ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 24, Number 1, Winter 1994

AUTOMORPHISMS AND DERIVATIONS OF DIFFERENTIAL EQUATIONS AND ALGEBRAS

MICHAEL K. KINYON AND ARTHUR A. SAGLE

Dedicated to Paul Waltman on the occasion of his 60th birthday

1. Introduction and main results. We consider autonomous differential equations

$$\dot{X} = F(X) \tag{1}$$

in $\mathbf{R}^n$ where $F : \mathbf{R}^n \to \mathbf{R}^n$ is smooth and $\dot{X} = (dX/dt)$. An automorphism of F is an invertible linear transformation $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ satisfying

$$F(\varphi X) = \varphi F(X) \tag{2}$$

for all $X \in \mathbf{R}^n$. The set $\operatorname{Aut} F$ of all automorphisms of F is a closed (Lie) subgroup of $GL(n, \mathbf{R})$. Equivalently, one can define $\operatorname{Aut} F$ to be the largest closed subgroup of $GL(n, \mathbf{R})$ relative to which F is $(\operatorname{Aut} F)$-equivariant.

If $\phi_t(X)$ denotes the flow associated with equation (1), then for each $\varphi \in \operatorname{Aut} F$ we have

$$\phi_t \circ \varphi = \varphi \circ \phi_t. \tag{3}$$

Conversely, any invertible linear transformation satisfying (3) is an automorphism of F (see [**5**, Lemma 5.2]).

A derivation of F is a linear transformation $D : \mathbf{R}^n \to \mathbf{R}^n$ satisfying

$$DF(X) = F'(X) \cdot DX \tag{4}$$

for all $X \in \mathbf{R}^n$; here $F'(X) \cdot Y = (dF(X + sY)/ds)|_{s=0}$. The set $\operatorname{Der} F$ of all derivations of F is a Lie subalgebra of $gl(n, \mathbf{R})$. If $D \in \operatorname{Der} F$,

Received by the editors on March 3, 1993.

then $\exp D = I + D + (1/2!)D^2 + \cdots \in \operatorname{Aut} F$; thus $\operatorname{Der} F$ is the Lie algebra of $\operatorname{Aut} F$ [**6**].

In terms of the flow $\phi_t(X)$, the condition equivalent to (4) is

$$D\phi_t(X) = \phi_t'(X) \cdot DX. \tag{5}$$

(Here $\phi_t'(X) \cdot Y = (d\phi_t(X + sY)/ds))|_{s=0}$.)

The importance of the existence of automorphisms for the study of the dynamics of (1) is indicated in the following result. Let $\mathcal{E}$ denote the set of equilibria of F, i.e., the set of all points P for which $F(P) = 0$. Let $\mathcal{P}_\tau$ denote the set of periodic trajectories of (1) with period τ, i.e., the set of all solution curves $\{\phi_t(P)\}$ such that $\phi_\tau(P) = P$ and $\phi_t(P) \neq P$ for $0 < t < \tau$.

Proposition 1.1.

(i) $(\operatorname{Aut} F)\mathcal{E} = \mathcal{E}$;

(ii) $(\operatorname{Aut} F)\mathcal{P}_\tau = \mathcal{P}_\tau$.

Proof. See [**5**, Proposition 5.7]. □

Automorphisms also preserve domains of attraction; see [**5**, Theorem 5.13].

In [**5**], the authors focused their attention on automorphisms and derivations of *quadratic* differential equations

$$\dot{X} = TX + B(X, X), \tag{6}$$

where $T : \mathbf{R}^n \to \mathbf{R}^n$ is linear, and $B : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is bilinear. In particular, we usually restricted ourselves to the homogeneous case $T \equiv 0$. (See [**5**, Section 2] or [**7**] for a discussion of how equations like (6) can always be "homogenized.") In this paper we continue to draw on quadratic equations for examples, but the main results and discussions will hold in general.

The terms "automorphism" and "derivation" come from our algebraic perspective. In [**5**], our view of (6) was driven by the fact that the bilinear map B defines a multiplication on $\mathbf{R}^N$, thus making

$A = (\mathbf{R}^n, B)$ into a commutative, *nonassociative* algebra. The theme of [**5**] was that the study of the structure of A can help determine the behavior of solutions of (6) (at least in the homogeneous case). Now an automorphism φ of an algebra A is an invertible linear transformation satisfying $\varphi B(X,Y) = B(\varphi X, \varphi Y)$ for all $X, Y \in A$, and a derivation D of A is a linear transformation satisfying $DB(X,Y) = B(DX,Y) + B(X,DY)$ for all $X, Y \in A$. The Lie group of all automorphisms of A is denoted by $\operatorname{Aut} A$, and the Lie algebra of all derivations of A is denoted by $\operatorname{Der} A$. The connection between our two notions of automorphism and derivation is made in the following result [**5**, Theorem 5.1].

Proposition 1.2. *Let* $\dot{X} = F(X) = TX + B(X,X)$ *occur in* $A = (\mathbf{R}^n, B)$. *Then*

(i) $\operatorname{Aut} F = \{\varphi \in \operatorname{Aut} A : \varphi T = T\varphi\}$, *and*

(ii) $\operatorname{Der} F = \{D \in \operatorname{Der} A : DT = TD\}$.

More generally, for $F(X) = \sum_{k\geq 0} F_k(X)$ with $F_k : \mathbf{R}^n \to \mathbf{R}^n$ a homogeneous polynomial map of degree k, we have $\operatorname{Aut} F = \cap_{k\geq 0} \operatorname{Aut}(\mathbf{R}^n, F_k)$ and $\operatorname{Der} F = \cap_{k\geq 0} \operatorname{Der}(\mathbf{R}^n, F_k)$, where $(\mathbf{R}^n, F_k)$ denotes the *k-ary algebra* structure defined on $\mathbf{R}^n$ by F_k [**7**].

If a given map $F : \mathbf{R}^n \to \mathbf{R}^n$ has a nontrivial automorphism group, it is natural to ask when the action of a one-parameter subgroup on an initial point coincides with the solution through that point; that is, given a derivation D, for which initial points P does one have $\phi_t(P) = (\exp tD)P$? A characterization of such initial points is given as follows [**5**, Proposition 5.3].

