $\rho_2(x)$ to denote the largest and the smallest characteristic values of $P(x)$, respectively. For a given scalar-valued positive function $\rho(x)$ we use the notation $\nu_n[\rho]$ to denote the n^{th} eigenvalue of the *string equation with density function* $\rho(x)$:

$$\begin{gathered} u''(x) + \nu\rho(x)u(x) = 0, \\ u(-1/2) = u(1/2) = 0. \end{gathered} \tag{1.3}$$

We prove the following two theorems among other results:

Theorem 2.2. *Suppose $P(x)$ is of the form* (1.2) *and is positive definite with positive entries for each x in $[-1/2, 1/2]$. If $\nu_2[\rho_1] > \nu_1[\rho_2]$, then the second eigenvalue of* (1.1) *is nondegenerate.*

Theorem 3.1. *Suppose $b(x)$ is a real analytic symmetrically decreasing function in $[-1/2, 1/2]$, $a(x)$ and $c(x)$ in* (1.2) *are constants a and c, and $a > c > b(x) > 0$ for all x in $[-1/2, 1/2]$. Let $v(x) = \operatorname{col}(v_1(x), v_2(x))$ denote a second eigenfunction of* (1.1). *If $\nu_2[a+b] \ge \nu_1[\rho_2]$, then the second eigenvalue of* (1.1) *is nondegenerate and $v_1(x)v_2(x) < 0$ for all x in the interval $(-1/2, 1/2)$.*

In Section 2 we present the proof of Theorem 2.2. In Section 3, by using the Ahmad-Lazer comparison theorem and some detailed analysis about the second eigenfunction, we prove Theorem 3.1.

2. The multiplicity of the second eigenvalue. Let $P(x)$ be the 2×2 positive definite matrix

$$\begin{bmatrix} a(x) & b(x) \\ b(x) & c(x) \end{bmatrix},$$

where $a(x)$, $b(x)$ and $c(x)$ are C^1-positive functions on $[-1/2, 1/2]$. The (geometric) multiplicity of the second eigenvalue λ_2 of (1.1), or more precisely, of

$$\begin{gathered} y_1'' + \lambda a(x) y_1 + \lambda b(x) y_2 = 0, \\ y_2'' + \lambda b(x) y_1 + \lambda c(x) y_2 = 0, \\ y_j(-1/2) = y_j(1/2) = 0, \qquad j = 1, 2, \end{gathered} \tag{2.1}$$

is not as clear as its Sturm-Liouville analogy, even in the case when $a(x)$, $b(x)$ and $c(x)$ are constants. The situation is explained in the following theorem.

Theorem 2.1. *P as above. Suppose a, b and c are positive constants and $\rho_1 > \rho_2$ are the two characteristic values of P. Then*

(i) *the second eigenvalue λ_2 of (2.1) is nondegenerate if $4\rho_2 > \rho_1$;*

(ii) *the multiplicity of λ_2 is 2 if $4\rho_2 = \rho_1$.*

Proof. Let u, v be the normalized characteristic vectors of P belonging to the characteristic values ρ_1, ρ_2, respectively. Using u and v to construct an orthogonal matrix U whch diagonalizes P:

$$U^{-1} P U = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}.$$

Let $z(x) = U^{-1} y(x)$. Then (3.1) is diagonalized into the following equations

$$\begin{gathered} z_1'' + \lambda \rho_1 z_1 = 0, \qquad z_2'' + \lambda \rho_2 z_2 = 0, \\ z_i(-1/2) = z_i(1/2) = 0, \qquad i = 1, 2. \end{gathered} \tag{2.2}$$

Now it is clear that the sequence of eigenvalues of λ_n of (2.1) is just the sequence which consists of $\nu_n[\rho_1], \nu_m[\rho_2]$, which are arranged in ascending order. From this sequence we can imagine the possibility of degeneracy of the eigenvalues of (2.1). Notice that $\nu_n[\rho_1] = n^2\pi^2/\rho_1$, $\nu_m[\rho_2] = m^2\pi^2/\rho_2$.

