ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 24, Number 2, Spring 1994

WEAK COMPACTNESS IN THE SPACE OF VECTOR-VALUED MEASURES OF BOUNDED VARIATION

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ABSTRACT. Let X be a Banach space and (Ω, Σ) a measure space. A characterization of relatively weak compact subset of the space of X-valued countably additive vector measures of bounded variation defined on Σ is given.

1. Introduction. Let X be a Banach space, (Ω, Σ) a measure space. We denote by $M(\Omega, X)$ the space of X-valued-countably additive measure on (Ω, Σ) of bounded variation.

Recently Ülger [**6**], Diestel, Ruess and Schachermayer [**2**] gave a characterization of weakly compact subsets of $L^1(X)$. The only known characterization of weakly compact subsets of $M(\Omega, X)$ is given by the following theorem:

Theorem A (Bartle-Dunford-Schwartz) [**3**, p. 105]. *Suppose X and X^* have the Radon-Nikodym property* (RNP). *A subset K of $M(\Omega, X)$ is relatively weakly compact if and only if*

(i) *K is bounded,*

(ii) *K is uniformly countably additive,*

(iii) *For each $A \in \Sigma$, the set $\{G(A); G \in K\}$ is a relatively weakly compact subset of X.*

It turns out that one can show, using similar methods as in [**4**], that if (i), (ii) and (iii) are to characterize relatively weakly compact subsets of $M(\Omega, X)$, then X and X^* must have the Radon-Nikodym property.

The use of the Radon-Nikodym derivative was essential in the proof of Theorem A to reduce the study of weakly relative compact subsets of

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Received by the editors on January 2, 1992.
AMS (MOS) *Subject Classification* (1980). Primary 46E40, 46G10, Secondary 28B05, 28B20.
This work will constitute a portion of the Ph.D. thesis of the first named author at the University of Missouri-Columbia.

$M(\Omega, X)$ to those of $L^1(\lambda, X)$. When X does not have the Radon-Nikodym property one cannot hope to represent a measure by its Bochner derivative, but the next best thing to a Bochner derivative is a weak*-derivative valued in X^{**}. The lifting of $L^\infty(\lambda)$ comes in handy to construct such a weak*-derivative. That is, the approach we will take to give a characterization of relatively weakly compact subsets of $M(\Omega, X)$ is similar to those given by Ülger [**6**] for $L^1(\lambda, X)$. This approach was used to Talagrand in [**7**]. In this paper we follow and adopt Talagrand's techniques to obtain our result.

2. Definitions and some preliminary results. Let (Ω, Σ) be a measure space and X a Banach space. For $m \in M(\Omega, X)$, we denote by $|m|$ its variation. Let λ be a probability measure on Ω with $|m| \leq \lambda$, and let ρ be a lifting of $L^\infty(\lambda)$ [**5**]. For $x^* \in X^*$, the scalar measure $x^* \circ m$ has density $(d/d\lambda)(x^* \circ m) \in L^\infty(\lambda)$. We define $\rho(m)(\omega)$ to be the element in X^{**} defined by

$$\rho(m)(\omega)(x^*) = \rho \frac{d}{d\lambda}(x^* \circ m)(\omega).$$

It is known (see for instance [**1**]) that $x^*(m(A)) = \int_A < \rho(m)(\omega), x^* > d\lambda(\omega)$ for each measurable subset A of Ω and each $x^* \in X^*$. Similarly, it can be shown that $|m|(A) = \int_A ||\rho(m)(\omega)||\, d\lambda(\omega)$ for each measurable subset A of Ω. In case $X = Y^*$ is a dual space, $\rho(m)(\omega)$ will be the element in $X = Y^*$ defined by

$$\rho(m)(\omega)(y) = \rho \frac{d}{d\lambda}(y \circ m)(\omega) \qquad \text{for every } y \in Y.$$

