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ON CONVERGENT $(0,3)$ INTERPOLATION PROCESSES

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Dedicated to Professor A. Sharma on his 70th birthday

ABSTRACT. It is proved that the "pure" and "modified" (0,3) interpolation operators based on the roots of the integral of the Legendre polynomials uniformly converge for all continuous functions. Until now, no such algebraic Birkhoff interpolation was known. Error estimates in terms of moduli of continuity, as well as the optimal order of uniform convergence are determined.

Introduction. As the pioneering work of J. Balázs and P. Turán [**2**] has shown, the (0,2) interpolation (i.e., when function values and zero second derivatives are prescribed) based on the roots

$$-1 = x_1 < x_2 < \cdots < x_n = 1 \tag{1}$$

of the polynomial

$$\pi_n(x) = (1-x^2)P'_{n-1}(x), \tag{2}$$

converges uniformly for *some* continuous functions in the interval $[-1,1]$. (Here $P_n(x)$ is the Legendre polynomial of degree n normed such that $P_n(1)=1$.) The condition of convergence was later improved by G. Freud [**4**] and H. Gonska [**5**], but as P. Vértesi [**8**] has shown, the procedure is not uniformly convergent for *all* continuous functions, the reason being that the Lebesgue constant of this type of interpolation is of order exactly $O(n)$. The situation is similar for other classical systems of nodes, and the conjecture is that whenever the (0,2) interpolating polynomials exist, they always diverge for some properly chosen continuous function.

Thus, looking for Birkhoff interpolation procedures that are uniformly convergent for all continuous functions, one may turn to higher order

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$(0, M)$ interpolation. This has been done in a recent work of M.R. Akhlaghi, A.M. Chak and A. Sharma [**1**], where they considered "modified" and "pure" (0,3) interpolation. It turned out that these problems are uniquely solvable for all $n \geq 3$, and in order to obtain the so-called fundamental polynomials of pure (0,3) interpolation, first it is more convenient to consider a modified procedure. However, the fundamental polynomials of first kind (being the most important) are given in integral form in [**1**], which practically prevents us from applying the results to convergence problems.

In this paper we overcome this difficulty by giving a simpler form to the fundamental polynomials of first kind of the modified (0,3) interpolation (Section 1.3). This will be applied to get uniform error estimates (Section 1.2). The results will provide Birkhoff interpolation operators *convergent for all continuous functions* in $[-1, 1]$, similar to the classical result of L. Fejér for the (0,1) interpolation on the Chebyshev nodes. In Section 1.6 we give an example of a function for which the error estimate is better than that given by Corollary 1, and in Section 1.7 we prove that this error is optimal. In Part II the corresponding questions for the pure (0,3) interpolation will be considered.

1. Modified (0,3) interpolation.

1.1. Definitions. Modified (0,3) interpolation means that at the endpoints ± 1, instead of third derivatives we prescribe *first* derivatives. It is proved in [**1**] that for $n \geq 3$ there exist polynomials $r_\nu(x)$ $(\nu = 1, \dots, n)$, $\sigma_1(x), \sigma_n(x), \rho_\nu(x)$ $(\nu = 2, \dots, n-1)$ of degree $2n-1$ such that (see (1))

(3)
$$r_\nu(x_j) = \delta_{\nu j}, \quad j = 1, \dots, n, \qquad r'_\nu(\pm 1) = r'''_\nu(x_j) = 0, \quad j = 2, \dots, n-1;$$

$$\begin{gathered} \sigma_1(x_j) = 0, \qquad j = 1, \dots, n; \\ \sigma'_1(-1) = 1, \qquad \sigma'_1(1) = \sigma'''_1(x_j) = 0, \qquad j = 2, \dots, n-1; \end{gathered} \tag{4}$$

$$\begin{gathered} \sigma_n(x_j) = 0, \qquad j = 1, \dots, n; \\ \sigma'_n(-1) = \sigma'''_n(x_j) = 0, \qquad j = 2, \dots, n-1, \qquad \sigma'_n(1) = 1; \end{gathered} \tag{5}$$

$$(6)\qquad \begin{aligned} &\rho_\nu(x_j) = \rho_\nu'(\pm 1) = 0, \qquad j = 1, \dots, n,\\ &\rho_\nu'''(x_j) = \delta_{\nu j}, \qquad j = 2, \dots, n-1. \end{aligned}$$

These polynomials give rise to the definition of the modified (0,3) interpolation operator

$$(7)\qquad R_n(f,x) = \sum_{j=1}^{n} f(x_j) r_j(x)$$

defined for all $f(x) \in C[-1,1]$ and having the properties

$$\begin{aligned} &R_n(f,x_j) = f(x_j), \qquad j = 1, \dots, n;\\ &R_n'(f,\pm 1) = R_n'''(f,x_j) = 0, \qquad j = 2, \dots, n-1. \end{aligned}$$

We note that sometimes the seemingly more general operator

$$(8)\qquad \overline{R}_n(f,x) = R_n(f,x) + \alpha_1\sigma_1(x) + \alpha_n\sigma_n(x) + \sum_{j=2}^{n-1} \beta_j \rho_j(x)$$

is considered, where the numbers $\alpha_1, \alpha_n, \beta_j$ $(j = 2, \dots, n-1)$ are subject to certain growth conditions in order to ensure the uniform convergence of (8). However, these conditions are independent of the structural properties of $f(x)$, and they guarantee only that $\overline{R}_n - R_n$ should converge to zero uniformly as $n \to \infty$, thus not contributing anything to the convergence of R_n to f. Therefore, we prefer to consider the operator (7).

1.2. Uniform error estimates. Let $\omega_s(g,h)$ denote the modulus of smoothness of order s of $g \in C[-1,1]$, and let $||g|| = \max_{|x|\le 1} |g(x)|$.

Theorem 1. *We have*

$$||f(x) - R_n(f,x)|| = \begin{cases} O\Big(\omega_1\Big(f, \dfrac{\log^{1/3} n}{n}\Big)\Big), & \text{if } f \in C[-1,1],\\ O\Big(\omega_3\Big(f, \dfrac{\log^{1/3} n}{n}\Big)\Big) + O(n^{-2}), & \text{if } f' \in C[-1,1]. \end{cases}$$

Here the "O" signs indicate absolute constants. It will be clear from the proof that the following corollary holds:

Corollary 1. *If* $f'(x) \in C[-1,1]$ *and* $f'(\pm 1) = 0$, *then*

$$||f(x) - R_n(f,x)|| = O\left(\frac{\log^{1/3} n}{n}\omega_2\left(f', \frac{\log^{1/3} n}{n}\right)\right).$$

We do not know that, in general, the $\log^{1/3} n$ terms can be dropped from these estimates. In some particular cases this is possible (see Section 1.6). On the other hand, the term $O(n^{-2})$ in Theorem 1 cannot be omitted, since then we would have for $f(x) = x$

$$0 = x - R_n(f,x) = \sigma_1(x) + \sigma_n(x),$$

a contradiction, since by (4)–(5), $\sigma_1'(1) + \sigma_n'(1) = 1$.

The proof of Theorem 1 is long and therefore it is broken into a series of lemmas.