For the λ_1, it follows from the works of Ahmad and Lazer ([**3**, Proposition 2], [**1**, Theorem 1]) or, under our assumption, just observing the ascending sequence consists of $\nu_n[\rho_1]$ and $\nu_m[\rho_2]$, that $\lambda_1 = \pi^2/\rho_1$ is nondegenerate. As for λ_2, the Minmax principle tells us that

$$\nu_n[\rho_1] < \nu_n[\rho_2], \qquad n = 1, 2, \dots . \tag{2.3}$$

Thus, if $\nu_1[\rho_2] < \nu_2[\rho_1]$, then (2.3) implies that

$$\nu_1[\rho_1] < \nu_1[\rho_2] < \nu_n[\rho_1]$$

for all $n \geq 2$. Hence, if $\nu_1[\rho_2] < \nu_2[\rho_1]$, then $\lambda_2 = \nu_1[\rho_2]$ and it is nondegenerate. Since $\nu_1[\rho_2] = \pi^2/\rho_2$, $\nu_2[\rho_1] = 4\pi^2/\rho_1$, (i) follows from the last statement. As for (ii), simply notice that $4\rho_2 = \rho_1$ implies that $\nu_1[\rho_2] = \nu_2[\rho_1]$. □

The result in (the proof of) Theorem 2.1 (i) can be generalized as follows.

Theorem 2.2. *For each x in $[-1/2, 1/2]$, let $\rho_1(x)$, respectively $\rho_2(x)$, denote the largest, respectively smallest, characteristic value of the 2×2 positive definite matrix-valued function $P(x)$. Suppose $a(x), b(x)$ and $c(x)$ are strictly positive in $[-1/2, 1/2]$ and*

$$\nu_1[\rho_2] < \nu_2[\rho_1].$$

Then the second eigenvalue λ_2 of (3.1) is nondegenerate. In particular, if $\max \rho_1 < 4 \min \rho_2$, then λ_2 is nondegenerate, where $\max \rho_1$ (respectively, $\min \rho_2$) denotes the maximum of ρ_1 (respectively, minimum of ρ_2) in the interval $-1/2 \leq x \leq 1/2$.

Proof. For $\xi = \mathrm{col}\,(\xi_1, \xi_2)$ in $\mathbf{R}^2$, we have

$$\rho_2(x)[\xi_1^2 + \xi_2^2] \leq \langle P(x)\xi, \xi\rangle \leq \rho_1(x)[\xi_1^2 + \xi_2^2], \tag{2.4}$$

where $\langle u, v\rangle$ denotes the inner product of the vectors u and ν. Thus, if we use $D_i(x)$ to denote the 2×2 diagonal matrix with diagonal elements equal to $\rho_i(x)$, and use the notation $\alpha_n^{(i)}$ to denote the nth eigenvalue of the equation

$$v'' + \alpha^{(i)} D_i(x) v = 0, \qquad v(-1/2) = v(1/2) = 0,$$

then

$$\langle D_2(x)\xi, \xi\rangle \le \langle P(x)\xi, \xi\rangle \le \langle D_1(x)\xi, \xi\rangle.$$

Hence, by the minmax principle, we have

$$\alpha_n^{(2)} \ge \lambda_n \ge \alpha_n^{(1)}.$$

Since $\alpha_{2n-1}^{(i)} = \alpha_{2n}^{(i)} = \nu_n[\rho_i]$, we have

$$\lambda_2 \le \nu_1[\rho_2], \qquad \nu_2[\rho_1] \le \lambda_3. \tag{2.5}$$

Thus, if $\nu_1[\rho_2] < \nu_2[\rho_1]$, we have $\lambda_1 \le \lambda_2 < \lambda_3$. On the other hand, as $a(x), b(x)$ and $c(x)$ are strictly positive, Ahmad and Lazer's theorem ([**3**, Proposition 2], [**1**, Theorem 1]) tells us that λ_1 is nondegenerate. Thus, $\lambda_1 < \lambda_2 < \lambda_3$, i.e., λ_2 is nondegenerate. This proves the first part of the theorem.

