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ON THE EXISTENCE OF MULTIPLE SOLUTIONS OF A BOUNDARY VALUE PROBLEM ARISING FROM FLOWS IN FLOATING CAVITIES

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Dedicated to Paul Waltman on the occasion of his 60th birthday

ABSTRACT. Existence of multiple solutions of the similarity equation $f''' + Q[Aff'' - f'^2] = \beta$ satisfying $f(0) = f(1) = f''(0) + 1 = f''(1) = 0$ is proved using the shooting method. Here Q, A and β are parameters, $Q > 0$ and $A = 1$.

1. Introduction. The third order nonlinear differential equation

$$f''' + Q[Aff'' - (f')^2] = \beta, \qquad f = f(\eta), \qquad 0 \le \eta < 1$$

with boundary condition $f(0) = f(1) = f''(1) = f''(0) + 1 = 0$, where $Q > 0$, $A > 0$, and β are parameters, governs the velocity of boundary layer flow in a low Prandtl number fluid zone having the shape either of rectangular ($A = 1$) or a circular disk ($A = 2$) [**1, 2**]. Existence of solutions to the boundary value problem has been proved in [**4**] and [**5**] for the following cases:

(1) for given $A > 0$ and for $\beta \in [0, 1]$, there exists at least one $Q > 0$ for which the equation has at least one convex solution;

(2) Given $Q > 0$ and $A \in [1, 2]$, there exists at least one β for which the equation has a convex solution. Moreover, $\beta < 0$ if Q is sufficiently large;

(3) If $A = 2$, there exists a unique solution for every $Q > 0$;

(4) If $A = 1$, there may exist multiple solutions for some $Q > 0$.

In this paper we improve the result in (4). We present a proof of the existence of multiple solutions for $A = 1$ as long as Q is sufficiently large, i.e., if $A = 1$, then there exists a number $Q_0 > 0$ such that there are at least three solutions for any given $Q > Q_0$. Since Q

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is proportional to the Reynolds number, the result means that, for floating rectangular cavities, there always exist two-cell and three-cell flows for sufficiently larger Reynolds numbers.

The method of proof is a topological shooting argument based on a technique of McLeod and Serrin [**6**]. As in our previous work [**5**], we differentiate the third order equation with respect to the independent variable η, and let $g(\eta) = Qf(1-\eta)$. Then, the given equation takes the following form:

$$g^{(iv)} = gg''' - g'g'' \tag{1}$$

and the boundary conditions become

$$g(0) = 0, \qquad g''(0) = 0, \tag{1a}$$

$$g(1) = 0, \qquad g''(1) = -Q. \tag{1b}$$

The main result in this paper is the following theorem.

Theorem. *There exists a $Q_0 > 0$ such that for $Q > Q_0$, the boundary value problem* (1)–(1a)–(1b) *has at least three solutions.*

2. Proof of theorem. The technique employed in the proof is the shooting argument. Consider the initial value problem consisting of (1)–(1a) and

$$g'(0) = \lambda, \qquad g'''(0) = \mu, \tag{1c}$$

where λ and μ are parameters which are to be found so that the solution $g = g(\eta) = g(\eta; \lambda, \mu)$ of (1)–(1a)–(1c) also satisfies condition (1b). For simplicity, we will usually suppress the dependence of g on pairs of (λ, μ) in the context. Before we start shooting, we list the lemma that was proved in [**5**] for completeness.