1.3. Fundamental polynomials of modified (0,3) interpolation. In [**1**] (4.2) it is proved that the fundamental polynomials $\rho_\nu(x)$, $\nu = 2, \dots, n-1$, satisfying (6) can be given in the form

$$\rho_\nu(x) = \frac{\pi_n(x)(1-x_\nu^2)}{n^2(n-1)^2 P_{n-1}^3(x_\nu)} \sum_{k=2}^{n-1} \frac{(2k-1)\pi_k(x) P_{k-1}'(x_\nu)}{k(k-1)\lambda_{k,n}}, \quad \nu = 2, \dots, n-1, \tag{9}$$

where

$$\lambda_{k,n} = 3k(k-1) + n(n-1), \qquad k = 2, \dots, n-1. \tag{10}$$

As we mentioned above, the formulas of $r_\nu(x)$ and $\sigma_1(x)$, $\sigma_n(x)$ given in [**1**] are not suitable for our purpose. In this connection we prove

Lemma 1. *We have*

$$r_1(x) = l_1^2(x) + \frac{n(n-1)}{2}\sigma_1(x) - 6\sum_{j=2}^{n-1} l_1'(x_j) l_1''(x_j)\rho_j(x) = r_n(-x), \tag{11}$$

$$(12)\quad r_\nu(x) = l_\nu^2(x) + \frac{2n(n-1)x_\nu}{(1-x_\nu^2)^2}\rho_\nu(x) - 6\sum_{j=2}^{n-1} l'_\nu(x_j)l''_\nu(x_j)\rho_j(x), \qquad \nu = 2,\dots,n-1,$$

$$(13)\quad \sigma_1(x) = \frac{(-1)^n}{n(n-1)}\Big[\pi_n(x)l_1(x) - 3\sum_{j=2}^{n-1}\pi'_n(x_j)l''_1(x_j)\rho_j(x)\Big] = -\sigma_n(-x),$$

where

$$(14)\quad \begin{cases} l_1(x) = \dfrac{(1-x)P'_{n-1}(x)}{(-1)^n n(n-1)} = l_n(-x), \\ l_\nu(x) = \dfrac{\pi_n(x)}{\pi'_n(x_\nu)(x-x_\nu)} = -\dfrac{(1-x^2)P'_{n-1}(x)}{n(n-1)P_{n-1}(x_\nu)(x-x_\nu)}, \end{cases}$$

$\nu = 2,\dots,n-1$ *are the fundamental polynomials of Lagrange interpolation.*

Proof. In order to save space, we do not give the details of deriving these formulae (11)–(13). On the other hand, to check that these polynomials satisfy conditions (3)–(5) is a routine work; therefore we only indicate that in doing so we may use the differential equation

$$(15)\quad (1-x^2)P''_{n-1}(x) - 2xP'_{n-1}(x) + n(n-1)P_{n-1}(x) = ((1-x^2)P'_{n-1}(x))' + n(n-1)P_{n-1}(x) = 0$$

of the Legendre polynomials. □

1.4. Estimates of the fundamental polynomials. From now on we shall use the convenient notations

$$x = \cos t, \qquad x_j = \cos t_j, \qquad 0 = t_n < t_{n-1} < \dots < t_2 < t_1 = \pi.$$

Lemma 2. *We have*

$$|P_n(x)+P_{n+1}(x)| = O\left(\sqrt{\frac{\sin t}{n}}\right), \qquad -1 \le x \le 1-\varepsilon < 1 \tag{16}$$

and

$$|P_n'(x)+P_{n+1}'(x)| = O\left(\sqrt{\frac{n}{\sin t}}\right), \qquad -1 < x \le 1-\varepsilon < 1 \tag{17}$$

where the "O" depends on ε*.*

Proof. Starting from the formula

$$P_n(x)+P_{n+1}(x) = (1+x)P_n(x) + n\int_{-1}^{x} P_n(\xi)\,d\xi \tag{18}$$

(see G. Sansone [**6**, page 201]), we obtain by differentiation

$$P_n'(x)+P_{n+1}'(x) = (n+1)P_n(x) + (1+x)P_n'(x),$$

i.e., using the estimates

$$\begin{aligned} |P_n(x)| &= O\left(\frac{1}{\sqrt{n\sin t}}\right), \\ |P_n'(x)| &= O\left(\sqrt{\frac{n}{\sin^3 t}}\right), \qquad -1 < x < 1 \end{aligned} \tag{19}$$

([**6**, III.10(8)] and G. Szegö [**7**, (7.33.7)] we obtain

$$\begin{aligned} |P_n'(x)+P_{n+1}'(x)| &\le (n+1)|P_n(x)| + \sin^2 t\cdot|P_n'(x)| \\ &= O\left(\sqrt{\frac{n}{\sin t}}\right) + O(\sqrt{n\sin t}) \\ &= O\left(\sqrt{\frac{n}{\sin t}}\right), \qquad -1 < x \le 1-\varepsilon. \end{aligned}$$

To get (16), we use (18), (15) and (19) again

$$\begin{aligned} P_n(x)+P_{n+1}(x) &= (1+x)P_n(x) - \frac{(1-x^2)P_n'(x)}{n+1}, \\ |P_n(x)+P_{n+1}(x)| &\le \sin^2 t\cdot|P_n(x)| + n^{-1}\sin^2 t|P_n'(x)| \\ &= O\left(\sqrt{\frac{\sin^3 t}{n}}\right) + O\left(\sqrt{\frac{\sin t}{n}}\right) \\ &= O\left(\sqrt{\frac{\sin t}{n}}\right), \qquad -1 \le x \le 1-\varepsilon. \qquad \square \end{aligned}$$

Lemma 3. *We have for an arbitrary $c > 0$*

$$(20)\quad \left|\sum_{s=1}^{m}(2s+1)P_s'(x)P_s(x_\nu)\right| = \begin{cases} O\left(\dfrac{n}{\sin^{3/2} t \sin^{1/2} t_\nu \sin \frac{|t-t_\nu|}{2}}\right), & \text{if } |t-t_\nu| \geq c/n \\ O\left(\dfrac{n^2}{\sin^{3/2} t \sin^{1/2} t_\nu}\right), & \text{if } |t-t_\nu| < c/n, \end{cases}$$

$$\nu = 2,\dots,n-1; \quad m = 1,2,\dots,n; \quad -1 < x < 1.$$

Proof. By $P_s(-x) = (-1)^s P_s(x)$ and $x_{n-\nu+1} = -x_\nu$, $\nu = 1,\dots,n$, we may assume that $-1 < x \leq 0$, i.e., $\pi/2 \leq t < \pi$. By the Christoffel-Darboux formula (see [**6**, III.5(18)])

$$(21)\quad \frac{1}{m+1}\sum_{s=1}^{m}(2s+1)P_s'(x)P_s(x_\nu)$$
$$= \frac{P_m'(x)P_{m+1}(x_\nu) - P_{m+1}'(x)P_m(x_\nu)}{x_\nu - x} + \frac{P_m(x)P_{m+1}(x_\nu) - P_{m+1}(x)P_m(x_\nu)}{(x_\nu - x)^2}$$
$$= \frac{[P_m'(x) + P_{m+1}'(x)]P_{m+1}(x_\nu) - [P_m(x_\nu) + P_{m+1}(x_\nu)]P_{m+1}'(x)}{x_\nu - x} + \frac{[P_m(x) + P_{m+1}(x)]P_{m+1}(x_\nu) - [P_m(x_\nu) + P_{m+1}(x_\nu)]P_{m+1}(x)}{(x_\nu - x)^2},$$

$x_\nu \neq x$. Applying Lemma 2, (19) and $\max(\sin t, \sin t_\nu) \leq 2\sin((t+t_\nu)/2)$ we get

$$\left|\sum_{s=1}^{m}(2s+1)P_s'(x)P_s(x_\nu)\right| = O\Bigg(m\frac{(\sin t \sin t_\nu)^{-1/2} + \sin^{-3/2} t \sin^{1/2} t_\nu}{\sin\frac{|t-t_\nu|}{2}\sin\frac{t+t_\nu}{2}}$$

$$
\begin{aligned}
&\qquad\qquad + \frac{\sqrt{\frac{\sin t}{\sin t_\nu}}+\sqrt{\frac{\sin t_\nu}{\sin t}}}{\sin^2\frac{t-t_\nu}{2}\sin^2\frac{t+t_\nu}{2}}\Bigg)\\
&= O\Bigg(\frac{m}{\sin^{3/2}t\sin^{1/2}t_\nu\sin\frac{|t-t_\nu|}{2}}\\
&\qquad + \frac{1}{(\sin t\sin t_\nu)^{1/2}\sin^2\frac{t-t_\nu}{2}\sin\frac{t+t_\nu}{2}}\Bigg)\\
&= O\Bigg(\frac{n}{\sin^{3/2}t\sin^{1/2}t_\nu\sin\frac{|t-t_\nu|}{2}}\Bigg),\\
&\frac{c}{n}\le|t-t_\nu|,\quad \frac{\pi}{2}\le t<\pi,\quad t_\nu\ge\frac{\pi}{3}.
\end{aligned}
$$