As for the second part of the theorem, we notice that $\min \rho_2 \le \rho_2(x)$, $\rho_1(x) \le \max \rho_1$ imply

$$\nu_1[\rho_2] \le \nu_1[\min \rho_2], \qquad \nu_2[\max \rho_1] \le \nu_2[\rho_1].$$

Since $\nu_1[\min \rho_2] = \pi^2/\min \rho_2$, $\nu_2[\max \rho_1] = 4\pi^2/\max \rho_1$. Thus, if $\max \rho_1 < 4 \min \rho_2$, then $\nu_1[\rho_2] < \nu_2[\rho_1]$, and the second part follows from the result in the first part. □

The readers might notice that the matrix P we discussed in Theorem 2.2 is an *oscillatory* matrix. As for oscillatory matrices we have very precise information about the sign of the components of their characteristic vectors (see [**4, 5**]), for example, the two components of the characteristic vector corresponding to the smallest characteristic value of a 2×2 oscillatory matrix have opposite signs.

Now let P be as that in Theorem 2.1 (i) and u, v, U be as those in the proof of Theorem 2.1(i). Let $u = \mathrm{col}\,(u_1, u_2)$, $v = \mathrm{col}\,(v_1, v_2)$, where $u_1, u_2 > 0$, $v_1 = -u_2$, $v_2 = u_1$. Then, for the constant matrix P, the second eigenvalue of (2.1) is π^2/ρ_2, and the corresponding eigenfunction is

$$y(x) = \begin{bmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{bmatrix} \begin{pmatrix} 0 \\ \cos \pi x \end{pmatrix} = \begin{pmatrix} -u_2 & \cos \pi x \\ u_1 & \cos \pi x \end{pmatrix};$$

the two components of $y(x)$ have opposite signs for $-1/2 < x < 1/2$. Does this phenomenon also appear when $P(x)$ is not a constant matrix? We shall study this question in the next section.

3. The components of the second eigenfunction. To answer the question at the end of the last section, we shall prove the following theorem.

Theorem 3.1. *Suppose a and c are positive constants, b is a positive real analytic function on $-1/2 \le x \le 1/2$ such that*

(i) $a > c > b(x) > 0$, $-1/2 \le x \le 1/2$,

(ii) $b(-x) = b(x)$, $-1/2 \le x \le 1/2$, *and $b(x)$ is decreasing in* $[0, 1/2]$

(iii) $\nu_2[a+b] \ge \nu_1[\rho_2]$, *where $\rho_2(x)$ is the smallest characteristic value of the matrix $P(x)$.*

Let $(\lambda_2, v(x))$ denote the second eigenpair of (3.1), $v(x) = \mathrm{col}\,(v_1(x), v_2(x))$. *Then $v_1(x)v_2(x) < 0$ in the interval $-1/2 < x < 1/2$.*

Notice that the condition (i) implies $a + b > \rho_1$, then the condition (iii) and Theorem 2.1 imply that the second eigenvalue λ_2 in Theorem 3.1 is nondegenerate.

To prove this theorem, we shall need a sequence of lemmas. The first one was due to S. Ahmad and A.C. Lazer (see [**3**, Theorem A, Theorem C and Proposition 2]) but is reformulated according to our purpose.

Proposition 3.2. *Suppose $a(x), b(x), c(x)$ are positive functions on* $[-1/2, 1/2]$ *as those in* (2.1). *Let λ_1 be the first eigenvalue of* (2.1). *Then, for $\mu > \lambda_1$, there is a nontrivial solution* $z(x) = \mathrm{col}\,(z_1(x), z_2(x))$ *of*

$$z''(x) + \mu P(x) z(x) = 0 \tag{3.1}$$

such that $z(-1/2) = z(\beta) = 0$ where

(i) $-1/2 < \beta < 1/2$,

(ii) $z_i(x) > 0$, $-1/2 < x < \beta$, $i = 1, 2$,

(iii) $z_i'(-1/2) > 0$, $z_i'(\beta) < 0$, $i = 1, 2$.

Furthermore, β is the first conjugate point of $-1/2$ relative to (3.1).

