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A QUASILINEAR TWO POINT BOUNDARY VALUE PROBLEM

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1. Introduction. With $Du = du/dx$ and $\Omega = (0,1)$ the open unit interval, let

$$Lu = -D[(a_1 + a_2)Du] \tag{1.1}$$

In this representation of $L, a_1(x)$ and $a_2(x)$ will both satisfy $(a-1)$ and $(a-2)$ where with $W^{1,\infty}(\Omega)$ the usual Sobolev space of functions with bounded derivatives in Ω, these two conditions are given as follows:

(a-1) $a(x)$ is a real-valued function in $C(\bar{\Omega}) \cap C^1(\Omega) \cap W^{1,\infty}(\Omega)$;

(a-2) $$\exists \varepsilon_0 > 0 \quad \text{s.t. } a(x) \geq \varepsilon_0 \qquad \forall\, x \in \bar{\Omega}.$$

To L, we associate the quasilinear differential operator

$$\begin{aligned} Qu = -D\left[\sum_{j=1}^{2} a_j(x)\sigma_{ij}(u)Du\right] &+ \sigma_{21}(u)b_1(x,u)[Du]^+ \\ &+ \sigma_{22}(u)b_2(x,u)[Du]^- \end{aligned} \tag{1.2}$$

where

$$\sigma_{ij} : W_0^{1,2}(\Omega) \to \mathbf{R} \quad \text{with } \sigma_{ij} \text{ continuous in the strong } W_0^{1,2}\text{-topology for } i,j = 1,2, \text{ and} \tag{1.3}$$

$$b_j(x,s) \in C[\bar{\Omega} \times \mathbf{R}] \quad \text{for } j = 1,2. \tag{1.4}$$

Also,

$$[Du(x)]^+ = \max[Du(x), 0], \qquad [Du(x)]^- = \max[-Du(x), 0].$$

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We set

$$(1.5) \qquad \lambda_1 = \inf \int_\Omega (a_1 + a_2)|Du|^2 \qquad u \in W_0^{1,2}(\Omega), \qquad ||u||_{L^2} = 1,$$

and observe from $(a-2)$, and the Poincare inequality that $\lambda_1 > 0$. Also, from [**5**, p. 198] and [**9**, p. 7] we see that

$$(1.6) \qquad \exists \phi_1 > 0 \quad \text{in } \Omega \quad \text{s.t. } \phi_1 \in C^1(\bar{\Omega}) \cap C^2(\Omega) \cap W^{2,\infty}(\Omega),$$

and

$$L\phi_1 = \lambda_1 \phi_1 \quad \text{in } \Omega \quad \text{with } ||\phi_1||_{L^2} = 1 \text{ and } \phi_1(0) = \phi_1(1) = 0.$$

A well-known result of Aguinaldo and Schmidt [**2**] in the case $a_j(x) \equiv 1/2$ for $j = 1, 2$ and $h \in C(\bar{\Omega})$ states that a necessary and sufficient condition that the boundary value problem

$$\text{(A-S)} \qquad Lu = \lambda_1 u - \alpha u^- + h(x) \qquad u(0) = u(1) = 0$$

has a solution in $C^2(\bar{\Omega})$ where $\alpha > 0$ is that $\int_\Omega h\phi_1 \geq 0$. Castro [**4**] generalized the sufficiency condition of this result by adding $p(u)$ to the right-hand side of the first equation in (A-S) where p is a real-valued function in $C^0(\mathbf{R})$ with $p(s)$ sublinear for $s \leqq 0$ and $= 0$ for $s \geqq 0$. We intend to obtain similar results for our quasilinear operator Q given in (1.2). Our method of proof is very different from the techniques employed in [**2**] and [**4**] and depends upon some of the ideas used in [**10**] and also [**3**]. (Part of Castro's paper is discussed in the famous nonlinear survey of Nirenberg [**7**, p. 283]).

In order to get our quasilinear results in line with those of [**2**] and [**4**], we shall also assume the following.

(1.7)

(i) $\exists K$ and $\varepsilon_0 > 0$ s.t. $\varepsilon_0 \leqq \sigma_{1j}(u) \leqq K \qquad \forall\, u \in W_0^{1,2}(\Omega)$ and $j = 1, 2$,

(ii) $\sigma_{1j}(u) = 1$ for $\displaystyle\int_\Omega u\phi_1 > 0$,

$\sigma_{1j}(u) \leqq 1$ for $\displaystyle\int_\Omega u\phi_1 \leqq 0$ where ϕ_1 is given in (1.6) and $j = 1, 2$;

(iii) $\displaystyle\lim_{||u||_{L^2} \to \infty} \sigma_{1j}(u) = 1$ for $j = 1, 2$.