To extend the validity of this estimate for $t_\nu < \pi/3$, we apply the first relation in (21), as well as the estimates (19). We omit the easy calculations. Finally, the second relation in (20) is obtained by term-by-term estimates and on using (19) again. □

Lemma 4. *We have for $-1\le x\le 1$ and $\nu=2,\dots,n-1$*

(22)
$$
\rho_\nu(x)=\begin{cases}
-\dfrac{\pi_n^2(x)(1-x_\nu^2)}{n^2(n-1)^2\lambda_{n-1,n}P_{n-1}^2(x_\nu)(x-x_\nu)} \\
\qquad +O\bigg(\dfrac{\sin^2 t_\nu}{n^5\sin\frac{|t-t_\nu|}{2}}+\dfrac{\sin^3 t_\nu}{n^5\sin^2\frac{t-t_\nu}{2}}\bigg), & \text{if } t\ne t_\nu\\
O(n^{-3}\sin^3 t_\nu), & \text{if } |t-t_\nu|\le\frac{c}{n}.
\end{cases}
$$

Proof. Let

$$
K(x,x_\nu)=\sum_{m=2}^{n-1}\frac{(2m-1)P'_{m-1}(x)P'_{m-1}(x_\nu)}{m(m-1)\lambda_{m,n}},\qquad \nu=2,\dots,n-1. \tag{23}
$$

Applying Abel's summation, using the relations

$$
\frac{1}{\lambda_{m-1,n}}-\frac{1}{\lambda_{m,n}}=\frac{6(m-1)}{\lambda_{m-1,n}\lambda_{m,n}}
$$

(see (10)) and

$$(m-1)\sum_{s=2}^{m-1}\frac{(2s-1)P'_{s-1}(x)P'_{s-1}(x_\nu)}{s(s-1)} = \frac{P'_{m-1}(x)P'_{m-2}(x_\nu)-P'_{m-1}(x_\nu)P'_{m-2}(x)}{x-x_\nu}$$

(this follows from [**7**, (4.21.7) and (4.5.2)], we obtain

$$\begin{aligned}K(x,x_\nu) &= \frac{1}{\lambda_{n-1,n}}\sum_{m=2}^{n-1}\frac{(2m-1)P'_{m-1}(x)P'_{m-1}(x_\nu)}{m(m-1)} \\ &\quad+\sum_{m=3}^{n-1}\left(\frac{1}{\lambda_{m-1,n}}-\frac{1}{\lambda_{m,n}}\right)\sum_{s=2}^{m-1}\frac{(2s-1)P'_{s-1}(x)P'_{s-1}(x_\nu)}{s(s-1)} \\ &= \frac{P'_{n-1}(x)P'_{n-2}(x_\nu)}{\lambda_{n-1,n}(n-1)(x-x_\nu)} \\ &\quad+\frac{6}{x-x_\nu}\sum_{m=3}^{n-1}\frac{P'_{m-1}(x)P'_{m-2}(x_\nu)-P'_{m-1}(x_\nu)P'_{m-2}(x)}{\lambda_{m-1,n}\lambda_{m,n}}, \\ &\qquad x\neq x_\nu.\end{aligned}$$

Here we apply another Abel summation using

$$\frac{1}{\lambda_{m-2,n}\lambda_{m-1,n}}-\frac{1}{\lambda_{m-1,n}\lambda_{m,n}}=O(n^{-5}), \quad 3\le m\le n-1$$

and then make the substitutions

$$P'_{m-1}(x)=P'_{m-3}(x)+(2m-3)P_{m-2}(x)$$

(see [**6**, III.5(14)]) and

$$P'_{n-2}(x_\nu)=-(n-1)P_{n-1}(x_\nu)$$

(see [**7**, (4.7.28)]) to obtain

$$K(x,x_\nu)=-\frac{P'_{n-1}(x)P_{n-1}(x_\nu)}{\lambda_{n-1,n}(x-x_\nu)}+O\left(\frac{1}{n^4|x-x_\nu|}\right)$$

$$
\begin{aligned}
&\cdot \sum_{m=3}^{n-1} [P'_{m-1}(x)P'_{m-2}(x_\nu) - P'_{m-1}(x_\nu)P'_{m-2}(x)] \\
&\quad + O\left(\frac{1}{|x-x_\nu|}\right) \sum_{m=4}^{n-1} \left(\frac{1}{\lambda_{m-2,n}\lambda_{m-1,n}} - \frac{1}{\lambda_{m-1,n}\lambda_{m,n}}\right) \\
&\cdot \sum_{s=3}^{m-1} [P'_{s-1}(x)P'_{s-2}(x_\nu) - P'_{s-1}(x_\nu)P'_{s-2}(x)] \\
&= -\frac{P'_{n-1}(x)P_{n-1}(x_\nu)}{\lambda_{n-1,n}(x-x_\nu)} + O\left(\frac{1}{n^4|x-x_\nu|}\right) \\
&\cdot \left(|P'_{n-2}(x)P'_{n-3}(x_\nu)| + \left|\sum_{s=1}^{n-3}(2s+1)P'_s(x)P_s(x_\nu)\right|\right) \\
&\quad + O\left(\frac{1}{n^5|x-x_\nu|}\right) \sum_{m=4}^{n-1} \Bigg(|P'_{m-2}(x)P'_{m-3}(x_\nu)| \\
&\qquad\qquad + \left|\sum_{s=1}^{m-3}(2s+1)P'_s(x)P_s(x_\nu)\right|\Bigg), \qquad x \neq x_\nu.
\end{aligned}
$$

Now we apply Lemma 3 as well as (19):

$$
\begin{aligned}
K(x,x_\nu) &= -\frac{P'_{n-1}(x)P_{n-1}(x_\nu)}{\lambda_{n-1,n}(x-x_\nu)} + O\left(\frac{1}{n^4|x-x_\nu|}\right) \\
&\cdot \left(\frac{n}{(\sin t \sin t_\nu)^{3/2}} + \frac{n}{\sin^{3/2} t \sin^{1/2} t_\nu \sin \frac{|t-t_\nu|}{2}}\right) \\
&\quad + O\left(\frac{1}{n^5|x-x_\nu|}\right) \sum_{m=4}^{n-1} \Bigg(\frac{m}{(\sin t \sin t_\nu)^{3/2}} \\
&\qquad\qquad + \frac{m}{\sin^{3/2} t \sin^{1/2} t_\nu \sin \frac{|t-t_\nu|}{2}}\Bigg) \\
&= -\frac{P'_{n-1}(x)P_{n-1}(x_\nu)}{\lambda_{n-1,n}(x-x_\nu)} \\
&\quad + O\left(\frac{\sin\frac{|t-t_\nu|}{2} + \sin t_\nu}{n^3(\sin t \sin t_\nu)^{3/2} \sin^2 \frac{t-t_\nu}{2} \sin \frac{t+t_\nu}{2}}\right), \qquad x \neq x_\nu.
\end{aligned}
$$

Hence by (9) and (23) we obtain the first relation in (22). The second relation in (22) is obtained from (23) by using (2) and (19) term-by-term. We omit the details. □

Lemma 5. *We have*

$$\left\| \sum_{\nu=2}^{n-1} (1-x_\nu^2)^{-3/2} |\rho_\nu(x)| \right\| \sim \frac{\log n}{n^3}.$$

Proof. Lemma 4, (2) and (19) and the relation

$$0 < c_1 \le P_{n-1}^2(x_\nu) n \sin t_\nu \le c_2, \qquad \nu = 2, \dots, n-1 \tag{24}$$

(see [**7**, (7.3.8)] and [**2**, pages 202-203]) imply

$$\begin{aligned} |\rho_\nu(x)| &= O\left(\frac{\sin^3 t_\nu}{n^4 \sin \frac{|t-t_\nu|}{2}} + \frac{\sin^2 t_\nu}{n^5 \sin \frac{|t-t_\nu|}{2}} + \frac{\sin^3 t_\nu}{n^5 \sin^2 \frac{t-t_\nu}{2}} \right) \\ &= O\left(\frac{\sin^3 t_\nu}{n^4 \sin \frac{|t-t_\nu|}{2}} \right), \qquad |t-t_\nu| \ge c/n. \end{aligned} \tag{25}$$