Proposition 1.3. *Let* $D \in \operatorname{Der} F$. *Then* $(\exp tD)$ *is a solution to* $\dot{X} = F(X)$ *if and only if* $DP = F(P)$.

Proof. We have

$$\begin{aligned} DP = F(P) &\quad \text{if and only if} \\ (\exp tD)DP = (\exp tD)F(P) &\quad \text{if and only if} \\ \frac{d}{dt}(\exp tD)P = F((\exp tD)P), &\quad \text{since } \exp tD \in \operatorname{Aut} F. \qquad \square \end{aligned}$$

Remark. The equation $F(P) = DP$ that must be solved should be compared with the reduced equation obtained from Lyapunov-Schmidt reduction when F is in Birkhoff normal form [**3**, Theorem XVI.10.1]. In the latter setting, $F'(0)$, assumed to be nonzero, is used instead of D. Later we will list some advantages to our approach; here we note that Proposition 1.3 applies to all types of solutions, not just periodic.

Proposition 1(ii) and Proposition 3 suggest that when a given F has a nontrivial derivation algebra, it may be the case that $(\exp tD)P$ gives a *periodic* solution. The next proposition shows that this is usually the case [**5**, Theorem 5.8(1)].

Theorem 1.4. *Suppose $\mathcal{P}_\tau$ consists of isolated orbits*; *i.e., for each $\gamma \in \mathcal{P}_\tau$, there exists a neighborhood of γ containing no other trajectory in $\mathcal{P}_\tau$. If there is a $\gamma \in \mathcal{P}_\tau$, $P \in \gamma$, and $D \in \operatorname{Der} F$ such that $DP \neq 0$, then there is a nonzero $a \in \mathbf{R}$ such that $\phi_t(P) = (\exp taD)P$.*

Proof. Let $\gamma \in \mathcal{P}_\tau$, $P \in \gamma$, and $D \in \operatorname{Der} F$ be as in the hypothesis. Define the map

$$k : \mathbf{R} \to \mathbf{R}^n : s \to (\exp sD)P.$$

By Proposition 1.1(ii), $k(s)$ is a periodic point of period τ for each $s \in \mathbf{R}$. Since k is continuous, the image of k is connected. But since the orbits in $\mathcal{P}_\tau$ are isolated, there is $\tilde{\gamma} \in \mathcal{P}_\tau$ such that $k(s) \in \tilde{\gamma}$ for all $s \in \mathbf{R}$. Since $k(0) = P \in \gamma$, we have $\tilde{\gamma} = \gamma$. Thus there exists $u : \mathbf{R} \to \mathbf{R} : s \mapsto u(s)$ so that

$$(\exp sD)P = \phi_{u(s)}(P). \tag{7}$$

The function $u(s)$ is differentiable and we may assume without loss that $u(0) = 0$. (Write $u(s) = \tilde{u}(s) + u(0)$ where $\tilde{u}(0) = 0$,for then $(\exp sD)P = \phi_{\tilde{u}(s)+u(0)}(P) = \phi_{\tilde{u}(s)}(\phi_{u(0)}(P))$. Setting $s = 0$, we obtain $P = \phi_{u(0)}(P)$ and thus $(\exp sD)P = \phi_{\tilde{u}(s)}(P)$.)

Differentiating (7) we obtain

$$\begin{aligned}(\exp sD)DP &= \frac{d}{ds}((\exp sD)P)\\ &= \frac{d}{ds}\phi_{u(s)}(P)\\ &= \frac{d}{du}\phi_u(P)\frac{du}{ds}, \quad \text{chain rule}\\ &= F(\phi_u(P))u'(s)\end{aligned}$$

and setting $s = 0$, we have $DP = F(\phi_{u(0)}(P))u'(0) = u'(0)F(P)$. Since $DP \neq 0$, we have $u'(0) \neq 0$. Setting $a = 1/u'(0)$, we obtain $aDP = F(P)$. By Proposition 1.2, $\phi_t(P) = (\exp taD)P$. □

Next we extend this result to invariant tori. Let γ be a trajectory for (1). Then γ is said to be *quasiperiodic* if there exists a subspace $V \subseteq \mathbf{R}^n$ with basis $\{X_1, \ldots, X_k\}$, a rationally independent set $\{\tau_1, \ldots, \tau_k\}$ of nonzero real numbers, and a function $G : V \times \tau \to \mathbf{R}^n$ such that

(i) $G(Y + \tau_i X_i, P) = G(Y, P)$, $i = 1, \ldots, k$, for all $Y \in V$ and $P \in \gamma$, and

(ii) $G(tE, P) = \phi_t(P)$ for all $t \in \mathbf{R}$ and $P \in \gamma$, where $E = \sum_i X_i$.

This is an adaptation of [**9**, p. 30].

The numbers $\tau_1, \ldots, \tau_k$ are called the *quasiperiods* of γ. Let $\mathcal{Q}_{\tau_1,\ldots,\tau_k}$ denote the set of all quasiperiodic trajectories of (1) with quasiperiods $\tau_1, \ldots, \tau_k$.