Before stating the first proposition, let us introduce the following notation: For λ a probability measure on (Ω, Σ), we denote by

$$W(\lambda, X) = \{m \in M(\Omega, X) \mid |m| \leq \lambda\}.$$

Proposition 1. *Let Y be a Banach space and $(G_p)_p$ a sequence in $W(\lambda, Y^*)$ so that*

(i) *$G_p(A)$ converges weak* to 0 for each $A \in \Sigma$.*

(ii) *There is a lifting ρ of $L^\infty(\lambda)$, such that $\rho(G_p)(\omega)$ converges weakly in Y^* for λ a.e. ω then G_p converges to 0 weakly in $M(\Omega, Y^*)$.*

Proof. The idea is contained in the proof of Theorem 15 in Talagrand's paper [**7**]. The only difference is that the y_i chosen below are taken from the predual while the corresponding ones in Talagrand's paper are taken from the triple dual. Let us introduce the following subset B of the unit ball $M(\Omega, Y^*)_1^*$ of $M(\Omega, Y^*)^*$ as follows: an element $\varphi \in B$ if and only if there exist a finite partition of measurable subsets $A_1, A_2, \dots, A_n$ of Ω and $y_1, \dots, y_n$ in Y_1 the unit ball of Y so that

$$\varphi(m) = \sum_{i \le n} y_i(m(A_i)) \qquad \text{for all } m \in M(\Omega, Y^*).$$

The set B is clearly convex, and for each $m \in M(\Omega, Y^*)$ one has

$$||m|| = |m| = \sup\{\varphi(m), \varphi \in B\}$$

so B is weak*-dense in $M(\Omega, Y^*)_1^*$. For each $m \in W(\lambda, Y^*)$, let

$$Z(., m) : B \to L^1(\lambda)$$

given by

$$Z(\varphi, m) = \sum_{i \le n} y_i(\rho(m)(.))\mathcal{X}_{A_i}(\cdot).$$

Notice now that, for each $A \in \Sigma$,

$$(*) \qquad \langle \varphi, m^A \rangle = \int_A Z(\varphi, m)(\omega)\, d\lambda(\omega)$$

where $m^A : \Sigma \to Y^*$, $m^A(A') = m(A \cap A')$ for all $A' \in \Sigma$, in particular

$$\langle \varphi, m \rangle = \int Z(\varphi, m)(\omega)\, d\lambda(\omega).$$

From $(*)$, one can deduce that $Z(., m) : B \to L^1(\lambda)$ is weak* to weak uniformly continuous. So the map $\varphi \to Z(\varphi, m)$ has a continuous extension (still denoted by $Z(\varphi, m)$)

$$Z(., m) : M(\Omega, Y^*)_1^* \to L^1(\lambda).$$

Before we proceed, we need the following lemma:

Lemma 1. *For a fixed $\varphi \in M(Y^*)^*$, $||\varphi|| \leq 1$ and a sequence $(G_p)_p$ in $W(\lambda, Y^*)$, there exists a countable subset D of the unit ball of Y and a map $\omega \to g(\omega) \in \overline{D}^{\sigma(Y^{**},Y^*)}$, so that for each $p \in \mathbf{N}$, $Z(\varphi, G_p)(\omega) = g(\omega)(\rho(G_p)(\omega))$, λ almost everywhere.*

Proof of the lemma. Here we adopt the methods in [**7**] to our situation. Since B is weak*-dense in $M(\Omega, Y^*)^*_1$, choose as in [**7**] a sequence (φ_n) in B so that

$$||Z(\varphi_n, G_p) - Z(\varphi, G_p)||_1 \leq 2^{-n}$$

for each $p \leq n$.