Lemma 1. *Any solution $g(\eta; \lambda, \mu)$ of* (1)–(1a)–(1c) *satisfies $g^{(iv)} \leq 0$ for all $\eta \geq 0$ for which it exists, and if $\mu \neq 0$, then $g^{(iv)}(\eta; \lambda, \mu) < 0$ for all $\eta > 0$ for which it exists.*

Proof of Lemma 1. Differentiating (1) with respect to η, we obtain

$$g^{(v)} = gg^{(iv)} - g''^2. \tag{2}$$

Multiplying (2) by an integrating factor $\exp(-\int_0^\eta g\,ds)$ and integrating the resulting equation, we get

$$g^{(iv)}(\eta)e^{-\int_0^\eta g\,d\eta} = -\int_0^\eta g''^2 e^{-\int_0^t g\,ds}\,dt, \tag{3}$$

which implies the lemma. □

Now we start the shooting argument by varying λ and μ. To do this, we define two subsets of the $\lambda\mu$-plane as follows

$$P = \{(\lambda,\mu) \mid g(1;\lambda,\mu) < 0 \text{ or } g(\eta;\lambda,\mu) \text{ blows up before } \eta = 1\},$$
$$S = \{(\lambda,\mu) \mid g(1;\lambda,\mu) > 0\}.$$

Here by blow up we mean the solution becomes unbounded. According to Lemma 1, solutions of (1) can only blow up by tending to $-\infty$. By the definitions of P and S and the theorem on continuous dependence of solutions on initial values, it is clear that P and S are open subsets on the $\lambda\mu$-plane. Then the following lemma is obtained using the existence and uniqueness of solutions to the initial value problem (1)–(1a)–(1c).

Lemma 2. *The positive λ axis, $\{(\lambda,\mu) \mid \lambda > 0, \mu = 0\}$ is contained in S and the negative λ axis, $\{(\lambda,\mu) \mid \lambda < 0, \mu = 0\}$ is contained in P.*

Proof of Lemma 2. If $\mu = 0$, then $g(\eta) = \lambda\eta$ is the unique solution, which implies the result. □

By Lemma 1 we see that if $\lambda < 0$ and $\mu < 0$, then $g' < 0$ for $\eta > 0$ so long as the solution exists. Therefore, in order to hit the boundary condition (1b), we need only consider the following three cases: (i) $\lambda > 0$ and $\mu < 0$; (ii) $\lambda > 0$ and $\mu > 0$; (iii) $\lambda < 0$ and $\mu > 0$. To complete the proof of the theorem, we shall prove for each case that $g'(1) = 0$, $g''(1) < 0$, and $\lim_{|\lambda|\to\infty} \inf g''(1) = -\infty$.

A. *Existence of solutions with $g'(0) > 0$ and $g'''(0) < 0$.*

Lemma 3. *For each $\lambda > 0$, there exists a $\mu(\lambda) < 0$ such that if $\mu < \mu(\lambda)$, then $(\lambda, \mu) \in P$.*

Proof of Lemma 3. We first note that any solution of (1) can be differentiated with respect to the independent variable η infinitely many times as long as it exists. Applying Taylor's theorem to the solution of (1)–(1a)–(1c), one sees that $g(\eta) < \lambda\eta + \mu\eta^3/3!$ for $\eta > 0$ as long as $g(\eta)$ exists. Therefore, for given $\lambda > 0$ there must be a $\mu(\lambda)$ such that if $\mu < \mu(\lambda)$, then either $g(1) < 0$ or $g(\eta)$ blows up before $\eta = 1$. □

Let Σ_1 be the fourth quadrant of $\lambda\mu$-plane, i.e., $\Sigma_1 = \{(\lambda, \mu) \mid \lambda > 0 \text{ and } \mu < 0\}$. By the continuous dependence of solutions on initial values, we see from Lemma 2 that the set $R_1 = S \cap \Sigma_1 \neq \varnothing$. It also follows from Lemma 3 that the set $P_1 = P \cap \Sigma_1 \neq \varnothing$. Since P_1 and R_1 are disjoint, open subsets of Σ_1, it follows that the set $[\Sigma_1 - (P_1 \cup R_1)]$ is not empty. Furthermore, it follows from the result in [**4**] that there exists a continuum Ω_1 in Σ_1 such that $g(1; \lambda, \mu) = 0$ for any pair $(\lambda, \mu) \in \Omega_1$. From Lemmas 2 and 3, in fact, we see that $\Omega_1 = \{(\lambda, \mu) \mid \lambda > 0,\ g(1; \lambda, \mu) = 0\}$. Since $g''(1; \lambda, \mu)$ is a continuous function of (λ, μ) on Ω_1, the existence of this convex solution for any $Q > 0$ will follow if we show that $\limsup_{\lambda\to 0} g''(1; \lambda, \mu) = 0$, and $\liminf_{\lambda\to+\infty} g''(1; \lambda, \mu) = -\infty$ for (λ, μ) is in Ω_1. These limits are established in the next two lemmas.