Hence

$$\sum_{\nu=2}^{n-1} (1-x_\nu^2)^{-3/2} |\rho_\nu(x)| = O(n^{-4}) \sum_{|t-t_\nu| \ge c/n} \frac{1}{\sin \frac{|t-t_\nu|}{2}} + O(n^{-3}).$$

Now we use the asymptotic formula

$$t_{n+1-\nu} = \frac{4\nu+3}{2n-1} \cdot \frac{\pi}{2} + O\left(\frac{1}{n \cdot \min(\nu, n-\nu)} \right), \qquad \nu = 2, \dots, n-1 \tag{26}$$

(see P. Vértesi [**9**]), as well as the relation

$$|t_\mu - t_\nu| \ge c \cdot \frac{|\mu - \nu|}{n}, \qquad \mu, \nu = 1, \dots, n \tag{27}$$

(see [**7**, (6.21.7)]). With the notation

$$|t - t_s| = \min_{1 \le k \le n} |t - t_k| \tag{28}$$

we obtain

$$\sum_{|t-t_\nu|\geq c/n} \frac{1}{\sin\frac{|t-t_\nu|}{2}} = O\bigg(\sum_{|t-t_\nu|\geq c/n} \frac{n}{|s-\nu|}\bigg) = O(n\log n);$$

thus the upper estimate is proved. To prove the lower estimate we note that the main contribution of $\rho_\nu(x)$ comes from the first term in (22). Let, e.g., n be even, then

$$\begin{aligned}\bigg\|\sum_{\nu=2}^{n-1}(1-x_\nu^2)^{-3/2}|\rho_\nu(x)|\bigg\| &\geq \sum_{\nu=2}^{n-1}\sin^{-3} t_\nu |\rho_\nu(0)| \\ &\geq \frac{c}{n^4}\sum_{\nu=2}^{n-1}\frac{1}{|x_\nu|} + O(n^{-3}) \geq c_1\frac{\log n}{n^3}.\end{aligned}$$

Lemma 6. *We have*

$$||\sigma_1(x)|| = ||\sigma_n(x)|| \sim n^{-2}.$$

Proof. We obtain from (2), (14), (15) and (24)

$$\begin{aligned}|\pi_n(x)l_1(x)| &= O\bigg(\frac{(1-x^2)P_{n-1}'(x)^2}{n^2}\bigg) = O(1),\\ |\pi_n'(x_j)l_1''(x_j)| &= O\bigg(\frac{n}{\sin^5 t_j}\bigg), \qquad j=2,\dots,n-1.\end{aligned}$$

Thus (13) and (25)–(28) yield the upper estimate

$$\begin{aligned}|\sigma_1(x)| &= O(n^{-2})\bigg(1+n^{-3}\sum_{|t-t_\nu|\geq c/n}\frac{1}{\sin^2 t_\nu \sin\frac{|t-t_\nu|}{2}} + \frac{1}{n^2\sin^2 t_s}\bigg)\\ &= O(n^{-2}).\end{aligned}$$

Now, instead of proving the lower estimate for $\sigma_1(x)$, we shall prove more, namely that there exists a $y_n \in (-1,1)$ such that for n even

$$(29) \qquad \sigma_1(y_n) \pm \sigma_n(y_n) \sim \mp n^{-2} \quad \text{and} \quad \sigma_1(-y_n) \pm \sigma_n(-y_n) \sim n^{-2}.$$

We use the formula (n even)

$$\begin{aligned}\sigma_1(x) &= \frac{\pi_n(x)(1-x)}{2n(n-1)} \\ &\quad - \pi_n(x)\sum_{k=2}^{n-1}\frac{(2k-1)(n-k)(n+k-1)(-1)^k\pi_k(x)}{2n(n-1)k(k-1)\lambda_{k,n}} \\ &= \frac{\pi_n(x)(1-x)}{2n(n-1)} + O\left(\frac{|\pi_n(x)|\sqrt{\sin t}}{n^{3/2}}\right)\end{aligned}$$

(see [**1**, (4.3)]). Hence

$$\sigma_n(x) = -\frac{\pi_n(x)(1+x)}{2n(n-1)} + O\left(\frac{|\pi_n(x)|\sqrt{\sin t}}{n^{3/2}}\right),$$

i.e.,

$$\begin{aligned}\sigma_1(x)+\sigma_n(x) &= -\pi_n(x)\left[\frac{x}{n(n-1)} + O\left(\frac{\sqrt{\sin t}}{n^{3/2}}\right)\right], \\ \sigma_1(x)-\sigma_n(x) &= \pi_n(x)\left[\frac{1}{n(n-1)} + O\left(\frac{\sqrt{\sin t}}{n^{3/2}}\right)\right].\end{aligned}$$

It is easy to see from the properties of Legendre polynomials that

$$\pi_n(x) \geq c_1(n\sin t)^3 \quad \text{if} \quad 0 \leq t \leq \frac{1}{2}t_{n-1} \quad \text{or} \quad \pi - \frac{1}{2}t_2 \leq t \leq \pi.$$

Thus choosing $x = y_n = \cos a/n$ with a sufficiently small $a > 0$ we get (29). For odd n's the proof is analogous. □

Lemma 7. *We have*

$$\begin{aligned}\sum_{\substack{j=2 \\ j\neq\nu}}^{n-1}\frac{1-x_j^2}{(x_\nu-x_j)^3(x-x_j)} &= \frac{x_\nu n(n-1)}{6(1-x_\nu^2)(x-x_\nu)} - \frac{n(n-1)}{3(x-x_\nu)^2} \\ &\quad + \frac{n(n-1)P_{n-1}(x)}{(x-x_\nu)^3P'_{n-1}(x)} + \frac{1-x^2}{(x-x_\nu)^4}, \\ &x \neq x_i,\ i=2,\dots,n-1;\ \nu=2,\dots,n-1.\end{aligned}$$

Proof. Our starting point is the formula

$$\pi_n(x)\sum_{\substack{j=1\\ j\neq\nu}}^{n}\frac{1}{x-x_j}=\pi'_n(x)-\pi'_n(x_\nu)l_\nu(x). \tag{30}$$

Differentiating and substituting $x=x_\nu$, we obtain

$$\pi'_n(x_\nu)\sum_{\substack{j=1\\ j\neq\nu}}^{n}\frac{1}{x_\nu-x_j}=\pi''_n(x_\nu)-\pi'_n(x_\nu)l'_\nu(x_\nu)=0,$$

i.e.,

$$\sum_{\substack{j=1\\ j\neq\nu}}^{n}\frac{1}{x_\nu-x_j}=0,\qquad \nu=2,\dots,n-1.$$

Differentiating (30) twice and substituting $x=x_\nu$:

$$\begin{aligned}-2\pi'_n(x_\nu)\sum_{\substack{j=1\\ j\neq\nu}}^{n}\frac{1}{(x_\nu-x_j)^2}&=\pi'''_n(x_\nu)-\pi'_n(x_\nu)l''_\nu(x_\nu)\\ &=\frac{2}{3}\pi'''_n(x_\nu),\\ \sum_{\substack{j=1\\ j\neq\nu}}^{n}\frac{1}{(x_\nu-x_j)^2}&=-\frac{\pi'''_n(x_\nu)}{3\pi'_n(x_\nu)}=\frac{(n(n-1)}{3(1-x_\nu^2)},\\ \nu&=2,\dots,n-1.\end{aligned}$$

Finally, differentiating (30) three times and putting $x=x_\nu$:

$$\begin{aligned}6\pi'_n(x_\nu)\sum_{\substack{j=1\\ j\neq\nu}}^{n}\frac{1}{(x_\nu-x_j)^3}&=\pi^4_n(x_\nu)-\pi'_n(x_\nu)l'''_\nu(x_\nu)=\frac{3}{4}\pi^{(4)}_n(x_\nu),\\ \sum_{\substack{j=1\\ j\neq\nu}}^{n}\frac{1}{(x_\nu-x_j)^3}&=\frac{\pi^{(4)}_n(x_\nu)}{8\pi'_n(x_\nu)}=-\frac{x_\nu n(n-1)}{2(1-x_\nu^2)^2},\\ \nu&=2,\dots,n-1.\end{aligned}$$