Hence $\lim_{n\to\infty} Z(\varphi_n, G_p)(\omega) = Z(\varphi, G_p)(\omega)$ a.e. for all $p \in \mathbf{N}$. Let $\varphi_n(m) = \sum_{i\leq k_n} y_{i,n}\{m(A_{i,n})\}$ and $D = \{y_{i,n} | n \geq 1, i \leq k_n\}$ countable subset of the unit ball of Y. Now consider an ultrafilter $\mathcal{U}$ on $\mathbf{N}$ and for each $\omega \in \Omega$, let $g(\omega) \in \overline{D}^{\sigma(Y^{**},Y^*)}$ be the weak*-limit along $\mathcal{U}$ of the sequence $(y_{i(n,\omega),n})_n$ where $i(n,\omega)$ is the unique $i \leq k_n$ so that $w \in A_{i(n,\omega),n}$. We now have

$$\begin{aligned} Z(\varphi_n, G_p)(\omega) &= \sum_{i\leq k_n} < \rho(G_p)(\omega), y_{i,n} > \mathcal{X}_{A_{i,n}} \quad \text{a.e.} \\ &= \rho(G_p)(\omega)(y_{i(n,\omega),n}) \end{aligned}$$

and hence

$$Z(\varphi, G_p)(\omega) = g(\omega)(\rho(G_p)(\omega)) \quad \text{a.e.} \quad \square$$

We are now ready to prove the proposition.

Assume for the contrary that there exist $\varphi \in M(\Omega, Y^*)^*_1$, $\varepsilon > 0$ and a subsequence $(G'_p)_p \subseteq (G_p)$ so that $\varphi(G'_p) \geq \varepsilon$ for each $p \in \mathbf{N}$. Applying the lemma on $(G'_p)_{p\in\mathbf{N}}$, we have

$$\varepsilon \leq \int Z(\varphi, G'_p)(\omega)\, d\lambda(\omega) = \int g(\omega)(\rho(G'_p)(\omega))\, d\lambda(\omega)$$

taking the limit on p, we have

$$\varepsilon \leq \int g(\omega)(y(\omega))\, d\lambda(\omega)$$

where

$$y(\omega) = \text{weak-limit of } \rho(G_p)(\omega).$$

This implies that we can find $A \in \Sigma$, $\lambda(A) > 0$, so that $g(\omega)(y(\omega)) > 0$ for each $w \in A$. We claim that there exists $A' \subset A$, $\lambda(A') > 0$ and $v \in D$ such that $y(\omega)(v) > 0$ for $\omega \in A'$. To see this, let $D = \{x_n, n \geq 1\}$ and fix $A_n = \{\omega : y(\omega)(x_n) > 0\}$. We claim that $A \subset \cup_n A_n$. To see this, let $\omega \in A$, $y(\omega)(x(\omega)) > 0$, and since

$$g(\omega)(y(\omega)) = \lim_{\mathcal{U}} y(\omega)(y_{i(\omega,n),n}), \quad \text{and} \quad y_{i(\omega,n),n} \in D$$

we can find $x_n \in D$ so that $y(\omega)(x_n) > 0$. This shows that $\omega \in A_n$. Now notice that $0 < \lambda(A) \leq \lambda(\cup_n A_n)$, and hence there exists n_0, so that $\lambda(A_{n_0}) > 0$; fix $A' = A_{n_0}$ and $v = x_{n_0} \in Y$, and the claim is proved. Since

$$\langle v, G_p'(A')\rangle = \int_{A'} \rho(G_p')(\omega)(v)\, d\lambda(\omega)$$

converges to

$$\int_{A'} y(\omega)(v)\, d\lambda(\omega) > 0 \quad \text{as } p \text{ approaches } \infty,$$

we get a contradiction with condition (i). □

The next two lemmas are well known and can be found in [**1**] and [**6**], respectively.