Proposition 1.5.

$$(\operatorname{Aut} F)\mathcal{Q}_{\tau_1,\ldots,\tau_k} = \mathcal{Q}_{\tau_1,\ldots,\tau_k}$$

Proof. Let $\gamma \in \mathcal{Q}_{\tau_1,\ldots,\tau_k}$ and $\varphi \in \operatorname{Aut} F$. Let $V \subset \mathbf{R}^n$, $\{X_1, \ldots, X_k\} \subseteq V$, and $G : V \times \gamma \to \mathbf{R}^n$ be as in the definition. Define $\tilde{G} : V \times \varphi\gamma \to \mathbf{R}^n$ by $\tilde{G}(Y, \varphi P) \equiv \varphi G(Y, P)$ for all $Y \in V$ and $P \in \gamma$. Then for all $Y \in V$ and $P \in \gamma$, we have

$$\tilde{G}(Y + \tau_i X_i, \varphi P) = \varphi G(Y + \tau_i X_i, P) = \varphi G(Y, P) = \tilde{G}(Y, \varphi P),$$

$i = 1, \dots, k$. Further, we have

$$\phi_t(\varphi P) = \varphi\phi_t(P) = \varphi G(tE, P) = \tilde{G}(tE, \varphi P),$$

for all $t \in \mathbf{R}$ and $P \in \gamma$. Thus $\varphi\gamma \in \mathcal{Q}_{\tau_1,\dots,\tau_k}$, and so $\varphi\mathcal{Q}_{\tau_1,\dots,\tau_k} \subseteq \mathcal{Q}_{\tau_1,\dots,\tau_k}$. Since $\gamma = \varphi(\varphi^{-1}\gamma)$ and $\varphi^{-1}\gamma \in \mathcal{Q}_{\tau_1,\dots,\tau_k}$, we also have $\mathcal{Q}_{\tau_1,\dots,\tau_k} \subseteq \varphi\mathcal{Q}_{\tau_1,\dots,\tau_k}$. This completes the proof. □

Theorem 1.6. *Suppose $\mathcal{Q}_{\tau_1,\dots\tau_k}$ consists of isolated orbits. If there is a $\gamma \in \mathcal{Q}_{\tau_1,\dots,\tau_k}$, $P \in \gamma$, and $D \in \operatorname{Der} F$ such that $DP \neq 0$, then there is a nonzero $a \in \mathbf{R}$ such that $\phi_t(P) = (\exp taD)P$.*

Proof. Let $\gamma \in \mathcal{Q}_{\tau_1,\dots,\tau_k}$, $P \in \gamma$, and $D \in \operatorname{Der} F$ be as in the hypothesis. Define the map

$$k : \mathbf{R} \to \mathbf{R}^n : s \mapsto (\exp SD)P.$$

As in the proof of Theorem 1.4, $k(s) \in \gamma$ for all $s \in \mathbf{R}$, and thus there exists $u : \mathbf{R} \to \mathbf{R} : s \mapsto u(s)$ such that

$$(\exp sD)P = \phi_{u(s)}(P). \tag{8}$$

Repeating the argument of Theorem 1.4, we conclude that $P = \phi_{u(0)}(P)$, and thus $DP = u'(0)F(\phi_{u(0)}(P)) = u'(0)F(P)$. Since $DP \neq 0$, we have $u'(0) \neq 0$. Thus $(\exp taD)P = \phi_t(P)$ for $a = 1/u'(0)$, using Proposition 1.2. This completes the proof. □

Remark. (1) The key idea behind Theorems 1.4 and 1.6 is that for both periodic and quasiperiodic trajectories, there exist invariants preserved by the automorphism group; in the case of periodic trajectories, the relevant invariant is the period, while for the quasiperiodic trajectories, the set of quasiperiods is preserved. In Section 3, we give an example of a system with a solution given by an automorphism where the trajectory is hyperbola. If there were some invariant associated with such trajectories, then we could prove a theorem like Theorems 1.4 and 1.6 that would guarantee that such solutions *must* be given by automorphisms provided that trajectories with the same invariant were isolated and that the derivation algebra acted nontrivially on some point in some trajectory. However, we know of no such invariant.

(2) When actually trying to find explicit solutions of a given system that are given by automorphisms, the workhorse result is Proposition 1.3; that is, given $F : \mathbf{R}^n \to \mathbf{R}^n$ and $D \in \operatorname{Der} F$, we try to solve the equation $DP = F(P)$ for P. The sections that follow are devoted to examples of this.

(3) The main results of this section should be compared with results obtained for vector fields in Birkhoff normal form with $O(2)$ symmetry [**3**, Chapter XVII]. Rotating wave solutions for such vector fields are actually given by the action of $SO(2)$. Our approach has the following advantages: (i) we do not assume that the vector field is in Birkhoff normal form, (ii) we make no assumptions about the eigenvalues of the linearization of the vector field, (iii) we make no assumptions about the structure of the group (we do not even assume compactness!).

Next we consider linearizations about solutions of the form $\phi_t(P) = (\exp tD)P$. First we need a lemma about how automorphisms in $\operatorname{Aut} F$ interact with higher derivatives of F. We shall not actually use the full strength of the lemma, but it has independent interest.

Lemma 1.7. *Let $\varphi \in \operatorname{Aut} F$ and let $k \geq 1$ be an integer. For all $P \in \mathbf{R}^n$, and all $Y_1, \dots, Y_k \in \mathbf{R}^n$,*

$$F^{(k)}(\varphi P)(Y_1, \dots, Y_k) = \varphi F^{(k)}(P)(\varphi^{-1} Y_1, \dots, \varphi^{-1} Y_k)$$

where $F^{(k)}(P)$ denotes the k-th derivative of F evaluated at P.

Proof. The proof is by induction on k, and in fact we may start the induction at $k = 0$ because the definition of $\operatorname{Aut} F$ is just the assertion in this case. Thus assume the assertion holds for $k - 1$ derivatives of F ($k \geq 1$). Then using the induction hypothesis and the linearity of φ, we have

$$\begin{aligned} F^{(k)}(\varphi P)&(Y_1, \dots, Y_k) \\ &= \frac{d}{ds}|_{s=0} F^{(k-1)}(\varphi P + sY_k)(Y_1, \dots, Y_{k-1}) \\ &= \frac{d}{ds}|_{s=0} \varphi F^{(k-1)}(P + s\varphi^{-1} Y_k)(\varphi^{-1} Y_1, \dots, \varphi^{-1} Y_{k-1}) \end{aligned}$$

$$= \varphi \frac{d}{ds}|_{s=0} F^{(k-1)}(P + s\varphi^{-1}Y_k)(\varphi^{-1}Y_1, \ldots, \varphi^{-1}Y_{k-1})$$
$$= \varphi F^{(k)}(P)(\varphi^{-1}Y_1, \ldots, \varphi^{-1}Y_k).$$