We refer the proof of Proposition 3.2 to [**3**].

Lemma 3.3. *Let λ_1 be the first eigenvalue of* (2.1). *Then $\nu_1[\rho_1] \leq \lambda_1 \leq \nu_1[a]$.*

Proof. Let $u(x) = \mathrm{col}\,(u_1(x), u_2(x))$ denote the first eigenfunction of (2.1). Then

$$\frac{\int_{-1/2}^{1/2} (u_i')^2\, dx}{\int_{-1/2}^{1/2} \rho_1 u_i^2\, dx} \geq \nu_1[\rho_1], \qquad i = 1, 2.$$

Hence, the second inequality of (2.4) implies that $\lambda_1 \geq \nu_1[\rho_1]$.

Let $\varphi_1(x)$ be the first eigenfunction $\varphi'' + \nu a(x)\varphi = 0$, $\varphi(-1/2) = \varphi(1/2) = 0$. Let $u(x) = \mathrm{col}\,(\varphi_1(x), 0)$. Then

$$\nu_1[a] = \frac{\int_{-1/2}^{1/2} (\varphi_i')^2\, dx}{\int_{-1/2} a\varphi_1^2\, dx} = \frac{\int_{-1/2}^{1/2} |u_i'|^2\, dx}{\int_{-1/2}^{1/2} < Pu, u > dx} \geq \lambda_1. \qquad \square$$

We note that Lemma 3.3 is independent of the interval $[-1/2, 1/2]$, i.e., if we replace $[-1/2, 1/2]$ by $[\alpha, \beta]$, the result of Lemma 3.3 still holds.

Proposition 3.4. *We make the same assumptions as those in the first part of Theorem* 2.2, *and we also assume that $a(x), b(x)$ and $c(x)$ are even functions in $-1/2 \leq x \leq 1/2$. Then both components of the second eigenfunction of* (2.1) *are even functions.*

Proof. Let $(\lambda_2, v(x))$ denote the second eigenpair of (2.1), $v(x) = \mathrm{col}\,(v_1(x), v_2(x))$. Since $a(x), b(x)$ and $c(x)$ are even functions in

$-1/2 \le x \le 1/2$, the function $v(-x)$ is also an eigenfunction of the eigenvalue λ_2. Since λ_2 is nondegenerate, we have $v(-x) = \alpha v(x)$, wehre α is a real constant. $\alpha = 1$ or -1 since $\int_{-1/2}^{1/2} |v(-x)|^2\,dx = \int_{-1/2}^{1/2} |v(x)|^2\,dx$. Suppose that $\alpha = -1$. Then we have $v(0) = 0$. Let $w(x)$ be the restriction of $v(x)$ on $0 \le x \le 1/2$. Then (λ_2, w) is an eigenpair of the following problem:

$$z'' + \mu P(x) z = 0, \qquad z(0) = z(1/2) = 0.$$

Let μ_1 be the first eigenvalue of this problem. Then $\lambda_2 \ge \mu_1$. On the other hand, by Lemma 3.3 we have $\mu_1 \ge \nu_1[\rho_1|_{[0,1/2]}]$. Since $a(x), b(x)$ and $c(x)$ are even functions in $-1/2 \le x \le 1/2$, ρ_1 is an even function. Hence, $\nu_1[\rho_1|_{[0,1/2]}] = \nu_2[\rho_1]$, the latter is the second eigenvalue of the string equation $y'' + \nu\rho_1 y = 0$, $y(-1/2) = y(1/2) = 0$. Thus, $\lambda_2 \ge \nu_2[\rho_1]$, and by the assumptions and (2.5) we have $\lambda_2 \ge \nu_2[\rho_1] > \nu_1[\rho_2] \ge \lambda_2$, which is absurd. Thus, $v(x) = v(-x)$, i.e., both components of $v(x)$ are even functions. □

Lemma 3.5. *We make the assumptions of Proposition* 3.4. *If* $v(x) = \mathrm{col}\,(v_1(x), v_2(x))$ *is the second eigenfunction of* (3.1), *then* $\mathrm{col}\,(v_1'(-1/2), v_2'(-1/2)) \ne \mathrm{col}\,(0,0)$.