Lemma 2 (A. Grothendieck). *A subset A of a Banach space E is relatively weakly compact if and only if, given any $\varepsilon > 0$, there exists a relatively weakly compact subset H_ε of E such that $A \subseteq H_\varepsilon + \varepsilon B_1$.*

Lemma 3. *Let A be a bounded subset of a Banach space E. Then the set A is relatively weakly compact if and only if, given any sequence (y_n) in A, there exists a sequence $\tilde{y}_n$ with $\tilde{y}_n \in \operatorname{conv}(y_n, y_{n+1}, \dots)$ that converges weakly.*

3. Main results.

Theorem 1. *Let X be a Banach sapce, (Ω, Σ) a measure space and λ a probability measure on (Ω, Σ), and let H be a subset of $W(\lambda, X)$. Then the following are equivalent*

(1) *H is relatively weakly compact in $M(\Omega, X)$;*

(2) *Given any sequence $(m_n)_n$ in H and a lifting ρ of $L^\infty(\lambda)$, there exists $(\tilde{m}_n)_n$ with $\tilde{m}_n \in \operatorname{conv}(m_n, m_{n+1}, \dots)$, such that the sequence $\rho(\tilde{m}_n)(\omega)$ converges weakly in X^{**} for λ a.e. ω;*

(3) *Given any sequence $(m_n)_n$ in H and a lifting ρ of $L^\infty(\lambda)$, there exists $(\tilde{m}_n)_n$ with $\tilde{m}_n \in \operatorname{conv}(m_n, m_{n+1}, \dots)$ such that the sequence $\rho(\tilde{m}_n)(\omega)$ converges (in norm) in X^{**} for λ a.e. ω.*

Proof. (1) $\Rightarrow$ (3). Assume that H is relatively weakly compact and $(m_n)_n \subseteq H$; by Lemma 3, there is a sequence $\tilde{m}_n \in \operatorname{conv}(m_n, m_{n+1}, \dots)$, so that $(\tilde{m}_n)$ is weakly convergent in $M(\Omega, X)$ to a measure $m \in \overline{\operatorname{conv}}(H)$. By Mazur's theorem there exists $\tilde{m}'_n \in \operatorname{conv}(\tilde{m}_n, \tilde{m}_{n+1}, \dots)$ so that $||\tilde{m}'_n - m|| \to 0$ which means that $\int ||\rho(\tilde{m}'_n)(\omega) - \rho(m)(\omega)||\, d\lambda(\omega) \to 0$, and, by taking a subsequence $\tilde{m}''_n$ of $\tilde{m}'_n$ (if necessary), we have $\rho(\tilde{m}''_n)(\omega) \to \rho(m)(\omega)\lambda$ a.e. which shows (1) $\Rightarrow$ (3).

(3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Consider $(m_n)_n \subseteq H$ and $\tilde{m}_n \in \operatorname{conv}(m_n, m_{n+1}, \dots)$ as in (2). Let

$$y(\omega) = \begin{cases} \text{weak-limit of } \rho(\tilde{m}_n)(\omega), & \text{if this limit exists} \\ 0 & \text{otherwise} \end{cases}.$$

It is evident that $y(\omega) \in X^{**}$, $||y(\omega)|| \leq 1$ for λ a.e. ω and the map $\omega \to y(\omega)$ is weak*-scalarly measurable. Let $m : \Sigma \to X^{**}$ given by $m(A) = \text{weak*-}\int_A y(\omega)\, d\lambda(\omega)$. Now for each $x^* \in X^*$ and $A \in \Sigma$, we have

$$\begin{aligned} \langle m(A), x^* \rangle &= \int_A \langle y(\omega), x^* \rangle \, d\lambda(\omega) \\ &= \int_A \lim_{n\to\infty} \langle \rho(\tilde{m}_n)(\omega), x^* \rangle \, d\lambda(\omega) \\ &= \lim_{n\to\infty} \int_A \langle \rho(\tilde{m}_n)(\omega), x^* \rangle \, d\lambda(\omega) \\ &= \lim_{n\to\infty} \langle \tilde{m}_n(A), x^* \rangle \end{aligned}$$

so $m(A)$ = weak*-limit of $\tilde{m}_n(A)$ in X^{**}. Using Proposition 1 with $Y = X^*$, we must have $\tilde{m}_n - m$ converges to 0 weakly in $M(\Omega, X^{**})$, and since $M(\Omega, X)$ is a closed subspace of $M(\Omega, X^{**})$ and $(\tilde{m}_n)_n \subseteq M(\Omega, X)$, then $m \in M(\Omega, X)$ and $\tilde{m}_n \to m$ weakly in $M(\Omega, X)$. This proves that H is relatively weakly compact (Lemma 3). □

The next theorem shows that all cases can be reduced as in Theorem 1.