Now suppose $\phi_t(P) = (\exp tD)P$ is a solution to (1). We linearize the differential system around $\phi_t(P)$ and consider the nonautonomous problem

$$\dot{Y} = F'(\phi_t(P))Y.$$

Then $F'(\phi_t(P))Y = F'((\exp tD)P)Y = (\exp tD)F'(P)(\exp -tD)Y$, using Lemma 1.7. Thus we make a change of variables and set $Z(t) = (\exp -tD)Y(t)$. Then

$$\begin{aligned}\dot{Z} &= (\exp -tD)\dot{Y} - D(\exp -tD)Y \\ &= (\exp -tD)(\exp tD)F'(P)(\exp -tD)Y - D(\exp -tD)Y \\ &= (F'(P) - D)Z,\end{aligned}$$

and $Z(0) = Y(0)$. Thus $Z(t) = (\exp t(F'(P) - D))Y(0)$, and the solution of the linearized system is

$$Y(t) = (\exp tD)(\exp t(F'(P) - D))Y(0).$$

Note that in the case where $\phi_t(P) = (\exp tD)P$ is a periodic solution of period, say τ, we have found the Floquet decomposition of the principal matrix solution to $\dot{Y} = F'(\phi_t(P))Y : (\exp tD)Y(0)$ is nonsingular and periodic of period τ, and $F'(P) - D$ is a constant matrix; see [**1**]. In particular, the orbital stability of $\phi_t(P)$ can be determined by the eigenvalues of $F'(P) - D$; these are the characteristic exponents of the linearized system.

2. A periodic example. In this section we construct an example of a three-dimensional quadratic vector field $F(X)$ having a periodic solution given by the action of a one-parameter subgroup of the automorphism group on an initial point. Thus, we look for a vector field supporting a solution of the form $\phi_t(P) = \exp tbDP$ where $b \in \mathbf{R}$, D is a derivation, and P is a solution to $DP = F(P)$. Since we desire a *linear* group action to yield a periodic solution, we may suppose that relative to some basis $\{X_0, X_1, X_2\}$, D has the form

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

We determine all three-dimensional quadratic vector fields $F(X) = TX + B(X,X)$ having D as a derivation. This generalizes an example in [**5**, Section 5].

In the case of a homogeneous quadratic vector field, $F(X) = B(X,X)$ was found in [**5**] by first decomposing (the complexification of) the associated algebra into the root spaces of the derivation, and then using the basic rules of root space multiplication to determine a multiplication table for the algebra. We will use the same method on all of our later examples in this paper. In this example, for the sake of comparison, we give a different argument that is closer to the spirit of invariant theory. The techniques in this example are the same as those used in the calculation of normal forms (see [**8**] and the references therein), and in the symmetry group approach to bifurcation theory [**3**].

In abstract Lie theoretic terms, the idea is to consider the space $\operatorname{poly}(3,\mathbf{R})$ of polynomial mappings on $\mathbf{R}^3$ to be a ($\mathbf{Z}$-graded) Lie algebra under the Poisson bracket $[F,G](X) = F'(X)G(X) - G'(X)\cdot F(X)$, and to compute the centralizer $\mathcal{C}(D) = \{F \in \operatorname{poly}(3,\mathbf{R}) : [D,F] = 0\}$ of the element D. More concretely, if $\Gamma = \{\exp tD : t \in \mathbf{R}\}$ ($\cong SO(2,\mathbf{R})$) denotes the one-parameter group associated with D, then $\mathcal{C}(D)$ is simply the space of Γ-equivariant mappings $F : \mathbf{R}^3 \to \mathbf{R}^3$.

Let $\mathcal{I}(D)$ denote the commutative, associative algebra of all Γ-invariant polynomial maps $p : \mathbf{R}^3 \to \mathbf{R}$, meaning $p(\gamma X) = p(X)$ for all $\gamma \in \Gamma$ and all $X \in \mathbf{R}^3$. The following are well known [**3, 8**].

Lemma 2.1. (a) *$\mathcal{I}(D)$ is finitely generated as an $\mathbf{R}$-algebra.*

(b) *$\mathcal{C}(D)$ is finitely generated as an $\mathcal{I}(D)$-module.*

We now determine a set of homogeneous generators for $\mathcal{I}(D)$ as an $\mathbf{R}$-algebra, and a set of homogeneous generators for $\mathcal{C}(D)$ as an $\mathcal{I}(D)$-module. Let $X = \sum x_i X_i$ denote an element of $\mathbf{R}^3$.

Lemma 2.2. (a) *The polynomials $f_1(X) = x_0$ and $f_2(X) = x_1^2 + x_2^2$ generate $\mathcal{I}(D)$ as an $\mathbf{R}$-algebra.*

(b) *The mappings $F_0(X) = X_0$, $F_1(X) = x_1X_1 + x_2X_2$, and $F_2(X) = x_2X_1 - x_1X_2$ generate $\mathcal{C}(D)$ as an $\mathcal{I}(D)$-module.*

We now specialize this to the case of quadratic vector fields. For a *linear* Γ-equivariant map $T : \mathbf{R}^3 \to \mathbf{R}^3$, we take

$$\begin{aligned} TX &= af_1(X)F_0(X) + \alpha F_1(X) + \beta F_2(X) \\ &= \begin{pmatrix} a & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