Since (2.1) can be written as a first order system, Lemma 3.5 is just a consequence of $\mathrm{col}\,(v_1(-1/2), v_2(-1/2)) = \mathrm{col}\,(0,0)$ and the uniqueness theorem of first order system.

From now on, the notation $v(x) = \mathrm{col}\,(v_1(x), v_2(x))$ shall be reserved for the second eigenfunction of (3.1). The assumption in Proposition 3.4 will also apply to the following lemmas.

Lemma 3.6. *If the* $a(x), b(x), c(x)$ *in* (2.1) *are real analytic functions in* $-1/2 \le x \le 1/2$, *then there exists* $\varepsilon > 0$ *such that* $v_1(x) \ne 0$, $v_2(x) \ne 0$ *for all* x *in the interval* $(-1/2, -1/2 + \varepsilon)$.

Proof. By the assumptions on $a, b, c, v_1(x), v_2(x)$ can be expanded into Taylor series with respect to $x = -1/2$. If the assertion failed, then either $v_1(x)$ or $v_2(x)$ would vanish identically in $(-1/2, 1/2)$, which would imply that $v(x) = \mathrm{col}\,(0,0)$ for all x in $[-1/2, 1/2]$, which is absurd. □

Lemma 3.7. *We make the assumptions as in Proposition* 3.4 *and Lemma* 3.6. *If for the second eigenfunction $v(x)$ of* (2.1) *we know that there exists an $\varepsilon > 0$ so that*

$$v_1(x)v_2(x) < 0 \quad \text{in } -1/2 < x < -1/2 + \varepsilon,$$

then

$$v_1(x)v_2(x) < 0 \quad \text{in } -1/2 < x < 1/2.$$

Proof. By Proposition 3.4, both $v_1(x)$ and $v_2(x)$ are even functions. Thus, if the assertion failed, then there would exist $-1/2 < x_0 \le 0$ such that $v_1(x)v_2(x) < 0$ for $-1/2 < x < x_0$ and $v_1(x_0)v_2(x_0) = 0$. Assume that $v_1(x_0) = 0$, the other case can be treated similarly. We may also assume $v_1(x) > 0$, $v_2(x) < 0$ in $-1/2 < x < x_0$. Then, by (2.1), we have

$$\lambda_2 \ge \frac{\int_{-1/2}^{x_0} (v_i')^2\, dx}{\int_{-1/2}^{x_0} a v_1^2\, dx}.$$

Since $-1/2 < x_0 \le 0$, let $w(x) = v_1(x)$ for $-1/2 \le x \le x_0$, $w(x) = 0$ for $x_0 \le x \le 0$. Then

$$\frac{\int_{-1/2}^{x_0} (v_1')^2\, dx}{\int_{-1/2}^{x_0} a v_1^2\, dx} = \frac{\int_{-1/2}^{0} (w')^2\, dx}{\int_{-1/2}^{0} a w^2\, dx} \ge \tilde{\nu}_1[a],$$

where $\tilde{\nu}_1[a]$ is the first eigenvalue of $w'' + \tilde{\nu} a w = 0$, $w(-1/2) = w(0) = 0$. Since $a(x)$ is an even function on $[-1/2, 1/2]$, $\tilde{\nu}_1[a] = \nu_2[a]$ where $\nu_2[a]$ is the second eigenvalue of the string equation $w'' + \nu a w = 0$, $w(-1/2) = w(1/2) = 0$. Thus, if the x_0 exists, then we have

$$\lambda_2 \ge \nu_2[a].$$

But as $\rho_1 > a > \rho_2$, and by the assumption and (2.5) we have $\nu_2[a] > \nu_2[\rho_1] > \nu_1[\rho_2] \ge \lambda_2$, which lead to $\lambda_2 > \lambda_2$, a contradiction. Hence, $v_1(x)v_2(x) < 0$ in $-1/2 < x < 1/2$. □

Now we present the proof of Theorem 3.1.