Lemma 4. $\limsup_{\lambda\to 0} g''(1; \lambda, \mu) = 0$ *for* (λ, μ) *in* Ω_1.

Proof of Lemma 4. From Lemma 1 and Taylor's theorem, $g(1; \lambda\mu) < \lambda + \mu/3!$. Thus $\mu \to 0$ as $\lambda \to 0$. On the other hand, $g''(1; 0, 0) = 0$. Again, by the continuity of solutions in (λ, μ), we see that $|g''(1; \lambda, \mu)|$ is small if λ is sufficiently small if $(\lambda, \mu) \in \Omega_1$. □

Lemma 5. $\liminf_{\lambda\to\infty} g''(1; \lambda, \mu) = -\infty$ *while* (λ, μ) *is in* Ω_1.

Proof of Lemma 5. We prove the lemma by contradiction. Assume it is false. Then there exists a sequence of $\{\lambda_i\}$ such that $\lambda_i \to \infty$ and $g''(1; \lambda_i, \mu_i)$ is bounded as $i \to \infty$. Since $g''' < \mu_i < 0$ for all $i \geq 1$, it follows that $g''(1) < g''(\eta) < 0$ for $\eta \in (0, 1)$. Hence, $g''(\eta; \lambda_i, \mu_i)$ is bounded uniformly for $i \geq 1$. Therefore, there exists a constant $k > 0$ such that

$$-k \leq g''(\eta; \lambda_i, \mu_i) \leq 0$$

for all $\eta \in (0,1)$ and for all $i \geq 1$. Integrating the inequality twice shows that

$$-\frac{k}{2}\eta^2 + \lambda_i \eta \leq g(\eta; \lambda_i, \mu_i), \tag{4}$$

which implies that $g(1; \lambda_i, \mu_i) > 0$ for sufficiently large λ_i. This contradicts the fact that $(\lambda_i, \mu_i) \in \Omega_1$. □

B. *Existence of solutions with* $g'(0) > 0$ *and* $g'''(0) > 0$. Let Σ_2 denote the first quadrant of $\lambda\mu$-plane, i.e., $\Sigma_2 = \{(\lambda, \mu) \mid \lambda > 0 \text{ and } \mu > 0\}$. The following lemma shows that both sets $(\Sigma_2 \cap P)$ and $(\Sigma_2 \cap S)$ are nonempty.

Lemma 6. *For any given* $\lambda > 0$ *there exist two numbers* $b_1 = b_1(\lambda) > 0$ *and* $b_2 = b_2(\lambda) > 0$ *such that if* $\mu \in (0, b_1)$ *then* $(\lambda, \mu) \in S$, *while if* $\mu > b_2$ *then* $(\lambda, \mu) \in P$.

Proof of Lemma 6. Since $g(\eta; \lambda, 0) = \lambda\eta$, we see that the continuous dependence of solutions on initial values implies the existence of b_1 and that $(\lambda, \mu) \in S$ if $\mu < b_1$. To prove the existence of b_2 we first prove that if $\lambda > 0$ is given and if μ is sufficiently large, then there is an $\eta_\mu = \eta_\mu(\mu)$ in (0,1) such that $g'''(\eta_\mu) = \mu/2$. Since $g'''(0) = \mu > \mu/2$, it follows that $g'''(\eta) > \mu/2$ on the interval $[0, \eta_\mu)$. Then, as long as $g''' \geq \mu/2$,

$$g'' \geq \frac{\mu}{2}\eta, \qquad g > 0, \qquad g^{(v)} = gg^{(iv)} - (g'')^2 < -(g'')^2 \leq -\frac{\mu^2}{4}\eta^2.$$