Thus, partitioning with respect to x_j and using the above relations we obtain

$$\begin{aligned}\sum_{\substack{j=2\\ j\neq\nu}}^{n-1}\frac{1-x_j^2}{(x_\nu-x_j)^3(x-x_j)} &= \sum_{\substack{j=1\\ j\neq\nu}}^{n}\frac{1-x_j^2}{(x_\nu-x_j)^3(x-x_j)}\\
&= \frac{1-x_\nu^2}{x-x_\nu}\sum_{\substack{j=1\\ j\neq\nu}}^{n}\frac{1}{(x_\nu-x_j)^3}\\
&\quad+\left(\frac{2x_\nu}{x-x_\nu}-\frac{1-x_\nu^2}{(x-x_\nu)^2}\right)\sum_{\substack{j=1\\ j\neq\nu}}^{n}\frac{1}{(x_\nu-x_j)^2}\\
&\quad+\frac{1-x^2}{(x-x_\nu)^3}\sum_{\substack{j=1\\ j\neq\nu}}^{n}\frac{1}{x_\nu-x_j}-\frac{1-x^2}{(x-x_\nu)^3}\sum_{\substack{j=1\\ j\neq\nu}}^{n}\frac{1}{x-x_j}\\
&= -\frac{x_\nu n(n-1)}{2(1-x_\nu^2)(x-x_\nu)}+\frac{2x_\nu n(n-1)}{3(1-x_\nu^2)(x-x_\nu)}\\
&\quad-\frac{n(n-1)}{3(x-x_\nu)^2}-\frac{1-x^2}{(x-x_\nu)^3}\left(\frac{\pi_n'(x)}{\pi_n(x)}-\frac{1}{x-x_\nu}\right)\\
&= \frac{x_\nu n(n-1)}{6(1-x_\nu^2)(x-x_\nu)}-\frac{n(n-1)}{3(x-x_\nu)^2}\\
&\quad+\frac{n(n-1)P_{n-1}(x)}{(x-x_\nu)^3P_{n-1}'(x)}+\frac{1-x^2}{(x-x_\nu)^4},\end{aligned}$$

$\nu=2,\dots,n-1$. □

Lemma 8. *We have*

$$\left\|\sum_{\nu=1}^{n}|r_\nu(x)|\right\|=O(1).$$

Proof. Since $\sum_{k=1}^{n} l_k^2(x)\le 1$, we have $l_1^2(x)\le 1$, and an easy calculation shows (see (14)) that

$$|l_1'(x_j)l_1''(x_j)|=O(n^{-1}\sin^{-7}t_j),\qquad j=2,\dots,n-1.$$

Thus we obtain from (11), (25), (26), (28) and Lemma 6

$$\begin{aligned} |r_1(x)| &\le l_1^2(x) + O(n^2)||\sigma_1(x)|| \\ &\quad + O\bigg(\sum_{j=2}^{n-1} |l_1'(x_j) l_1''(x_j) \rho_j(x)|\bigg) \\ &= 1 + O(1) + O\bigg(\sum_{|t-t_j|\ge c/n} \frac{1}{n^5 \sin^4 t_j \sin \frac{|t-t_j|}{2}} + \frac{1}{n^4 \sin^4 t_s}\bigg) \\ &= O(1). \end{aligned}$$

The same estimate holds for $r_n(x)$.

In order to estimate $r_\nu(x)$, $\nu = 2, \dots, n-1$, we need the relation

$$(31)\qquad \begin{aligned} l_\nu'(x_j) l_\nu''(x_j) &= \frac{2P_{n-1}^2(x_j)}{(x_\nu - x_j)^3 P_{n-1}^2(x_\nu)} \\ &= O\bigg(\frac{\sin t_\nu}{|x_\nu - x_j|^3 \sin t_j}\bigg), \qquad \nu \ne j, \end{aligned}$$

which is easily checked from (14). Thus we obtain

$$\begin{aligned} &\sum_{\nu=2}^{n-1} \bigg| \sum_{|t-t_j|>c/n} l_\nu'(x_j) l_\nu''(x_j) \frac{\sin^{i+1} t_j}{n^t \sin^i \frac{|t-t_j|}{2}} \bigg| \\ &\qquad = O(n^{-5}) \sum_{|t-t_j|\ge c/n} \frac{1}{\sin^i \frac{|t-t_j|}{2}} \sum_{\substack{\nu=2 \\ \nu\ne j}}^{n-1} \frac{\sin t_\nu \sin^i t_j}{|\sin \frac{t_\nu - t_j}{2} \sin \frac{t_\nu + t_j}{2}|^3} \\ &\qquad = O(n^{-5}) \sum_{|t-t_j|\ge c/n} \frac{1}{\sin^{2-i} t_j \sin^i \frac{|t-t_j|}{2}} \sum_{\substack{\nu=2 \\ \nu\ne j}}^{n-1} \frac{1}{\sin^3 \frac{|t_\nu - t_j|}{2}} \\ &\qquad = O(n^{-2}) \sum_{|t-t_j|\ge c/n} \frac{1}{\sin^{2-i} t_j \sin^i \frac{|t-t_j|}{2}} \\ &\qquad = O\bigg(\sum_{\substack{j=2 \\ j\ne s}}^{n-1} \frac{1}{j^{2-i}|s-j|^i}\bigg) = O(1), \qquad i = 1, 2, \end{aligned}$$

where s has the same meaning as before. Thus we obtain from (12), (24), (25), (31) and Lemmas 4 and 7

$$\begin{aligned}
\sum_{\nu=2}^{n-1}|r_\nu(x)| &\le \sum_{\nu=2}^{n-1} l_\nu^2(x)+O(n^2)\sum_{\nu=2}^{n-1}\sin^{-4}t_\nu|\rho_\nu(x)| \\
&\quad + O\Big(\frac{\pi_n^2(x)}{n^6}\Big)\sum_{\nu=2}^{n-1}\frac{1}{P_{n-1}^2(x_\nu)}\Big|\sum_{\substack{j=2\\ j\ne\nu}}^{n-1}\frac{1-x_j^2}{(x_\nu-x_j)^3(x-x_j)}\Big| \\
&\quad + \sum_{\nu=2}^{n-1}\sum_{|t-t_j|\ge c/n}\Big|l_\nu'(x_j)l_\nu''(x_j)\Big(\frac{\sin^2 t_j}{n^5\sin\frac{|t-t_j|}{2}}+\frac{\sin^3 t_j}{n^5\sin^2\frac{t-t_j}{2}}\Big)\Big| \\
&\quad + \sum_{\substack{\nu=2\\ \nu\ne s}}^{n-1}\frac{\sin t_\nu}{|x_\nu-x_s|^3\sin t_s}\cdot\frac{\sin^3 t_s}{n^3} \\
&= O(n^{-2})\sum_{\substack{\nu=2\\ \nu\ne s}}^{n-1}\frac{1}{\sin t_\nu\sin\frac{|t-t_\nu|}{2}}+O\Big(\frac{\sin^4 t P_{n-1}'^2(x)}{n^5}\Big) \\
&\quad \cdot\sum_{\nu=2}^{n-1}\sin t_\nu\Big|\sum_{\substack{j=2\\ j\ne\nu}}^{n-1}\frac{1-x_j^2}{(x_\nu-x_j)^3(x-x_j)}\Big|+0(1) \\
&= O\Big(\frac{\sin t}{n^2}\Big)\sum_{\substack{\nu=2\\ \nu\ne s}}^{n-1}\Big(\frac{1}{\sin t_\nu|x-x_\nu|}+\frac{\sin t_\nu}{(x-x_\nu)^2} \\
&\qquad\qquad + \frac{\sin t\sin t_\nu}{n|x-x_\nu|^3}+\frac{\sin^2 t\sin t_\nu}{n^2(x-x_\nu)^4}\Big) \\
&\quad + O\Big(\frac{\sin^2 t}{n^4}\Big)\sum_{\substack{j=1\\ j\ne s}}^{n}\frac{1-x_j^2}{(x_s-x_j)^4}+O(1) \\
&= O\Big(\sum_{\substack{\nu=2\\ \nu\ne s}}^{n-1}\Big(\frac{1}{\nu|s-\nu|}+\frac{1}{(\nu-s)^2}+\frac{1}{|\nu-s|^3}+\frac{1}{(\nu-s)^4}\Big)\Big) \\
&\quad + O\Big(\sum_{\substack{j=1\\ j\ne s}}^{n}\frac{1}{(j-s)^4}\Big)+O(1)=O(1).
\end{aligned}$$