Proof of Theorem 3.1. If there exists $\varepsilon > 0$ such that $v_1(x)v_2(x) < 0$ in $-1/2 < x < -1/2 + \varepsilon$, then by Lemma 3.7 we are done. We prove

Theorem 3.1 by contradiction. Thus, we assume that $v_1(x)v_2(x) > 0$ near $-1/2$.

Let β be the first conjugate point of $-1/2$ relative to the equation $z'' + \lambda_2 Pz = 0$. We have $0 < \beta < 1/2$ by the evenness of $P(x)$ and the nondegeneracy of λ_2 for (2.1). Let $z(x) = \mathrm{col}\,(z_1(x), z_2(x))$ be the positive solution of $z'' + \lambda_2 Pz = 0$, $z(-1/2) = z(\beta) = 0$. Notice that, by Proposition 3.2, $z'(-1/2)$ is positive. $v'(-1/2)$ and $z'(-1/2)$ are linearly independent. Furthermore, since $z' \cdot v = v' \cdot z$ in $-1/2 \le x \le \beta$, $z(\beta) = 0$ and $z'(\beta)$ is negative, we have $v_1(\beta)v_2(\beta) < 0$, hence $v_1(-\beta)v_2(-\beta) < 0$ since both $v_1(x)$ and $v_2(x)$ are even functions by Proposition 3.4. For the vectors $z'(-1/2)$ and $v'(-1/2)$ there are two possible cases:

Case 1. $v_2'(-1/2)/z_2'(-1/2) > v_1'(-1/2)/z_1'(-1/2)$,

Case 2. $v_1'(-1/2)/z_1'(-1/2) > v_2'(-1/2)/z_2'(-1/2)$.

We only discuss Case 1, the argument for Case 2 is similar. Let

$$\xi = v_1'(-1/2)/z_1'(-1/2), \qquad \eta = v_2'(-1/2)/z_2'(-1/2).$$

Then $\eta > \xi \ge 0$. For γ in the interval (ξ, η), let

$$\begin{aligned} w_{\gamma,1}(x) &= \gamma z_1(x) - v_1(x), \\ w_{\gamma,2}(x) &= \gamma z_2(x) - v_2(x), \\ w_\gamma(x) &= \mathrm{col}\,(w_{\gamma,1}, w_{\gamma,2}). \end{aligned}$$

Notice that $w_{\gamma,1}(x) > 0$, $w_{\gamma,2}(x) < 0$, near $-1/2$.

If there is a γ in (ξ, η) such that there exists $-1/2 < x_0 \le 0$ so that $w_{\gamma,1}(x)w_{\gamma,2}(x) < 0$ in $(-1/2, x_0)$ and $w_{\gamma,1}(x_0)w_{\gamma,2}(x_0) = 0$, we may assume $w_{\gamma,1}(x_0) = 0$, the other case is similar. Then, since $b(x) > 0$, $w_{\gamma,2}(x) < 0$ in $-1/2 < x < x_0$, and

$$w_\gamma'' + \lambda_2 P w_\gamma = 0,$$

we have

$$w_{\gamma,1}'' + \lambda_2 a w_{\gamma,1} \ge 0, \qquad w_{\gamma,1}(-1/2) = w_{\gamma,1}(x_0) = 0.$$

This inequality and $w_{\gamma,1} > 0$ in $-1/2 < x < x_0$ imply that

$$\lambda_2 \ge \frac{\int_{-1/2}^{x_0} (w_{\gamma,1}')^2\,dx}{\int_{-1/2}^{x_0} a w_{\gamma,1}^2\,dx} \ge \nu_2[a],$$