Integrating the last inequality $g^{(v)} \leq -(\mu^2/4)\eta^2$ twice yields the conclusion that, as long as $g''' > \mu/2$,

$$g''' < \mu - \frac{\mu^2}{48}\eta^4. \tag{5}$$

Therefore, g''' becomes 0 before $\eta = (24/\mu)^{1/4}$; hence, $\eta_\mu < (24/\mu)^{1/4}$. This implies $\eta_\mu \to 0$ as $\mu \to \infty$. We repeatedly integrate (5) to obtain

$$g''(\eta_\mu) < \mu\eta_\mu - \frac{\mu^2}{240}\eta_\mu^5 < \mu\eta_\mu, \tag{6}$$

$$g'(\eta_\mu) < \lambda + \frac{\mu}{2}\eta_\mu^2, \tag{7}$$

and

$$g(\eta_\mu) < \lambda\eta_\mu + \frac{\mu}{6}\eta_\mu^3, \tag{8}$$

as long as $g''' > \mu/2$. Since $g^{(v)} < 0$ as long as $g > 0$, the derivative $g^{(iv)}$ is decreasing. By the Mean Value Theorem applied to g''', there exists a $\xi \in (0, \eta_\mu)$ such that $g^{(iv)}(\xi)\eta_\mu = -\mu/2$. Thus $g^{(iv)}(\eta_\mu) < g^{(iv)}(\xi)$, and it follows that

$$g^{(iv)}(\eta_\mu) < -\frac{\mu}{2\eta_\mu} < -\frac{\mu}{2}\left(\frac{\mu}{24}\right)^{1/4}. \tag{9}$$

Since $g^{(iv)}$ continues to decrease until $g = 0$, integrating (9) four times over the interval (η_μ, η) produces the inequality

$$\begin{aligned} g(\eta) < {} & \lambda\eta_\mu + \frac{\mu}{6}\eta_\mu^3 + \left(\lambda + \frac{\mu}{2}\eta_\mu^2\right)(\eta - \eta_\mu) \\ & + \frac{\mu\eta_\mu}{2}(\eta - \eta_\mu)^2 + \frac{\mu}{12}(\eta - \eta_\mu)^3 \\ & - \frac{\mu^{5/4}}{48(24)^{1/4}}(\eta - \eta_\mu)^4. \end{aligned} \tag{10}$$

which is valid as long as $g > 0$. Therefore, either $g(\eta)$ blows up before $\eta = 1$ or $g(1) < 0$ for all values of μ that are sufficiently large compared with given λ. The proof of Lemma 6 is therefore completed. □

Again, by the result in [**6**], there exists a continuum Δ_1 in $\Sigma_1 - (P \cup S)$ such that $\Delta_1 = \{(\lambda, \mu) \mid \lambda > 0, \mu > 0 \text{ and } g(1; \lambda, \mu) = 0\}$. Since $g(1; \lambda, \mu)$ is a continuous function of (λ, μ) on Δ_1, it suffices to prove that $\liminf g''(1; \lambda, \mu) = -\infty$ as $\lambda \to \infty$, in order to get the existence of solutions of this kind for sufficiently large Q. This is given in the next two lemmas.

Lemma 7. *For any* $M>0$, $\sup\{g''(1;\lambda,\mu)\mid \lambda\in[0,M],$ *and* $(\lambda,\mu)\in\Delta_1\}<0$.