1.5. Proof of Theorem 1. Let $f^{(i)}(x) \in C[-1,1]$, $m = [n/(\log^{1/3} n)]$, and consider polynomials $p_n(x)$ of degree at most m such that

$$||f^{(j)}(x) - p_m^{(j)}(x)|| = O(m^{j-i})\omega_3\left(f^{(i)}, \frac{1}{m}\right), \quad 0 \le j \le i \tag{32}$$

and

$$||p_m^{(j)}(x)|| = O(m^{j-i})\omega_{j-i}\left(f^{(i)}, \frac{1}{m}\right), \qquad i+1 \le j \le i+3 \tag{33}$$

(cf. H. Gonska [**5**, p. 165]). Since the problem of modified (0,3) interpolation is uniquely solvable, evidently

$$p_m(x) - R_n(p_m, x) = p_m'(-1)\sigma_1(x) + p_m'(1)\sigma_n(x) + \sum_{\nu=2}^{n-1} p_m'''(x_\nu)\rho_\nu(x). \tag{34}$$

Here, applying Lemma 5 and (33) with $i = 0$ and $j = 3$, we get

$$\begin{aligned} \left|\sum_{\nu=2}^{n-1} p_m'''(x_\nu)\rho_\nu(x)\right| &= O(m^3)\omega_3\left(f, \frac{1}{m}\right)\sum_{\nu=2}^{n-1}|\rho_\nu(x)| \\ &= O(m^3)\omega_3\left(f, \frac{1}{m}\right)\frac{\log n}{n^3} \\ &= O\left(\omega_3\left(f, \frac{\log^{1/3} n}{n}\right)\right). \end{aligned} \tag{35}$$

Further, by (33) (with $i = 0$, $j = 1$)

$$||p_m'|| = O(m)\omega_1\left(f, \frac{1}{m}\right)$$

and thus, by Lemma 6,

$$\begin{aligned} ||p_m'(-1)\sigma_1(x) + p_m'(1)\sigma_n(x)|| &= O(||p_m'|| \cdot ||\sigma_1||) \\ &= O\left(\frac{m}{n^2}\right)\omega_1\left(f, \frac{1}{m}\right). \end{aligned} \tag{36}$$

Hence, by (32) (with $i = j = 0$) and Lemma 8

$$\begin{aligned}||f - R_n(f)|| &\le ||f - p_m|| + ||p_m - R_n(p_m)|| + ||R_n(p_m - f)|\\ &= O\left(\omega_3\left(f, \frac{1}{m}\right)\right) + O\left(\frac{m}{n^2}\right)\omega_1\left(f, \frac{1}{m}\right)\\ &\quad + ||p_m - f|| \cdot \left\| \sum_{k=1}^{n} |r_k(x)| \right\|\\ &= O\left(\omega_3\left(f, \frac{1}{m}\right)\right) + O\left(\frac{m}{n^2}\right)\omega_1\left(f, \frac{1}{m}\right).\end{aligned}$$

Considering the value of m, this completes the proof of Theorem 1. □

Proof of Corollary 1. Now if $f' \in C[-1,1]$ and $f'(\pm 1) = 0$, then (32) with $i = j = 1$ yields the better estimate

$$||p'_m|| = O\left(\omega_3\left(f, \frac{1}{m}\right)\right),$$

whence

$$||p'_m(-1)\sigma_1(x) + p'_m(1)\sigma_n(x)|| = O(n^{-2})\omega_3\left(f', \frac{1}{m}\right).$$

Thus we obtain

$$||f - R_n(f)|| = O\left(\omega_3\left(f, \frac{1}{m}\right)\right) + O(n^{-2})\omega_3\left(f', \frac{1}{m}\right)$$

whence Corollary 1 follows. □

1.6. A nontrivial example for an $O(n^{-3})$ order of convergence. Theorem 1 cannot provide $O(n^{-3})$ as the order of convergence. Nevertheless, we show that for the polynomial

$$f_1(x) = x^3 - 3x$$

we have

$$||f_1 - R_n(f_1)|| = O(n^{-3}).$$

To prove this, we need a strengthening of (24):

Lemma 9. *We have*

$$\frac{1}{P_{n-1}^2(x_k)} = \frac{\pi}{2}(n-1)\sin t_k\left(1+O\left(\frac{1}{\log n}\right)\right),$$
$$\log n \le k \le n-\log n.$$

Proof. We use the asymptotic formula

$$P_{n-1}(x_k) = \frac{\cos[(n-\frac{1}{2})t_k-\frac{\pi}{4}]+O(\frac{1}{n\sin t_k})}{\sqrt{\frac{\pi}{2}(n-1)\sin t_k}},$$
$$k=2,\dots,n-1$$

[**7**, (8.21.18)], as well as (26) in the form

$$t_{n+1-k} = \frac{4k+3}{2(2n-1)}\pi + O\left(\frac{1}{n\log n}\right),$$
$$\log n \le k \le n/2$$

to obtain

$$\begin{aligned}\frac{1}{P_{n-1}^2(x_k)} &= O\left(\frac{\frac{\pi}{2}(n-1)\sin t_k}{\cos O(\frac{1}{\log n})+\frac{1}{\log n}}\right)\\ &= \frac{\pi}{2}(n-1)\sin t_k\left(1+O\left(\frac{1}{\log n}\right)\right).\end{aligned}$$

Returning to the proof of (37), (34) yields

$$f_1(x)-R_n(f_1,x) = 6\sum_{\nu=2}^{n-1}\rho_\nu(x). \tag{38}$$

We have seen in the proof of Lemma 5 that the contribution of the remainder terms in (22) to $\sum_{\nu=2}^{n-1}|\rho_\nu(x)|$ is $O(n^{-3})$. Thus, without loss of generality, assuming $0\le x\le 1$, with the notation (28), we get

$$f_1(x)-R_n(f_1,x) = O\left(\frac{\pi_n^2(x)}{n^6}\right)\sum_{\nu=2}^{n-1}\frac{1-x_\nu^2}{P_{n-1}^2(x_\nu)(x-x_\nu)}+O(n^{-3})$$

$$
\begin{aligned}
&= O\left(\frac{\sin^3 t}{n^5}\right)\sum_{\substack{\nu=2\\ \nu\neq s}}^{n-1}\frac{1}{P_{n-1}^2(x_\nu)(x-x_\nu)}\\
&\quad + O\left(\frac{\sin t}{n^5}\right)\sum_{\nu=2}^{n-1}\frac{1}{P_{n-1}^2(x_\nu)}\\
&\quad + O\left(\frac{\pi_n(x)\pi_n'(x)(1-x^2)}{n^5}\right) + O(n^{-3})\\
&= O\left(\frac{\sin^3 t}{n^5}\right)\Bigg[\sum_{1\le|\nu|\le s/2}\frac{1}{P_{n-1}^2(x_{s+\nu})(x-x_{s+\nu})}\\
&\qquad\qquad + n\sum_{\substack{1\le\nu\le s/2\\ \text{or } 3s/2\le\nu\le n}}\frac{\sin t_\nu}{|x-x_\nu|}\Bigg] + O(n^{-3})\\
&= O\left(\frac{\sin^3 t}{n^5}\right)\sum_{1\le|\nu|\le s/2}\frac{1}{P_{n-1}^2(x_{s+\nu})(x-x_{s+\nu})}\\
&\quad + O\left(\frac{\sin^2 t}{n^4}\right)\sum_{\substack{1\le\nu\le s/2\\ \text{or } 3s/2\le\nu\le n}}\frac{\sin t_\nu}{\sin\frac{|t-t_\nu|}{2}} + O(n^{-3})\\
&= O\left(\frac{\sin^3 t}{n^5}\right)\sum_{1\le|\nu|\le s/2}\frac{1}{P_{n-1}^2(x_{s+\nu})(x-x_{s+\nu})}\\
&\quad + O(n^{-3}).
\end{aligned}
$$