For a *bilinear* (i.e., homogeneous quadratic) Γ-equivariant map $B : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$, we take

$$\begin{aligned} B(X,X) &= (\lambda f_1(X)^2 + \mu f_2(X))F_0(X) \\ &\quad + 2c_1 f_1(X)F_1(X) + 2c_2 f_1(X)F_2(X) \\ &= \begin{pmatrix} \lambda x_0^2 + \mu(x_1^2 + x_2^2) \\ 2x_0(c_1x_1 - c_2x_2) \\ 2x_0(c_2x_1 + c_1x_2) \end{pmatrix}, \end{aligned}$$

where $\lambda, \mu, a, \alpha, \beta, c_1, c_2 \in \mathbf{R}$, and the factor of 2 is chosen for later convenience. Thus, the general quadratic Γ-equivariant quadratic mapping $F : \mathbf{R}^3 \to \mathbf{R}^3$ is

$$F(X) = TX + B(X,X).$$

In order to find solutions of the form $(\exp tbD)P$, we must solve the equation $F(P) = bDP$. In terms of coordinates $P = \sum p_i X_i$, we have the following system of equations:

$$ap_0 + \lambda p_0^2 + \mu(p_1^2 + p_2^2) = 0 \tag{9}$$

$$\alpha p_1 + \beta p_2 + 2p_0(c_1p_1 - c_2p_2) = bp_2 \tag{10}$$

$$-\beta p_1 + \alpha p_2 + 2p_0(c_2p_1 + c_1p_2) = -bp_1. \tag{11}$$

In this section we give an exhaustive analysis of this system. This is a very tedious task, because of all the unspecified parameters (akin to doing bifurcation theory with eight bifurcation parameters!). In the sections that follow, we will just indicate what the system $F(P) = bDP$ looks like in coordinates.

Note that we can rewrite equations (10) and (11) as a homogeneous matrix-vector equation

$$(12) \qquad \begin{pmatrix} \alpha + 2p_0c_1 & \beta - b - 2p_0c_2 \\ 2p_0c_2 - \beta + b & \alpha + 2p_0c_1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Now assume that $P \neq 0$. If $p_1 = p_2 = 0$, then by (9), $ap_0 + \lambda p_0^2 = 0$. Thus either $p_0 = 0$ (a contradiction) or $\lambda p_0 = -a$. If $\lambda = 0$, then $a = 0$, and it follows that $P = p_0X_0$ is an equilibrium. If $\lambda \neq 0$, then $P = (-a/\lambda)X_0$ is an equilibrium.

Thus we may assume that at least one of p_1 and p_2 is not zero. This means that (12) has a nontrivial solution, so computing the determinant, we find

$$(\alpha + 2p_0c_1)^2 + (\beta - b - 2p_0c_2)^2 = 0.$$

Thus $2p_0c_1 = -\alpha$ and $2p_0c_2 = \beta - b$; hence, we have a consistency condition $\alpha c_2 = (b - \beta)c_1$. Set $c_1 = c\alpha_2$ and $c_2 = c(b - \beta)$ for $c \in \mathbf{R}$. If $c = 0$, then using the determinant condition, we have $\alpha = 0$ and $\beta = b$. This gives us the differential system

$$\begin{aligned} \dot{x}_0 &= ax_0 + \lambda x_0^2 + \mu(x_1^2 + x_2^2), \\ \dot{x}_1 &= bx_2, \qquad \dot{x}_2 = -bx_1. \end{aligned}$$

In this case any $P = \sum p_iX_i$ satisfying

$$ap_0 + \lambda p_0^2 + \mu(p_1^2 + p_2^2) = 0$$

is a periodic point, and $\phi_t(P) = (\exp tbD)P$ is the solution through P, with period $2\pi/|b|$. We obtain the same differential system if $c \neq 0$, $\alpha = 0$, and $b = \beta$. If $c \neq 0$, $\alpha = 0$, and $\beta \neq b$, then the differential system has the form

$$\begin{aligned} \dot{x}_0 &= ax_0 + \lambda x_0^2 + \mu(x_1^2 + x_2^2) \\ \dot{x}_1 &= bx_2 - 2c(b - \beta)x_0x_2 \\ \dot{x}_2 &= -bx_1 + 2c(b - \beta)x_0x_1. \end{aligned}$$

On the other hand, if $c \neq 0$, $b = \beta$ and $\alpha \neq 0$, then we have

$$\begin{aligned}\dot{x}_0 &= ax_0 + \lambda x_0^2 + \mu(x_1^2 + x_2^2)\\ \dot{x}_1 &= bx_2 + 2c\alpha x_0 x_1\\ \dot{x}_2 &= -bx_1 + 2c\alpha x_0 x_2.\end{aligned}$$

In either case, any point P satisfying $ap_0 + \lambda p_0^2 + \mu(p_1^2 + p_2^2) = 0$ yields a periodic solution $\phi_t(P) = (\exp tbD)P$ (when this quadric surface has nontrivial solutions).

If $c \neq 0$, $\alpha \neq 0$, and $b \neq \beta$, then $p_0 = -1/2c$. Thus, using (9), $\mu(p_1^2 + p_2^2) = (2ac - \lambda)/4c^2$. If $\mu = 0$, then $\lambda = 2ac$, and we have the system

$$\begin{aligned}\dot{x}_0 &= ax_0 + 2cax_0^2\\ \dot{x}_1 &= (2cx_0 + 1)(\alpha x_1 + \beta x_2) - 2cbx_0x_2\\ \dot{x}_2 &= (2cx_0 + 1)(-\beta x_1 + \alpha x_2) - 2cbx_0x_1.\end{aligned}$$

In this case, any $P = (-1/2c)X_0 + p_1X_1 + p_2X_2$ is a periodic point with solution $\phi_t(P) = (\exp tbD)P$.

If $\mu \neq 0$, then $p_1^2 + p_2^2 = (2ac - \lambda)/4\mu c^2$, and thus we assume $2ac > \lambda$ and $\mu > 0$, or $2ac < \lambda$ and $\mu < 0$. In this case the system can be written

$$\begin{aligned}\dot{x}_0 &= ax_0 + \lambda x_0^2 + \mu(x_1^2 + x_2^2)\\ \dot{x}_1 &= (2cx_0 + 1)(\alpha x_1 + \beta x_2) - 2cbx_0x_2\\ \dot{x}_2 &= (2cx_0 + 1)(-\beta x_1 + \alpha x_2) - 2cbx_0x_1.\end{aligned}$$

We can describe the periodic points P using cylindrical coordinates $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$. In this case we have

$$P = P(\theta) = \frac{-1}{2c}X_0 + \left(\frac{2ac - \lambda}{4\mu c^2}\right)^{1/2}[(\cos\theta)X_1 + (\sin\theta)X_2].$$

These points lie on the surface $ax_0 + \lambda x_0^2 + \mu(x_1^2 + x_2^2) = 0$.