Proof of Lemma 7. We first prove that there is a $\rho>0$ such that the set $\{(\lambda,\mu)\mid 0<\sqrt{\lambda^2+\mu^2}<\rho,\ \lambda>0$ and $\mu>0\}$ lies in set S. Expanding a solution $g(\eta;\lambda,\mu)$ in a Taylor series in (λ,μ) around (0,0), one sees the lower order terms are

$$(11)\quad g(\eta;\lambda,\mu)=\phi(\eta)\mu+\psi(\eta)\lambda+\{\phi_1(\eta)\mu^2+\phi_2(\eta)\mu\lambda+\phi_3(\eta)\lambda^2\}+\dots,$$

where $\phi(\eta)=\partial g(\eta,\lambda,\mu)/\partial\mu|_{\lambda=0,\mu=0}$ and $\psi(\eta)=\partial g(\eta,\lambda,\mu)/\partial\lambda|_{\lambda=0,\mu=0}$. Noticing that $\phi(\eta)$ satisfies

$$\begin{aligned}&\phi^{(iv)}=\phi g'''+g\phi'''-\phi'g''-g'\phi'',\\ &\phi(0)=\phi'(0)=\phi''(0)=0,\qquad \phi'''(0)=1,\end{aligned}$$

one finds that $\phi(\eta)=\eta^3/6$. Similarly, $\psi(\eta)=\eta$. Therefore, if $\lambda>0$ and $\mu>0$ are sufficiently small, then $g(\eta;\lambda,\mu)\approx(\eta^3/6)\mu+\eta\lambda$; hence, $g(1;\lambda,\mu)\approx(1/6)\mu+\lambda>0$. This implies that if $\lambda>0$ is small, then the corresponding μ for $(\lambda,\mu)\in\Delta_1$ cannot be too small. Therefore, there is a $\rho>0$ such that if $\lambda^2+\mu^2<\rho$, $\lambda>0$, and $\mu>0$, then $(\lambda,\mu)\notin\Delta_1$, and hence $\inf\{\mu\mid(\lambda,\mu)\in\Delta_1$ for all $\lambda\in[0,M]\}>0$ for any fixed $M>0$. Since $g''(1,\lambda,\mu)$ is continuous in (λ,μ) on $[0,M]\times[\mu_1,\mu_2]$ for any $\mu_2>\mu_1>0$, we find that $\operatorname{Sup}\{g''(1;\lambda,\mu)\mid\lambda\in[0,M],(\lambda,\mu)\in\Delta_1\}<0$. □

Lemma 8. $\liminf_{\lambda\to+\infty}g''(1;\lambda,\mu)=-\infty$ *provided* (λ,μ) *is in* Δ_1.

Proof of Lemma 8. Introduce a new function $h(\eta)=g(\eta)/\lambda$ and set $\varepsilon=1/\lambda$. Then

$$(12)\qquad \varepsilon h^{(iv)}=hh'''-h'h'',$$

together with $h(0)=0$, $h'(0)=1$, $h''(0)=0$ and $h'''(0)=\varepsilon\mu$. It has been proved that, for each $\varepsilon>0$, there is at least one positive

μ such that $h(1;\varepsilon,\mu) = 0$. Since $g'' = h''/\varepsilon$, it suffices to prove $\liminf_{\varepsilon\to 0} h''(1;\varepsilon,\mu) \le -\alpha$ for $(1/\varepsilon,\mu) \in \Delta_1$ and some constant $\alpha > 0$ in order to prove the lemma. In other words, we shall prove that there exists a constant $\alpha > 0$ such that $h''(1) < -\alpha$ for infinitely many ε's. In fact, we can prove $h''(1) < -1$ for all $\varepsilon > 0$, which, of course, implies the lemma. Since $g'(0) > 0$ and $g'''(0) > 0$, we claim that there is no zero point of g in (0,1). If it is not so, then let $\eta_0 < 1$ be the first zero of g. Thus, g', g'', and g''' must be nonpositive, and therefore $g < 0$ for $\eta > \eta_0$. This shows that $\eta = 1$ is the only zero of h on $(0,1]$. From the profile of $h(\eta)$ it is apparent that there exist points two $\eta_3 < \eta_2$ with $h'''(\eta_3) = 0$, $h''(\eta_3) > 0$, $h'(\eta_3) > 1$, $h''(\eta_2) = 0$, $h'''(\eta_2) < 0$, and $h'(\eta_2) > 1$. Since $h(1) = 0$ and $h^{(iv)} < 0$, we see that there is a point $\eta_1 > \eta_2$ with $h'(\eta_1) = 0$. Noting that h'' is decreasing and concave down for $\eta \ge \eta_2$, we find

$$h'' < \frac{h'(\eta_1) - h'(\eta_2)}{\eta_1 - \eta_2} < 1 \qquad \text{for } \eta > \eta_1.$$

