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SMOOTH POINTS OF VECTOR VALUED FUNCTION SPACES

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ABSTRACT. If E is a Banach space, then an element $x \in E$, $||x|| = 1$ is called smooth if there is a unique $x^* \in E^*$, $||x^*|| = 1$ such that $\langle x^*, x\rangle = 1$. The object of this paper is to characterize the smooth points of $L^p(I, X)$, $l^p(X)$, $1 \leq p < \infty$, where X is some Banach space. Some other related results are presented.

0. Introduction. Let E be a Banach space and $B_1(E)$ the unit ball of E. A point $x \in B_1(E)$ is called a smooth point if there is a unique point $x^* \in E^*$, the dual of E, such that $||x^*|| = 1$ and $\langle x^*, x\rangle = 1$.

In [**4**] Holub studied the smooth points of the unit ball of compact operator and the nuclear operators on a Hilbert space. In [**1**] the authors characterized the smooth points of the unit ball of compact operators and the bounded operator of l^p. Singer [**6**] characterized the smooth points of the unit ball of $C(I, X)$, the space of continuous functions with values in the Banach space X. The object of this paper is to study the smooth points of the unit ball of $L^p(I, X)$ and $l^p(I, X)$ where I is a finite measure space and $1 \leq p < \infty$. Further, we show that $B_1(l^p)$, $0 < p < 1$, has no smooth points. Examples of smooth points of the nuclear operators on l^p, $1 \leq p$, are presented.

Throughout this paper, if X is a Banach space, X^* denotes the dual of X. The projective tensor product of l^p with l^q is denoted by $l^p \hat{\otimes} l^q$. The nuclear operators from l^p to l^q is denoted by $N(l^p, l^q)$, and for $T \in N(l^p, l^q)$ we let $||T||_1$ denote the nuclear norm of T. For an element x in l^p, we write $\text{supp}\,(x) = \{n : x(n) \neq 0\}$, and $\delta_i = (0, \dots, 0, 1, 0, \dots)$ where 1 appears in the ith-coordinate. We refer to [**3**] for basics of nuclear operators on Banach spaces and for the basic theory of Bochner integrable functions.

1. Smooth points in $L^p(I, X)$. Let I be the unit interval $[0, 1]$

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and m be the Lebesgue measure on I. For a Banach space X, let $L^p(I, X)$ denote the space of all functions (equivalence classes) defined on I with values in X which are Bochner p-integrable, $1 \le p < \infty$. For $f \in L^p(I, X)$, we let

$$||f||_p = \left(\int_I ||f(t)||^p \, dm(t) \right)^{1/p}, \qquad \text{for } 1 \le p < \infty,$$

and $||f||_\infty = \operatorname{ess\,sup}_t ||f(t)||$, [**3**].

If X has the Radon-Nikodym property [**3**], then $[L^p(I, X)]^* = L^{p^*}(I, X^*)$, $1/p + 1/p^* = 1$.

Now we prove the main result of this section.

Theorem 1.1. *Let X be a Banach space for which X^* is separable and $f \in B_1(L^p(I, X))$. The following are equivalent*:

(i) *f is smooth.*

(ii) *$f(t)/||f(t)||$ is smooth in X for a.e. t.*

Proof. We shall prove the theorem for $p = 1$. The proof for $1 < p < \infty$ is exactly the same as for the case $p = 1$.

(ii) $\Rightarrow$ (i). Let f be in $L^1(I, X)$ such that $||f||_1 = 1$. By the Hahn Banach theorem there exists $F \in (L^1(I, X))^*$, such that $||F|| = F(f) = 1$. Since X^* is separable, X has the Radon-Nikodym property [**3**]. Consequently, $F \in L^\infty(I, X^*)$ and $||F(t)|| \le 1$ a.e. t. Further:

$$(*) \qquad F(f) = \int_I \langle F(t), f(t) \rangle \, dm(t) = \int_I ||f(t)|| \, dm(t) \cdots .$$

Since $|\langle F(t), f(t) \rangle| \le ||f(t)||$, it follows from $(*)$ that $\langle F(t), f(t) \rangle = ||f(t)||$ for a.e. t.