Theorem 2. *A subset A of $M(\Omega, X)$ is relatively weakly compact if and only if there exists a probability measure λ on (Ω, Σ) so that, for each $(m_n)_n$ in A, there is a sequence $\tilde{m}_n \in \operatorname{conv}(m_n, m_{n+1}, \dots)$ which satisfies the following: given $\varepsilon > 0$, there exist an integer N and a relatively weakly compact subset H of $NW(\lambda, X)$ so that $\{\tilde{m}_n, n \geq 1\} \subseteq H + \varepsilon B$ where B denotes the unit ball of $M(\Omega, X)$.*

Proof. Assume that A is relatively weakly compact. $V(A) = \{|m|, m \in A\} \subseteq M(\Omega)$ is relatively weakly compact. It is well known (see, for instance, [**3**]) that there is a probability measure λ on (Ω, Σ) so that $V(A)$ is uniformly λ-continuous, i.e.,

$$\forall \varepsilon > 0, \exists \delta > 0 \quad \text{so that if} \quad \lambda(B) < \delta, |m|(B) < \varepsilon \quad \text{for all } m \in A.$$

Fix $(m_n)_n$ a sequence in A. Let f_n be the λ-density of $|m_n|$. There exists a subsequence f_{n_j} which converges weakly in $L^1(\lambda)$ to a function f. By Mazur's theorem, there is a sequence $g_j \in \operatorname{conv}(f_{n_j}, f_{n_{j+1}}, \dots)$, such that g_j converges to f in norm. By taking a subsequence, if necessary, we may assume that $g_j(\omega)$ converges to $f(\omega)\lambda$ a.e. Consequently $\sup_j g_j(\omega) < \infty$ for λ a.e., so we have $\Omega = (\cup_N \{\omega : \sup_j g_j(\omega) < N\}) \cup Z$ where Z is a set of measure zero. Now fix $\varepsilon > 0$, consider $\delta > 0$ from the definition of the uniform integrability and choose N so that

$$\lambda(\{\omega : \sup_j g_j(\omega) > N\} < \delta \quad \text{and let} \quad E = \{\omega : \sup_j g_j(\omega) \leq N\}$$

if

$$g_j = \sum_{l \geq j} \lambda_l^j f_{n_l} \quad \text{and} \quad \sum_{l \geq j} \lambda_l^j = 1.$$

Consider

$$\tilde{m}_j = \sum_{l \geq j} \lambda_l^j m_{n_l}, \qquad \tilde{m}_n \in \operatorname{conv}(m_n, m_{n+1}, \dots).$$

The sums above are, of course, finite sums. Let us denote by $\tilde{m}_n^E$ the measure $\sum \to X$ given by $\tilde{m}_n^E(B) = \tilde{m}_n(B \cap E)$. Let $H = \{\tilde{m}_n^E, n \geq 1\}$. Since $\{\tilde{m}_n, n \geq 1\} \subseteq \operatorname{conv}(A)$, which is relatively weakly compact. H is realtively weakly compact. Also, we have $H \subseteq NW(\lambda, X)$, in fact

$$||\tilde{m}_n^E(B)|| = ||\tilde{m}_n(E \cap B)|| \leq \int_{B \cap E} g_n \, d\lambda \leq N\lambda(B).$$

Finally, let us notice that $\tilde{m}_n = \tilde{m}_n^E + \tilde{m}_n^{E^c}$ and $||\tilde{m}_n^{E^c}|| = |\tilde{m}_n|(E^c) \leq \sum_{j \geq n} \lambda_j^n |m_{n_j}|(E^c) \leq \varepsilon$, so

$$\{\tilde{m}_n, n \geq 1\} \subseteq H + \varepsilon B.$$

The converse is an immediate consequence of Lemma 2. □

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