Therefore, $h''(1) < -1$ and the lemma is proved. □

C. *Existence of solutions with* $g'(0) < 0$, $g'''(0) > 0$.

Lemma 9. *If* $\lambda < 0$ *and* $\mu > 0$, *then* $g''' > 0$ *as long as* $g' < 0$.

Proof of Lemma 9. If the lemma is false, then there is a first zero η_3 of g''' at which $g'' > 0$ and $g' < 0$. From equation (1), we see that $g^{(iv)} = -g''g' > 0$. This contradicts Lemma 1. □

Lemma 10. *Let* $\phi(\eta;\lambda,\mu) = \partial g(\eta,\lambda,\mu)/\partial\mu$ *and* $\lambda < 0$, $\mu > 0$. *Then* ϕ, ϕ' *and* ϕ'' *are all positive as long as* $g' \le 0$.

Proof of Lemma 10. Integration of (1) gives

$$g''' = gg'' - g'^2 + \lambda^2 + \mu. \tag{20}$$

Differentiating (20) with respect to μ leads to

$$\phi''' = 1 - 2\phi' g' + \phi g'' + g\phi'', \tag{21}$$

with $\phi(0) = \phi'(0) = \phi''(0) = 0$, $\phi'''(0) = 1$. Initially, ϕ, ϕ', ϕ'', and ϕ''' are all positive. Suppose there is a first zero of ϕ'', say η_2. Then $\phi'''(\eta_2) \le 0$. On the other hand,

$$\phi''' = 1 - 2\phi' g' + \phi g'', \tag{22}$$

at η_2. By Lemma 9, we see that if $g' \le 0$, then $g''' > 0$ and hence $g'' > 0$. This yields $\phi''' > 0$ ($\phi' > 0, \phi > 0$), a contradiction. Therefore, $\phi'' > 0$ and hence ϕ and ϕ' are positive as long as $g' \le 0$. □

Taking a fixed $\lambda < 0$, we consider the solution for $\mu > 0$. If μ is small, then $g(\eta) \approx \lambda\eta$, $g'(\eta) \approx \lambda$. Then Lemmas 9 and 10 imply $\phi'(1; \lambda, \mu) > 0$ for small $\mu > 0$.

Lemma 11. *If μ is sufficiently large, then $g'(\eta) > 0$ and $g(\eta) > 0$ for some $\eta \in (0, 1)$.*

Proof of Lemma 11. On some initial interval, $g' < 0$ and, as long as this inequality persists, we have

$$\lambda < g' < 0, \qquad \lambda\eta \le g \le 0, \qquad g'' > 0, \quad \text{and} \quad g^{(iv)} \ge \lambda g'''. \tag{23}$$

Repeatedly integrating the last inequality in (23) shows that

$$g'' > \mu e^{\lambda\eta}/\lambda, \qquad g' > \frac{\mu e^{\lambda\eta}}{\lambda^2} + \lambda, \quad \text{and} \quad g > \frac{\mu e^{\lambda\eta}}{\lambda^3} + \lambda\eta.$$

The last two inequalities imply the lemma. □

From Lemma 11, we see that, for any given $\lambda < 0$ there is a $\mu = \mu_1(\lambda) > 0$ such that $g(\eta; \lambda, \mu) > 0$ for some $\eta \in (0, 1)$ provided $\mu > \mu_1$. Recall too that if $\mu = 0$, then $g(1) < 0$, so by the continuity of the solution in μ, we see $g(1) < 0$ for small $\mu > 0$. Therefore, there exists at least one $\mu > 0$ such that $g(1) = 0$ and $g''(1) > 0$. A similar argument shows that there is a continuum in the $\lambda\mu$-plane such that $g(1; \lambda, \mu) = 0$ for (λ, μ) in the continuum. Denote the intersection of such a continuum and $\Sigma_3 = \{(\lambda, \mu) \mid \lambda < 0, \mu > 0\}$ by Δ_3. In fact, any pair (λ, μ) in Δ_3 gives a concave up solution, which is proved in Lemma 12.