Now, if possible, assume that $G \in L^\infty(I, X^*)$ such that $\langle G, f \rangle = ||f|| = 1 = ||G||_\infty$. Then, as for F, we get $\langle G(t), f(t) \rangle = ||f(t)||$ for a.e. t. By (ii) we get $G(t) = F(t)$ a.e., and f is smooth.

Conversely (i) $\Rightarrow$ (ii). Assume $f \in B_1(L^1(I, X))$ is smooth. Thus, there exists a unique $g \in L^\infty(I, X^*)$ such that $\langle g, f \rangle = ||f|| = ||g|| = 1$.

If possible, assume that $f(t)/||f(t)||$ is not smooth for all $t \in E \subset I$, where $m(E) > 0$. Consider the set valued map

$$\varphi : I \to 2^{X^*}$$
$$\varphi(t) = \{x^* : ||x^*|| = 1,\ \langle x^*, f(t)\rangle = ||f(t)||\}.$$

By the Hahn Banach theorem, $\varphi(t)$ is not empty for all t, and $\varphi(t)$ is a closed convex set in X^*. Further, since X is separable (X^* being separable), we can show that $\varphi(t)$ is w^*-compact in X^*.

Now we claim that φ is weakly measurable in the sense that, for any closed subset K of $S_1(X^*)$, the unit sphere of X^*,

$$\varphi^{-1}(K) = \{t : \varphi(t) \cap K \text{ is not empty}\},$$

is measurable in I.

Let K be any closed set in X^*. Then, since $\varphi(t)$ is bounded in X^*, we can rewrite $\varphi^{-1}(K)$ as

$$\varphi^{-1}(K) = \{t : ||f(t)|| = \sup_{x^*} |\langle f(t), x^*\rangle|, x^* \in K\}.$$

Since X is separable, we get

$$\varphi^{-1}(K) = \{t : ||f(t)|| = \sup_{n} |\langle f(t), x_n^*\rangle|, x_n^* \in K\}.$$

Since $||f(t)||$ is a measurable function in t, and $\langle f(t), x_n^*\rangle$ is measurable for all n, we get that $\varphi^{-1}(K)$ is measurable, and so φ is weakly measurable. By the Kuratowski-Ryll-Nordzewski selection theorem [**5**], there exists a measurable selection for φ, say h. This means that $h : I \to X^*$, $h^{-1}(Q)$ is measurable in I for every w^*-closed set Q. This implies that h is w^*-measurable in the sense that $\langle h(t), x\rangle$ is measurable for every $x \in X$.

Since X^* is separable and X is a norming set for X^* it follows [**3**] that h is strongly measurable. Further, since $h(t) \in \varphi(t)$, we have $||h(t)|| = 1$ and $h \in L^\infty(I, X^*)$.

Finally, since $\varphi(t)$ is closed and convex, we can get another measurable selection $\hat{h}$ for φ. Thus, $\{g, h, \hat{h}\}$ is a set with at least two distinct elements. Further,

$$\langle h, f\rangle = \langle g, f\rangle = \langle \hat{h}, f\rangle = 1.$$

This contradicts the fact that f is smooth. Hence, $f(t)/||f(t)||$ is smooth in X for a.e. t. $\square$

2. Smooth points in $l^p(N,X)$. For a Banach space X and $1 \le p < \infty$, we let $l^p(N,X) = \{(x_n) : x_n \in X \text{ and } \sum_{n=1}^{\infty} ||x_n||^p < \infty\}$. For $f \in l^p(N,X)$, we let $||f||_p = (\sum_{n=1}^{\infty} ||f(n)||^p)^{1/p}$.

Theorem 2.1. *Let $f \in B_1$, $(l^p(N,X))$. The following are equivalent*:

(i) *f is smooth.*

(ii) *$f(n)/||f(n)||$ is smooth in X for all n.*

Proof. (ii) $\Rightarrow$ (i). If f is not smooth, then there exists $g_1, g_2 \in l^{p^*}(N,X^*)$, $||g_1||_{p^*} = ||g_2||_{p^*} = 1$ and $\langle g_1, f\rangle = \langle g_2, f\rangle = 1$. Since $g_1 \ne g_2$, then there is an $n_0 \in N$ such that $g_1(n_0) \ne g_2(n_0)$.