Next we turn our attention to the Floquet theory for this three-dimensional example. We will only consider the last case, leaving the others to the reader. Recall from Section 1 that the characteristic exponents of the periodic solution passing through P are exactly the eigenvalues of the matrix

$$F'(P) - bD$$
$$= \begin{pmatrix} a + 4cap_0 & 0 & 0 \\ 2c(\alpha p_1 + \beta p_2) - 2cbp_2 & \alpha(2cp_0 + 1) & \beta(2cp_0+1)-2cbp_0-b \\ 2c(-\beta p_1+\alpha p_2)+2cbp_1 & -\beta(2cp_0+1)+2cbp_0+b & \alpha(2cp_0 + 1) \end{pmatrix}.$$

Using $p_0 = -1/2c$, the characteristic polynomial is

$$\begin{aligned} \mathrm{char}\,(k) &= (a + 4cap_0 - k)((\alpha(2cp_0 + 1) - k^2) \\ &\quad + (\beta(2cp_0 + 1) - 2cbp_0 - b)^2) \\ &= -(a + k)k^2. \end{aligned}$$

Thus the eigenvalues are 0, 0, and $-a$.

From general theory, we always expect at least one of the characteristic exponents to be zero [**2**, p. 322]. Actually, we can see this directly because $F(P) = bDP$ is an eigenvector:

$$\begin{aligned} (F'(P) - bD)F(P) &= F'(P)F(P) - bDF(P) \\ &= F'(P)F(P) - bF'(P)DP \\ &= 0. \end{aligned}$$

Normally, the extra 0 would not allow us to conclude orbital stability/instability. However, because the automorphism group is nontrivial, we may use equivariant Floquet theory instead of traditional Floquet theory [**3**, Theorem XVI.6.2]. In particular, we check to see if Γ-equivariance forces $F'(P) - bD$ to have an extra 0 eigenvalue. We simply sketch the idea and leave it to the reader to check the details using [**3**]. From [**3**, Theorem XVI.6.1], the number of eigenvalues of $F'(P) - bDP$ that are 0 is the 1 plus dimension of the group minus the dimension of the *isotropy subgroup* of $\Gamma \times S^1$ where S^1 denotes the circle group acting on $\phi_t(P)$ (and all periodic solutions of the same period as $\phi_t(P)$) by phase shift. It is straightforward to check that the dimension of this subgroup is 1, and thus symmetry only forces one of the eigenvalues to be 0. Thus, the standard equivariant Floquet theorem does not apply, and we expect a center manifold to pass through $\phi_t(P)$.

3. A quasiperiodic example. Next we consider an example of a five-dimensional quadratic vector field $F(X) = TX + \beta(X, X)$ with a quasiperiodic solution given by an automorphism. For this example (as well as those in the next section), we construct the vector field by the methods of [**5**]. In order to have a quasiperiodic solution of the form $\phi_t(P) = (\exp tD)P$, we require that relative to some basis $\{X_0, \dots, X_4\}$, D has the form

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_1 & 0 & 0 \\ 0 & -b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & -b_2 & 0 \end{pmatrix},$$

for $b_i \in \mathbf{R}$, $i = 1, 2$. (Notice that unlike the previous example, we have absorbed the frequencies into our definition of D.) As before, we set $\Gamma = \{\exp tD : t \in \mathbf{R}\}$.

Unlike the "invariant theory" method of the previous example, the approach of [**5**] finds *homogeneous* Γ-equivariant polynomial mappings, i.e., Γ-equivariant polynomial mappings are determined one homogeneous term at a time. The idea is to work directly with the derivation D without dropping down to the automorphism group Γ. By Proposition 1.2, we first determine all linear mappings $T : \mathbf{R}^5 \to \mathbf{R}^5$ that commute with D. This is just an easy exercise in linear algebra; the general such T has the form (relative to $\{X_0, \dots, X_4\}$)

$$T = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & 0 & 0 \\ 0 & -\beta_1 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & -\beta_2 & \alpha_2 \end{pmatrix},$$

for $a, \alpha_i, \beta_i \in \mathbf{R}$, $i = 1, 2$.

Next we wish to determine the general bilinear mapping $B : \mathbf{R}^5 \times \mathbf{R}^5 \to \mathbf{R}^5$ relative to which D is a derivation. As remarked in Section 1, this would imply that D would be a derivation of the associated algebra $A = (\mathbf{R}^5, B)$. We can determine a multiplication table for A by determining how the eigenspaces of D multiply together. First we introduce the abbreviation $XY = B(X, Y)$. Since D has complex eigenvalues, we decompose the complexification $A_{\mathbf{C}}$ of the algebra A relative to D rather than A itself:

$$A_{\mathbf{C}} = A_{\mathbf{C}}(0) + A_{\mathbf{C}}(b_1 i) + A_{\mathbf{C}}(-b_1 i) + A_{\mathbf{C}}(b_2 i) + A_{\mathbf{C}}(-b_2 i),$$

a direct sum. Here $A_{\mathbf{C}}(u) = \{Z \in A_{\mathbf{C}} : DZ = uZ\}$. A well-known result (e.g., [**4**, p. 54]) shows that the eigenspaces satisfy $A_{\mathbf{C}}(u)A_{\mathbf{C}}(v) \subseteq A_{\mathbf{C}}(u+v)$ if $u+v$ is an eigenvalue; otherwise $A_{\mathbf{C}}(u)A_{\mathbf{C}}(v) = 0$. This gives us the following inclusions for products of eigenspaces:

$$A_{\mathbf{C}}(0)A_{\mathbf{C}}(0) \subseteq A_{\mathbf{C}}(0), \tag{13}$$

$$A_{\mathbf{C}}(0)A_{\mathbf{C}}(\pm b_k i) = A_{\mathbf{C}}(\pm b_k i)A_{\mathbf{C}}(0) \subseteq A_{\mathbf{C}}(\pm b_k i), \qquad k=1,2, \tag{14}$$

$$A_{\mathbf{C}}(b_k i)A_{\mathbf{C}}(-b_k i) = A_{\mathbf{C}}(-b_k i)A_{\mathbf{C}}(b_k i) \subseteq A_{\mathbf{C}}(0), \qquad k=1,2. \tag{15}$$