Lemma 12. *If* $g \leq 0$ *on* $[0,1]$ *solves equation* (1)–(1a)–(1c) *with* $\lambda < 0$ *and* $\mu > 0$, *then* g' *has exactly one zero and* $g'' > 0$ *on* $(0,1)$.

Proof of Lemma 12. It is observed from equation (1) that $g''' > 0$ when $g' = 0$ and $g'' \geq 0$. Then $g'' > 0$ before $g' = 0$. Noting g''' reaches zero before g'' does, we see that $g'' > 0$ wherever $g''' = 0$ with $g' > 0$ and $g < 0$. Therefore, $g'' < 0$ and g' has only one zero on (0,1). □

Remark. The equation only reflects the case $Q > 0$ from the original model. However, it is mathematically interesting to know whether any solution exists for $Q < 0$. It is easy to see that $\limsup g''(1;\lambda,\mu) \to 0$ as $\lambda \to 0$ for $(\lambda,\mu) \in \Delta_3$ from the continuous dependence of the solution on the initial conditions. This means that there exists at least a constant, say Q_1, such that solutions exist for $Q \in (-Q_1, 0)$. Also, the solutions found in this region are concave up. It is a conjecture that $\sup g''(1;\lambda,\mu) < \infty$ as $\lambda \to -\infty$, which was found by Wang and Chen numerically [**7**].

Finally, we consider the existence of solutions with two wiggles for $\lambda < 0$.

Recall the function $\phi(\eta;\lambda,\mu) = \partial g(\eta,\lambda,\mu)/\partial\mu$ and that if $\lambda = \mu = 0$, $g = 0$, then $\phi(\eta;0,0) = \eta^3/6$. Hence $\phi(1;0,0) = 1/2$ and $\phi(1;\lambda\mu) \geq 1/4$ in some neighborhood of $(0,0)$. Also, the functions ϕ, $\partial\phi/\partial\lambda$ and $\partial\phi/\partial\mu$ are uniformly bounded for $\eta \in [0,1]$ and (λ,μ) in some neighborhood of $(0,0)$. Therefore, there are $\mu_0 > 0$ and $\lambda_0 < 0$ such that $\phi(1,\lambda,\mu) > 0$ for $(\lambda,\mu) \in [0,\lambda_0] \times [0,\mu_0]$. Since $g(1;\lambda,\mu) = 0$, we see that $g(1,\lambda,\mu) > 0$ if $|\lambda_0|$ is small enough to ensure that $\mu(\lambda_0) < \mu_0$, $\lambda_0 \leq \lambda \leq 0$, and $\mu(\lambda) < \mu < \mu_0$. This proves that S contains an open set bounded below by $\mu = \mu_1(\lambda)$ for $\lambda \leq 0$ and by $\mu = 0$ for $\lambda > 0$.

Lemma 13. *For each* $\lambda < 0$ *there is a* $\mu_3(\lambda) > 0$ *such that if* $\mu > \mu_3(\lambda)$, *then* $(\lambda,\mu) \in P$.