Here, first let $1 \le s \le \log n$, then

$$
\begin{aligned}
O\left(\frac{\sin^3 t}{n^5}\right)&\sum_{1\le|\nu|\le s/2}\frac{1}{P_{n-1}^2(x_{s+\nu})|x-x_{s+\nu}|}\\
&= O\left(\frac{\log^3 n}{n^5}\right)\sum_{1\le|\nu|\le s/2}\frac{s+\nu}{\sin\frac{t_\nu}{2}\sin\frac{t_s+t_{s+\nu}}{2}}\\
&= O\left(\frac{\log^3 n}{n^6}\right)\sum_{1\le|\nu|\le s/2}\frac{s+\nu}{\nu(2s+\nu)}\\
&= O\left(\frac{\log^3 n\log\log n}{n^6}\right).
\end{aligned}
$$

In case $\log n \le s \le n/2$, applying Lemma 9 we get

$$
\begin{aligned}
O\left(\frac{\sin^3 t}{n^5}\right) & \sum_{1\le|\nu|\le s/2} \frac{1}{P_{n-1}^2(x_{s+\nu})|x-x_{s+\nu}|} \\
&= O\left(\frac{\sin^3 t}{n^5}\right) \sum_{\nu=1}^{s/2} \biggl(\frac{1}{P_{n-1}^2(x_{s+\nu})(x_s-x_{s+\nu})} \\
&\qquad + \frac{1}{P_{n-1}^2(x_{s-\nu})(x_s-x_{s-\nu})}\biggr) \\
&= O\left(\frac{\sin^3 t}{n^5}\right) \sum_{\nu=1}^{s/2} \biggl(\frac{2x_s-x_{s+\nu}-x_{s-\nu}}{P_{n-1}^2(x_{s+\nu})(x_s-x_{s\pm\nu})^2} \\
&\qquad + \frac{1}{|x_s-x_{s-\nu}|}\left|\frac{1}{P_{n-1}^2(x_{s+\nu})} - \frac{1}{P_{n-1}^2(x_{s-\nu})}\right|\biggr) \\
&= O\left(\frac{\sin^3 t}{n^5}\right) \sum_{\nu=1}^{s/2} \biggl(\frac{(1-\cos t_\nu)(s+\nu)}{\sin^2\frac{t_\nu}{2}\sin^2\frac{t_s+t_{s+\nu}}{2}} \\
&\qquad + \frac{n|\sin t_{s+\nu}-\sin t_{s-\nu}| + \frac{s+\nu}{\log n}}{\sin\frac{t_\nu}{2}\sin\frac{t_s+t_{s-\nu}}{2}}\biggr) \\
&= O\left(\frac{\sin t_s}{n^5}\right)\sum_{\nu=1}^{s/2} s + O\left(\frac{\sin^2 t_s}{n^4}\right)\sum_{\nu=1}^{s/2} 1 \\
&\qquad + O\left(\frac{\sin^2 t_s}{n^4\log n}\right)\sum_{\nu=1}^{s}\frac{s+\nu}{\nu} = O(n^{-3}).
\end{aligned}
$$

Collecting these estimates, we get (37).

1.7. The optimal order of convergence. The operator R_n is saturated as it is shown by the following

Theorem 2. *We have*

$$||f(x) - R_n(f,x)|| = o(n^{-3})$$

if and only if $f(x) = \text{const}$.

Proof. Since R_n reproduces constants, the "if" part is obvious. Assume now that

$$||f - R_{2^n}(f)|| \le \frac{\varepsilon_n}{8^n}, \quad \text{where} \quad \varepsilon_n \to 0 \quad \text{as } n \to \infty.$$

Then, by Bernstein's inequality

$$\begin{aligned} |R_{2^n}^{(4)}(f,x)| &\le \sum_{k=2}^{n} |R_{2^k}^{(4)}(f,x) - R_{2^{k-1}}^{(4)}(f,x)| \\ &\le \sum_{k=2}^{n} \left(\frac{2^{k+1}}{\sqrt{1-x^2}} \right)^4 8^{1-k}(\varepsilon_{k-1} + \varepsilon_k) \\ &\le \frac{48}{(1-x^2)^2} \sum_{k=1}^{n} 2^k \varepsilon_k \\ &= O((1-a^2)^{-2})(2^{n/2} \max_{k \ge 1} \varepsilon_k + 2^n \max_{k \ge n/2} \varepsilon_k) \\ &= o\left(\frac{2^n}{(1-a^2)^2} \right), \qquad |x| \le a < 1. \end{aligned}$$

Hence, using the notation (28),

$$\begin{aligned} |R_{2^n}'''(f,x)| &= |R_{2^n}'''(f,x) - R_{2^n}'''(f,x_s)| \\ &\le \max_{|x| \le a} |R_{2^n}^{(4)}(f,x)| \cdot |x - x_s| \\ &= o\left(\frac{2^n}{(1-a^2)^2} \right) \frac{\sqrt{1-a^2}}{2^n} \\ &= o((1-a^2)^{-3/2}) \to 0 \end{aligned}$$

as $n \to \infty$, $|x| \le b < a < 1$. Thus

$$\begin{aligned} |\Delta_h^3 f(x)| &\le |\Delta_h^3 (f(x) - R_{2^n}(f,x))| + |\Delta_h^3 R_{2^n}(f,x)| \\ &\le \frac{4\varepsilon_n}{8^n} + h^3 \max_{-b \le x \le b-3h} |R_{2^n}'''(f,x)| \to 0 \end{aligned}$$

as $n \to \infty$, $-b \le x \le x + 3h \le b$. Hence $\Delta_h^3 f(x) \equiv 0$ for any $h > 0$, i.e., $f(x)$ is a quadratic polynomial in $[-b, b]$. But $b < a < 1$ are arbitrary, whence

$$f(x) \equiv \alpha x^2 + \beta x + \gamma, \qquad |x| \le 1.$$

Using (34) we obtain

$$\begin{aligned} f(x) - R_{2^n}(f,x) &= f'(-1)\sigma_1(x) + f'(1)\sigma_{2^n}(x) \\ &= -2\alpha(\sigma_1(x) - \sigma_{2^n}(x)) + \beta(\sigma_1(x) + \sigma_{2^n}(x)). \end{aligned}$$

Now if $\operatorname{sgn}\alpha = \operatorname{sgn}\beta$, then by (29) we get

$$\frac{\varepsilon_n}{8^n} \geq |f(y_{2^n}) - R_{2^n}(f, y_{2^n})| \geq \frac{c(2|\alpha| + |\beta|)}{4^n}$$

whence $\alpha = \beta = 0$, i.e., $f(x) \equiv \gamma = \text{const.}$ Similarly, when $\operatorname{sgn}\alpha = -\operatorname{sgn}\beta$.