The only products yet to be determined are $A_{\mathbf{C}}(\pm b_k i)A_{\mathbf{C}}(\pm b_k i)$, $A_{\mathbf{C}}(\pm b_1 i)A_{\mathbf{C}}(\pm b_2 i)$, and $A_{\mathbf{C}}(\pm b_1 i)A_{\mathbf{C}}(\mp b_2 i)$. These depend on whether or not the sums $\pm b_k \pm b_k$, $\pm b_1 \pm b_2$ and $\pm b_1 i \mp b_2 i$ are themselves eigenvalues. We observe that if any of these sums are indeed eigenvalues, then b_1 and b_2 are rationally dependent. This implies that $\exp tD$ would give a *periodic* orbit. Since we desire our solution to be truly quasiperiodic, we assume that none of these sums is an eigenvalue. Thus, in addition to the above eigenspace products, we also have

$$A_{\mathbf{C}}(\pm b_k i)A_{\mathbf{C}}(\pm b_k i) = \{0\}, \tag{16}$$

$$A_{\mathbf{C}}(\pm b_1 i)A_{\mathbf{C}}(\pm b_2 i) = \{0\}, \tag{17}$$

$$A_{\mathbf{C}}(\pm b_2 i)A_{\mathbf{C}}(\mp b_1 i) = \{0\}. \tag{18}$$

Next we use this information to construct a multiplication table for the real algebra A. Relative to the basis $\{X_0, \dots, X_4\}$, we have

$$\begin{aligned} A_{\mathbf{C}}(0) &= \mathbf{C} \cdot X_0 \\ A_{\mathbf{C}}(b_1 i) &= \mathbf{C} \cdot (X_1 + iX_2) \\ A_{\mathbf{C}}(-b_1 i) &= \mathbf{C} \cdot (x_1 - iX_2) \\ A_{\mathbf{C}}(b_2 i) &= \mathbf{C} \cdot (X_3 + iX_4) \\ A_{\mathbf{C}}(-b_2 i) &= \mathbf{C} \cdot (X_3 - iX_4). \end{aligned}$$

Using (13), $X_0^2 = \lambda X_0^2$ for $\lambda \in \mathbf{C}$, but since $X_0^2 \in A$, λ is real. Next, using (14), $X_0(X_{1+j} + iX_{2+j}) = z_j(X_{1+j} + iX_{2+j})$ and $X_0(X_{1+j} -$

$iX_{2+j}) = \bar{z}_j(X_{1+j} - iX_{2+j})$ for some $z_j \in \mathbf{C}$, $j = 0$ or 2. Adding and subtracting these equations, we have $X_0X_{1+j} = c_{1+j}X_{1+j} + c_{2+j}X_{2+j}$ and $X_0X_{2+j} = -c_{2+j}X_{1+j} + c_{1+j}X_{2+j}$, where $c_{1+j} = (1/2)(z_j + \bar{z}_j)$ and $c_{2+j} = (1/2)(z_j - \bar{z}_j)$, $j = 0$ or 2. Using (15) and (16), $X_{1+j}^2 + X_{2+j}^2 = (X_{1+j} + iX_{2+j})(X_{1+j} - iX_{2+j}) = \mu_{1+j/2}X_0$ for $\mu_{1+j/2} \in \mathbf{R}$ and $X_{1+j}^2 - X_{2+j}^2 = (1/2)(X_{1+j} + iX_{2+j})(X_{1+j} + iX_{2+j}) + (X_{1+j} - iX_{2+j})(X_{1+j} - iX_{2+j}) - (X_{1+j} - iX_{2+j})(X_{1+j} - iX_{2+j}) = 0$ so $X_{1+j}X_{2+j} = 0$. Similarly, (17) and (18) show $X_1X_3 = X_1X_4 = X_2X_3 = X_2X_4 = 0$. Summarizing this information, we obtain the following multiplication table for $A = (\mathbf{R}^n, B)$:

B	X_0	X_1	X_2	X_3	X_4
X_0	λX_0	$c_1X_1 + c_2X_2$	$-c_2X_1 + c_1X_2$	$c_3X_3 + c_4X_4$	$-c_4X_3 + c_3X_4$
X_1	$c_1X_1 + c_2X_2$	μ_1X_0	0	0	0
X_2	$c_2X_1 + c_1X_2$	0	μ_1X_0	0	0
X_3	$c_3X_3 + c_4X_4$	0	0	μ_2X_0	0
X_4	$-c_4X_3 + c_3X_4$	0	0	0	μ_2X_0

Thus, the general Γ-equivariant quadratic vector field $F(X)$ for which there might be a quasiperiodic solution given by the action of Γ has the form

$$\begin{aligned} F(X) &= TX + B(X,X) \\ &= \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & 0 & 0 \\ 0 & -\beta_1 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & -\beta_2 & \alpha_2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \\ &\quad + \begin{pmatrix} \lambda x_0^2 + \mu_1(x_1^2 + x_2^2) + \mu_2(x_3^2 + x_4^2) \\ 2x_0(c_1x_1 - c_2x_2) \\ 2x_0(c_2x_1 + c_1x_2) \\ 2x_0(c_3x_3 - c_4x_4) \\ 2x_0(c_4x_3 + c_3x_4) \end{pmatrix}. \end{aligned}$$

Finally, we indicate in coordinates what the system $F(P) = DP$ turns out to be:

$$\begin{aligned}\lambda p_0^2 + \mu_1(p_1^2 + p_2^2) + \mu_2(p_3^2 + p_4^2) &= 0\\ 2p_0(c_1p_1 - c_2p_2) &= b_1p_2\\ 2p_0(c_2p_1 + c_1p_2) &= -b_1p_1\\ 2p_0(c_3p_3 - c_4p_4) &= b_2p_4\\ 2p_0(c_4p_3 + c_3p_4) &= -b_2p_3.\end{aligned}$$

A tedious analysis like that of the previous section yields cases when this system of equations has nontrivial solutions; we omit this. As desired, the points lie on invariant tori, i.e., the solution trajectories $\phi_t(P) = (\exp tD)P$ are quasiperiodic.