Now, since $\langle g_1, f\rangle = 1$, we get

$$\left|\sum_{n=1}^{\infty} \langle g_1(n), f(n)\rangle\right| \le \sum_{n=1}^{\infty} ||f(n)|| ||g_1(n)|| \le 1,$$

(by Hölder's inequality). Hence, $\langle g_1(n), f(n)\rangle = ||f(n)|| ||g_1(n)||$ for all n. Similarly for g_2. Consequently,

$$\langle g_1(n_0), f(n_0)\rangle = ||f(n_0)|| ||(g_1(n_0)||,$$

and

$$\langle g_2(n_0), f(n_0)\rangle = ||f(n_0)|| ||g_2(n_0)||.$$

This implies that

$$\left\langle \frac{f(n_0)}{||f(n_0)||}, x_1^*\right\rangle = \left\langle \frac{f(n_0)}{||f(n_0)||}, x_2^*\right\rangle = 1,$$

where $x_1^* = (g_1(n_0)/||g_1(n_0)||)$, $x_2^* = g_2(n_0)/||g_2(n_0)||$. This contradicts the fact that $f(n)/||f(n)||$ is smooth for all n. Hence, f is smooth.

Conversely (i) $\Rightarrow$ (ii). Assume $f \in B_1(l^p(N,X))$ is smooth and $f(n_0)/||f(n_0)||$ is not smooth for some n_0. Thus, there exists $x_1^*, x_2^* \in B_1(X^*)$ such that $x_1^* \neq x_2^*$ and

$$\left\langle x_1^*, \frac{f(n_0)}{||f(n_0)||} \right\rangle = \left\langle x_2^*, \frac{f(n_0)}{||f(n_0)||} \right\rangle = 1.$$

Let $g \in B_1(l^{p^*}(N,X^*))$ such that $\langle g, f\rangle = 1$. Define

$$g_i : N \to X^*,$$
$$g_i(n) = \begin{cases} g(n), & \text{if } n \neq n_0 \\ x_i^*||g(n_0)||, & \text{if } n = n_0 \end{cases}$$

for $i = 1, 2$. Then the set $\{g, g_1, g_2\}$ has at least two distinct elements. Further,

$$\langle g, f\rangle = \langle g_1, f\rangle = \langle g_2, f\rangle = 1.$$

This contradicts the assumption on f. Hence, $f(n)/||f(n)||$ is smooth for all n. □

Remark. In [**4**] Holub proved that a nuclear operator T on l^2, $||T||_1 = 1$, is smooth in the space of nuclear operators on l^2, $l^2\hat{\otimes}l^2$, if and only if either T is (1–1) or T^* is (1–1). Theorem 2.1 shows that the result of Holub is not true for $p \neq 2$. Indeed, let us consider the projective tensor product $l^1\hat{\otimes}l^p$, $1 \leq p < \infty$ [**3**]. It is known [**3**] that $l^1\hat{\otimes}l^p = l^1(N,l^p)$. Hence, $T \in l^1\hat{\otimes}l^p$ is smooth if and only if T is smooth as an element in $l^1(N,l^p)$. Thus, we consider $T = u \otimes v$, where $u \in l^1$, $u(n) \neq 0$, for all n and $v \in l^p$, $v(n) \neq 0$ for all n. By Theorem 2.1, T is smooth. But since T is a 1-rank operator, it follows that neither T nor T^* is (1–1).

We conclude this section by considering smooth points of l^p, $0 < p < 1$, noticing that $(l^p)^* = l^\infty$, where $(l^p)^*$ is the space of all continuous linear functionals on l^p [**2**].

Theorem 2.2. *$B_1(l^p)$ has no smooth points for $0 < p < 1$.*

Proof. Let $x \in l^p$, $||x||_p = \sum |x(n)|^p = 1$. Hence, $|x(n)| < 1$ for all n if $x \neq \delta_i$ for any i, where (δ_i) is the natural basis of l^p. Hence $|x(n)| < |x(n)|^p$ for all n and $||x||_1 < ||x||_p$.