2. Pure (0,3) interpolation.

2.1. The fundamental polynomials and the main result. It is proved in [**1**] that the polynomials

$$(39) \qquad r_\nu^*(x) = \frac{1}{\Delta}\begin{vmatrix} r_\nu(x) & \sigma_1(x) & \sigma_n(x) \\ r_\nu'''(1) & \sigma_1'''(1) & \sigma_n'''(1) \\ r_\nu'''(-1) & \sigma_1'''(-1) & \sigma_n'''(-1) \end{vmatrix}, \qquad \nu = 1, \dots, n$$

and

$$(40) \quad \rho_\nu^*(x) = \frac{1}{\Delta}\begin{vmatrix} \rho_\nu(x) & \sigma_1(x) & \sigma_n(x) \\ \rho_\nu'''(1) & \sigma_1'''(1) & \sigma_n'''(1) \\ \rho_\nu'''(-1) & \sigma_1'''(-1) & \sigma_n'''(-1) \end{vmatrix}, \qquad \nu = 2, \dots, n-1,$$

$$(41) \quad \rho_1^*(x) = -\frac{1}{\Delta}\begin{vmatrix} \sigma_1(x) & \sigma_n(x) \\ \sigma_1'''(1) & \sigma_n'''(1) \end{vmatrix}, \qquad \rho_n^*(x) = \frac{1}{\Delta}\begin{vmatrix} \sigma_1(x) & \sigma_n(x) \\ \sigma_1'''(-1) & \sigma_1'''(1) \end{vmatrix}$$

of degree at most $2n-1$, where

$$\Delta = \sigma_1'''(1)^2 - \sigma_1'''(-1)^2,$$

satisfy

$$\begin{cases} r_\nu^*(x_j) = \rho_\nu^{*\prime\prime\prime}(x_j) = \delta_{\nu j}, \\ \rho_\nu^*(x_j) = r_\nu^{*\prime\prime\prime}(x_j) = 0, \end{cases} \qquad \nu, j = 1, \dots, n.$$

Thus we can define the pure (0,3) interpolation operator

$$R_n^*(f,x) = \sum_{j=1}^{n} f(x_j) r_j^*(x)$$

for any $f(x) \in C[-1,1]$. This operator has the properties

$$R_n^*(f,x_j) = f(x_j), \qquad R_n^{*'''}(f,x_j) = 0, \qquad j = 1,\dots,n.$$

Theorem 3. *If* $f(x) \in C[-1,1]$, *then*

$$||f(x) - R_n^*(f,x)|| = O\left(\omega_3\left(f, \frac{\log^{1/3} n}{n}\right)\right),$$

where $\omega_3(f,h)$ *is the modulus of smoothness of order* 3 *of* $f(x)$.

For the proof we need two lemmas.

2.2. Estimates of the fundamental polynomials.

Lemma 10. *We have*

$$\left\| \sum_{\nu=1}^{n} |\rho_\nu^*(x)| \right\| = O\left(\frac{\log n}{n^3}\right), \qquad |x| \le 1.$$

Proof. In [**1**] it is proved that $\Delta \neq 0$, but here we need more. It follows from [**1**], Section 5 that for n even

$$\begin{aligned} \sigma_1'''(-1) + \sigma_1'''(1) = {} & \frac{(n+1)n(n-1)(n-2)}{8} \\ & + \frac{3n(n-1)}{2} \sum_{k=1}^{n/2} \frac{(4k-3)(n-2k+1)(n+2k-2)}{\lambda_{2k-1,n}} \\ & + 3 \sum_{k=1}^{n/2} \frac{(4k-3)(n-2k+1)(n+2k-2)(2k-1)(k-1)}{\lambda_{2k-1,n}} \\ & \ge cn^4 \end{aligned}$$

and

$$\sigma_1'''(-1) - \sigma_1'''(1) = \frac{(n+1)n(n-1)(n-2)}{8} + \frac{3n(n-1)}{2} \cdot \sum_{k=1}^{n/2-1} \frac{(4k-1)(n-2k)(n+2k-1)}{\lambda_{2k,n}} + 3\sum_{k=1}^{n/2-1} \frac{(4k-1)(n-2k)(n+2k-1)k(2k-1)}{\lambda_{2k,n}} \geq cn^4.$$

Similar estimates hold when n is odd. Hence

$$|\Delta| \geq c^2 n^8.$$

On the other hand, by Lemma 6 and Markov's inequality

$$|\sigma_1'''(\pm 1)| = |\sigma_n'''(\pm 1)| = O(n^4), \tag{43}$$

whence and by (40)

$$|\rho_\nu^*(x)| = O(|\rho_\nu(x)| + n^{-6}[|\rho_\nu'''(-1)| + |\rho_\nu'''(1)|]), \quad |x| \leq 1;\ \nu = 2, \dots, n-1,$$

$$||\rho_1^*(x)|| = ||\rho_n^*(x)|| = O(n^{-6}).$$

Using Lemma 5,

$$\sum_{\nu=1}^{n} |\rho_\nu^*(x)| = O(n^{-6}) + O\bigg(\sum_{\nu=2}^{n-1} |\rho_\nu(x)|\bigg) + O(n^{-6}) \sum_{\nu=2}^{n-1} (|\rho_\nu'''(-1)| + |\rho_\nu'''(1)|) = O\bigg(\frac{\log n}{n^3}\bigg) + O(n^{-6}) \sum_{\nu=2}^{n-1} (|\rho_\nu'''(-1)| + |\rho_\nu'''(1)|).$$

But, using Markov's inequality, it is easy to see that Lemma 5 implies

$$\sum_{\nu=2}^{n-1} |\rho_\nu'''(x)| = O(n^3 \log n), \qquad |x| \leq 1. \qquad \square$$

Applying this with $x = \pm 1$, the statement of the lemma follows.

Lemma 11. *We have*

$$\left\| \sum_{\nu=1}^{n} |r_\nu^*(x)| \right\| = O(1).$$

Proof. It follows from (39), Lemma 6 and (43) that

$$|r_\nu^*(x)| = O(|r_\nu(x)| + n^{-6}|r_\nu'''(1)| + n^{-6}|r_\nu'''(-1)|),$$
$$|x| \le 1;\ \nu = 1, \dots, n,$$

whence and by Lemma 8

$$\sum_{\nu=1}^{n} |r_\nu^*(x)| = O\biggl(\sum_{\nu=1}^{n} |r_\nu(x)|\biggr) + O(n^{-6}) \sum_{\nu=1}^{n} (|r_\nu'''(1)| + |r_\nu'''(-1)|),$$
$$|x| \le 1.$$

Just as in the previous proof, it is easily seen from Lemma 8 that

$$\left\| \sum_{\nu=1}^{n} |r_\nu'''(x)| \right\| = O(n^6),$$

and hence Lemma 11 follows. □

2.3. Proof of Theorem 3. Let first $g''(x)$ be absolutely continuous and $g'''(x) \in L_\infty$. Then applying again Gonska's result (32) with $i = 3$ and $j = 0$, respectively $j = 3$, we obtain polynomials p_n of degree at most $m = 2n - 1$ such that

$$||g(x) - p_n(x)|| = O\biggl(\frac{||g'''||}{n^3}\biggr) \quad \text{and} \quad ||p_n'''|| = O(||g'''||).$$

Similarly to (34), we now have by Lemma 10,

$$||p_n(x) - R_n^*(p_n, x)|| = \left\| \sum_{\nu=1}^{n} p_n'''(x_\nu) |\rho_\nu^*(x)| \right\| = O\biggl(\frac{\log n}{n^3}\biggr) ||g'''||,$$

i.e., by Lemma 11,

$$\begin{aligned}||g(x) - R_n^*(g,x)|| &\le ||g(x) - p_n(x)|| + ||p_n(x) - R_n^*(p_n,x)|| \\ &\quad + ||R_n^*(g-p_n,x)|| \\ &= O\left(\frac{\log n}{n^3}\right)||g'''||.\end{aligned}$$

Since R_n^* is a linear operator, applying Theorem 2.3 of R. DeVore [**3**], Theorem 3 follows. □

Theorem 4. *We have*

$$||f(x) - R_n^*(f,x)|| = o(n^{-3})$$

if and only if $f(x)$ is a quadratic polynomial.

Since the proof is almost the same as that of Theorem 2, we omit the details.

2.4. Open problems. (i) Eliminate the $\log^{1/3} n$ factor in Theorems 1 and 3; at least for $f \in \operatorname{Lip}\alpha$, $0 < \alpha < 1$.

(ii) Give pointwise estimates instead of the norm estimates.

(iii) Solve the saturation problem of the operators R_n and R_n^*.

(iv) Investigate, in general, $(0, M)$ interpolation (M odd) for different systems of nodes.

We hope to return to these problems in subsequent papers.

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ENDNOTES

1. Note that there is an error in formula (4.2) in [**1**].

2. $a_n \sim b_n$ means that there exist positive constants $c_1 < c_2$ independent of n such that $c_1 \le a_n/b_n \le c_2$.

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