Proof of Lemma 13. For large μ, as long as $g'' > 0$ and $g' \leq \sqrt{\mu/2}$, it follows from (20) that

$$\mu\eta \geq g'' > 0, \qquad g' > 1, \qquad g > \lambda\eta \quad \text{and} \quad g''' > \mu/2 + \lambda\mu\eta^2. \tag{24}$$

If $\eta < 1/(2\sqrt{|\lambda|})$, then $g''' \geq \mu/4$. Integrating the last inequality three times and using the initial conditions on g, we conclude that either $g = 0$ or $g' = \mu/2$ before $\eta = \sqrt{24|\lambda|/\mu}$. On the other hand, $g'' \leq \mu\eta$, and hence $g' \leq \lambda + \mu\eta^2/2$, so

$$g'\left(\sqrt{\frac{24|\lambda|}{\mu}}\right) \leq 11|\lambda| < \sqrt{\frac{\mu}{2}} \tag{25}$$

for large μ. This shows that, for large μ, $g = 0$ before $\eta = \sqrt{24|\lambda|/\mu}$. Furthermore, $g''' \geq \lambda^2 - \mu/2 + \mu \geq \mu/2$ and $0 \leq g' \leq 11|\lambda|$ at $g = 0$. From here the proof that $g(1) < 0$ proceeds with estimates which are very similar to those in Lemma 6. □

Applying the result in [**6**], we see that there is a continuum Δ_4 such that

$$\Delta_4 = \{(\lambda, \mu) \mid \lambda < 0, \mu > 0, g(1) = 0 \text{ and } g''(1) < 0\}.$$

The proof of existence of solutions with two wiggles for large Q is completed by showing that $\liminf g''(1; \lambda, \mu) = -\infty$ as $\lambda \to -\infty$. Once again, let $h = g/|\lambda|$, and integrate equation (12). Then

$$\varepsilon h''' = hh'' - h'^2 + 1 + \varepsilon^2\mu \tag{26}$$

with $h(0) = -1$, $h''(0) = 0$. It is sufficient to show that $h''(1) \leq -1$ as $\varepsilon \to 0$. This is proved in the following lemma.

Lemma 14. *$h''(1) \leq -1$.*

Proof of Lemma 14. The solution $h(\eta)$ is nonconcave and has three zeros in $[0, 1]$. Thus, there are two zeros of h' and one zero of h'' in (0,1). In fact, $h(\eta)$ has the following properties:

$$\begin{aligned}
h^{(iv)} &< 0, \\
h' &< 0 \quad \text{on some interval } [0, \eta_1), \\
h' &> 0 \quad \text{on } (\eta_1, \eta^*), \qquad h' < 0 \quad \text{on } (\eta^*, 1), \\
h'' &> 0 \quad \text{on } (0, \eta_2), \qquad h'' < 0 \quad \text{on } (\eta_2, 1], \\
h''' &> 0 \quad \text{on } [0, \eta_3), \qquad h''' < 0 \quad \text{on } (\eta_3, 1], \\
h &< 0 \quad \text{on } (0, \eta_4), \qquad h > 0 \quad \text{on } (\eta_4, 1),
\end{aligned}$$

where $0 < \eta_1 < \eta_3 < \eta_2$. Since h''' becomes zero before h'' does, we see by Lemma 1 that h'' is decreasing and concave down on $[\eta_2, 1]$. From (26) we see $h'(\eta_2) \geq \sqrt{1+\varepsilon^2\mu} > 1$. Hence, there is an $\hat{\eta} > \eta_2$ with $h'(\hat{\eta}) = 1$. Furthermore, there is an $\eta_* \in (\eta_2, \eta^*)$ such that $h''(\eta_*) = (0 - h'(\eta_2))/(\eta^* - \eta_2) < -1$. Since $h'''(\eta) < 0$ for $\eta > \eta_2$, it follows that $h''(\eta) < h''(\eta^*)$ for $\eta > \eta^*$, which implies $h''(1) < -1$. This proves Lemma 14. □

Summing up, we see from Lemmas 5, 8 and 14 that the boundary value problem

$$\begin{aligned} &g^{(iv)} = gg''' - g'g'', \\ &g(0) = 0, \qquad g''(0) = 0, \\ &g(1) = 0, \qquad g''(1) = -Q \end{aligned}$$

has at least three solutions for all sufficiently large Q. The proof of the theorem in the paper is complete.

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