4. A "hyperbolic" example. In this section we construct a three-dimensional example of a quadratic vector field $F(X)$ having a solution $\phi_t(P) = (\exp tbD)P$ where, relative to a basis $\{X_0, X_1, X_2\}$, the derivation D has the form

$$D = \begin{pmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

This trajectory is a hyperbola. As indicated in Section 1, we know of no invariant associated with this type of solution that is preserved by the automorphism group. Nevertheless, we can use Proposition 1.3 to find such solutions once we have determined the general quadratic mapping $F(X)$ which is Γ-equivariant; here $\Gamma = \{\exp tD : t \in \mathbf{R}\}$.

We follow the procedure outlined in the last section. First we determine the linear mappings $T : \mathbf{R}^3 \to \mathbf{R}^3$ commuting with D. Again, this is just linear algebra; such mappings have the form

$$T = \begin{pmatrix} a_0 & 0 & 0\\ 0 & a_1 & 0\\ 0 & 0 & a_2\end{pmatrix},$$

for $a_i \in \mathbf{R}$, $i = 0, 1, 2$.

Next, we determine the bilinear mappings $B : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$ such that D is a derivation of the associated algebra $A = (\mathbf{R}^3, B)$. We decompose A relative to D as follows:

$$A = A(0) + A(1) + A(-1),$$

a direct sum. Note that $A(0) = \mathbf{R} \cdot X_0$, $A(1) = \mathbf{R} \cdot X_1$, and $A(-1) = \mathbf{R} \cdot X_2$. Using the multiplication rules $A(u)A(v) \subseteq A(u+v)$ if $u+v$ is an eigenvalue, and $A(u)A(v) = 0$ otherwise, we can find the multiplication table for A. The arguments are similar to those in the previous section (in fact, easier, because we are only working over the reals). We obtain the following table:

B	X_0	X_1	X_2
X_0	λX_0	$c_1 X_1$	$c_2 X_2$
X_1	$c_1 X_1$	0	μX_0
X_2	$c_2 X_2$	μX_0	0

Thus the general quadratic vector field having D as a derivation has the form

$$F(X) = \begin{pmatrix} a_0 x_0 \\ a_1 x_1 \\ a_2 x_2 \end{pmatrix} + \begin{pmatrix} \lambda x_0^2 + 2\mu x_1 x_2 \\ 2c_1 x_0 x_1 \\ 2c_2 x_0 x_2 \end{pmatrix}.$$

Finally, we of course must solve the system of equations $F(P) = bDP$. In coordinates, this is the following:

$$a_0 p_0 + \lambda p_0^2 + 2\mu p_1 p_2 = 0 \tag{19}$$

$$a_1 p_1 + 2c_1 p_0 p_1 = b p_1 \tag{20}$$

$$a_2 p_2 + 2c_2 p_0 p_2 = -b p_2. \tag{21}$$

As in the previous section, we omit the tedious analysis of these equations. Let us simply state that the expected hyperbolic solutions do indeed occur.

Remark. (1) In this paper, we have been using quadratic examples to maintain continuity with [**5**]. However, our technique for finding homogeneous equivariant polynomial mappings extends to higher order terms as well. The idea is to compute multiplication tables for the associated k-ary algebra. This is straightforward because the multiplication rules for the eigenspaces of the derivation extend to the k-ary case (i.e., the eigenvalues must add correctly) [**7**].

(2) As we have seen, the hard work in using our method to construct solutions of the form $\phi_t(P) = (\exp tD)P$ is in solving the equation

$F(P) = DP$. We think that two tools may prove to be quite useful in the analysis of such equations: (i) the use of k-linear forms that are preserved by the automorphism group (see [**5**] and [**7**] for how such forms can be used), and (ii) techniques from algebraic geometry.

(3) In closing, we make comparisons between our technique for computing equivariant vector fields and the "invariant theory method" sketched in Section 2. The invariant theory technique yields the general polynomial (in fact analytic) vector field all at once, while our method only yields one homogeneous term at a time. However, there are no general methods for finding generating sets of homogeneous polynomials for the algebra of polynomial invariants, or for finding generators for the module of equivariant polynomial mappings (unless the group is well-behaved, like $SO(2)$). By working directly with the derivations (i.e., staying at the Lie algebra level), we are able to automate the procedure of finding homogeneous equivariant polynomial mappings; we simply construct multiplication tables for the associated k-ary algebras. We think that this procedure could be programmed using a symbolic manipulation package.

REFERENCES

1. F. Brauer and J.A. Nohel, *The qualitative theory of ordinary differential equations*, Dover Publications, New York, 1989.

2. E.A. Coddington and N.L. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York-Toronto-London, 1955.

3. M. Golubitsky, I. Stewart and D.G. Schaffer, *Singularities and groups in bifurcation theory*, Vol. II, Applied Mathematical Sciences, vol. 69, Springer-Verlag, Berlin-Heidelberg-New York, 1988.

4. N. Jacobson, *Lie algebras*, Dover Publications, New York, 1979.

5. M.K. Kinyon and A.A. Sagle, *Quadratic dynamical systems and algebras*, to appear in J. Differential Equations.

6. A.A. Sagle and R.Walde, *Introduction to Lie groups and Lie algebras*, Academic Press, New York, 1973.

7. S. Walcher, *Algebras and differential equations*, Hadronic Press, Palm Harbor, 1991.

8. ———, *On differential equations in normal form*, Mat. Ann. **291** (1991), 293–314.

9. T. Yoshizawa, *Stability theory and the existence of periodic solutions and almost periodic solutions*, Applied Mathematical Sciences, vol. 14, Springer-Verlag, New York, 1975.

Department of Mathematics, University of Utah, Salt Lake City, Utah 84112

Department of Mathematics, University of Hawaii at Hilo, Hilo, Hawaii 96720-4091