Now, if $x^* \in l^\infty$, $||x^*||_\infty = 1$ and $\langle x^*, x\rangle = 1$, then $||x^*||_\infty > 1$ since $(l^1)^* = l^\infty$, and

$$||x^*||_\infty > x^*\left(\frac{x}{||x||_1}\right) = \frac{1}{||x_1||} > 1.$$

Hence, there is no $x^* \in B_1(l^\infty)$ such that $x^*(x) = 1$, if $x \neq \delta_i$ for some i. But if $x = \delta_i$, then there are many $x^* \in B_1(l^\infty)$ such that $x^*(x) = 1$. $\square$

3. Further results. In this section we give an example of smooth points in $N(l^p, l^p)$, the space of nuclear operators on l^p.

We recall that an operator $T : l^p \to l^p$ is called nuclear if T has a representation $T = \sum_{n=1}^\infty u_n \otimes v_n$, $u_n \in l^{p^*}$, $v_n \in l^p$ and $\sum_{n=1}^\infty ||u_n||_{p^*} ||v_n||_p < \infty$. Holub [**4**] characterized smooth points of $B_1(N(l^2, l^2))$. We were not able to characterize smooth points of $B_1(N(l^p, l^p))$, $1 < p \neq 2 < \infty$. However, we give an example of smooth points in $B_1(N(l^p, l^p))$, $1 < p < \infty$.

Theorem 3.1. *Let* $T = \sum_{i=1}^\infty y_i^* \otimes \delta_i$, $\sum_{i=1}^\infty ||y_i^*||_{p^*} = 1$, *and* $\operatorname{supp}(y_i^*) \cap \operatorname{supp}(y_j^*) = \phi$ *for* $i \neq j$. *Then* T *is a smooth point of* $B_1(N(l^p, l^p))$.

Proof. From the definition of the nuclear norm of operators [**3**], we get $||T||_1 \leq 1$.

Consider the operator $A = \sum_{i=1}^\infty y_i \otimes \delta_i$, where y_i is the unique point in l^p such that $\langle y_i^*, y_i\rangle = ||y_i^*||$. Thus, $||y_i|| = 1$. Since $[N(l^p, l^p)]^* = L(l^{p^*}, l^{p^*})$, the bounded linear operators on l^{p^*} [**3**], we have

$$\langle A, T\rangle = \tau_r(A^*T) = \sum_{i=1}^\infty \langle y_i, y_i^*\rangle = \sum_{i=1}^\infty ||y_i^*|| = 1.$$

Further, for any $x \in l^{p^*}$, we have

$$\begin{aligned} ||A^* x||_p^p &= \left\| \sum_{i=1}^{\infty} \langle \delta_i, x \rangle y_i \right\| \\ &= \sum_{i=1}^{\infty} |\langle \delta_i, x \rangle|^p ||y_i||^p, \quad \text{(from the assumption on } y_i^*) \\ &= \sum_{i=1}^{\infty} |\langle \delta_i, x \rangle|^p = ||x||_p^p. \end{aligned}$$

Hence A^* is an isometry, and so $||A|| = 1$. Consequently, $||T||_1 = 1$.

Now, if B is another operator in $B_1(L(l^{p^*}, l^{p^*}))$ such that $\langle B, T \rangle = 1$, then

$$\sum_{i=1}^{\infty} \langle y_i^*, B\delta_i \rangle = 1.$$

Since $|\langle y_i^*, B\delta_i \rangle| \leq ||y_i^*||$, we get

$$\langle y_i^*, B\delta_i \rangle = ||y_i^*||$$

for all i. By the uniform convexity of l^p, $1 < p < \infty$, we get $B^*\delta_i = y_i$ for all i. Hence, $B^* = A^*$ and $A = B$ and T is smooth. □

Remark. The operator T in Theorem 3.1 is not (1–1) but T^* is. We believe the following is true.

Conjecture. *An operator $T \in N(l^p, l^p)$, $1 < p < \infty$ with $||T||_p = 1$ is smooth if and only if either T or T^* is* (1–1).

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