where $\nu_2[a]$ is the second eigenvalue of the string equation with density $a(x)$ in $-1/2 \le x \le 1/2$. This is absurd since $\nu_2[a] > \nu_2[\rho_1]$ and $\lambda_2 < \nu_2[\rho_1]$ by the proof of Theorem 2.2. Thus, $w_{\gamma,1}(x)w_{\gamma,2}(x) < 0$ in $(-1/2, 0]$ for all γ in (ξ, η). We then have $w_{\gamma,1}(x) > 0$ and $w_{\gamma,2}(x) < 0$ in $(-1/2, 0]$. The latter implies that $v_2(x) > 0$ in $(-1/2, 0]$, hence $v_2(x) > 0$ in $(-1/2, 1/2)$ since it is an even function. Then $v_1(x) > 0$ near $-1/2$, $v_1(-\beta) = v_1(\beta) < 0$. Furthermore, $v_1(x)$ can only have one sign-change zero in $(-1/2, 0)$, otherwise the contradiction $\lambda_2 \ge \nu_2[a]$ would appear again. Let that zero be x_1. Let $x_M < x_1 < x_m \le 0$, x_M (respectively, x_m) be the local maximum point (respectively, local minimum point) of v_1 on the left side (respectively, on the right side) of x_1. Let $\varphi(x) = v_1'(x)$. Then φ is nonpositive in $[x_M, x_m]$ and satisfies the following equation:

$$\varphi'' + \lambda_2 a\varphi + \lambda_2 b v_2' + \lambda_2(a'v_1 + b'v_2) = 0,$$
$$\varphi(x_M) = \varphi(x_m) = 0.$$

Now we use the assumption that a is a constant to reduce the previous equation to the following

$$\varphi'' + \lambda_2 a\varphi + \lambda_2 b v_2' + \lambda_2 b' v_2 = 0,$$
$$\varphi(x_M) = \varphi(x_m) = 0.$$

Since $b(x)$ is increasing in $(-1/2, 0]$ and $v_2 > 0$, we have

$$\varphi'' + \lambda_2 a\varphi + \lambda_2 b v_2' \le 0, \qquad \varphi(x_M) = \varphi(x_m) = 0. \tag{3.2}$$

If we can show that $v_2'(x) \ge 0$ in $(-1/2, 0]$, then (3.2) will imply $\lambda_2 > \nu_2[a]$, which is a contradiction to $\nu_2[a] > \nu_2[\rho_1] > \lambda_2$. Hence, Case 1 will not happen. To prove that $v_2'(x) \ge 0$ in $(-1/2, 0]$, since $v_2(x)$ is even, $v_2'(0) = 0$, it suffices to show that $v_2''(x) \le 0$ in $(-1/2, 0]$. Otherwise there exists $-1/2 < -x_2 < 0$ such that $v_2''(-x_2) > 0$, hence $cv_2(-x_2) + b(-x_2)v_1(-x_2) < 0$. Since $v_2(-x) > 0$, $c > b(x) > 0$, we have

$$v_2(-x_2) + v_1(-x_2) < 0.$$

Thus, by $v_2(x) > 0$, $v_1(x)v_2(x) > 0$ near $-1/2$ and the above inequality, we can find $-1/2 < -x_3 < 0$, such that

$$\begin{aligned} &v_1(x) + v_2(x) > 0 \quad \text{in } -1/2 < x < -x_3 \\ &v_1(-x_3) + v_2(-x_3) = 0. \end{aligned} \tag{3.3}$$

Let $U(x) = v_1(x) + v_2(x)$. Then, by (2.1), U satisfies the equation

$$U'' + \lambda_2(a+b)U - \lambda_2(a-c)v_2 = 0,$$
$$U(-1/2) = U(-x_3) = 0,$$

which implies

$$U'' + \lambda_2(a+b)U \geq 0, \qquad U(-1/2) = U(-x_3) = 0. \tag{3.4}$$

(3.3) and (3.4) imply $\lambda_2 \geq \nu_1[a+b|_{(-1/2,-x_3)}] > \nu_2[a+b]$, a contradiction to the assumption that $\nu_2[a+b] \geq \nu_1[\rho_2]$, and $\nu_1[\rho_2] \geq \lambda_2$. Similarly, Case 2 will not happen. Thus $v_1(x)v_2(x)$ must be negative near $-1/2$. By Lemma 3.7, the proof of Theorem 3.1 is complete. □

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Institute of Mathematics, National Tsing Hua University, Hsinchu, Taiwan 30043, Republic of China.