(1.8)

$$(i)\ \exists K>0 \quad \text{s.t. } 0 \leqq \sigma_{2j}(u) \leqq K \qquad \forall\, u \in W_0^{1,2}(\Omega) \text{ and } j=1,2.$$

$$(ii)\ \lim_{||u||_{L^2}\to\infty} \sigma_{2j}(u) = 0 \text{ for } j=1,2.$$

(1.9)
$$(i)\ b_i(x,s) \text{ meets } (f-2) \text{ below for } i=1,2.$$
$$(ii)\ \exists K>0 \quad \text{s.t. } |b_i(x,s)| \leqq K \qquad \forall\,(x,s)\in\Omega\times\mathbf{R},\ i=1,2.$$

Our first theorem deals with the following boundary value problem:

$$Qu = \lambda_1 u - \alpha u^- - f_1(x,u) + h \qquad u(0)=u(1)=0 \tag{1.10}$$

with $f_1(x,s)$ meeting $(f-1)-(f-3)$ where

(f-1) $f(x,s) \in C(\bar{\Omega}\times\mathbf{R})$.

(f-2) $$f(x,s) = \begin{cases} =0 & \text{for } s \geqq 0 \text{ and } x \in \Omega \\ \geqq 0 & \text{for } s \leqq 0 \text{ and } x \in \Omega. \end{cases}$$

(f-3) $\forall\,\varepsilon>0,\ \exists h_\varepsilon^*(x) \in L^2(\Omega)$ s.t.
$$|f(x,s)| \leqq \varepsilon|s| + h_\varepsilon^*(x) \qquad \forall\,(x,s)\in\Omega\times\mathbf{R}.$$

Our theorem which generalizes [**2**] is the following

Theorem 1. *Let Qu be given by* (1.2) *where a_1 and a_2 meet* $(a-1)$, $(a-2)$, *and* (1.3)–(1.9) *hold. Suppose also that $f_1(x,s)$ meets* $(f-1)-(f-3)$, *that $\sigma_{11}\equiv\sigma_{12}$, that $h\in C(\Omega)\cap L^2(\Omega)$ and that $\alpha>0$. Then a necessary and sufficient condition that there exists $u\in C^1(\bar{\Omega})\cap C^2(\Omega)\cap W^{2,2}(\Omega)$ which satisfies the boundary value problem* (1.10) *is that*

$$\int_\Omega h\phi_1 \geqq 0. \tag{1.11}$$

To be specific about the meaning of a solution in the concluding statement in Theorem 1, given $u,v\in W_0^{1,2}(\Omega)$, we set

$$\underset{\sim}{Q}(u,v) = \sum_{j=1}^{2} \sigma_{1j}(u)\langle \alpha_j Du, Dv\rangle + \sigma_{21}(u)\langle b_1(\cdot,u)[Du]^+, v\rangle + \sigma_{22}(u)\langle b_2(\cdot,u)[Du]^-, v\rangle \tag{1.12}$$

where

$$\langle u, v\rangle = \int_\Omega uv \tag{1.13}$$

and say u is a solution of the BVP (1.10) provided

$$\begin{gathered}\underset{\sim}{Q}(u,v) = \lambda_1\langle u, v\rangle - \alpha\langle u^-, v\rangle - \langle f_1(\cdot, u), v\rangle + \langle h, v\rangle \\ \forall\, v \in W_0^{1,2}(\Omega).\end{gathered} \tag{1.14}$$

Remark. It is clear from Lemma 3 below that if (1.14) holds for $u \in C(\bar\Omega)\cap W^{1,2}(\Omega)$ then $u \in C^1(\bar\Omega)\cap C^2(\Omega)\cap W^{2,2}(\Omega)$ and furthermore u satisfies BVP (1.10) in the classical sense, i.e., $u(0) = u(1) = 0$ and

$$\begin{aligned}-D\Big[\sum_{j=1}^{2}\sigma_{1j}(u)a_{1j}(x)Du(x)\Big] &+ \sigma_{21}(u)b_1(x,u)[Du]^+(x) \\ &+ \sigma_{22}(u)b_2(x,u)[Du]^-(x) \\ &= \lambda_1 u(x) - \alpha u^-(x) - f_1(x,u) + h(x) \qquad \forall\, x \in \Omega.\end{aligned}$$

A similar situation also prevails for Theorem 2 and BVP (1.16).

Of course in Theorem 1, $\sigma_{11}(u) = \sigma_{12}(u)$ for all $u \in W^{1,2}(\Omega)$, but this will not be the case in Theorem 2. In the sequel, we will use $\underset{\sim}{Q}(u,v)$ in the manner in which it is presented in (1.12).

To see that the inequality in (1.11) is indeed a necessary condition, suppose that u is a solution of the BVP (1.10) with the properties delineated in Theorem 1. Take $v = \phi_1$ in (1.14). It then follows from (1.1), (1.2) and (1.6) that

$$\begin{aligned}&[\sigma_{12}(u) - 1]\lambda_1\hat u(1) + \sigma_{21}(u)\langle b_1(x,u)[Du]^+, \phi_1\rangle \\ &+ \sigma_{22}(u)\langle b_2(x,u)[Du]^-, \phi_1\rangle + \langle f_1(x,u), \phi_1\rangle + \langle \alpha u^-, \phi_1\rangle \\ &\qquad\qquad = \langle h, \phi_1\rangle\end{aligned} \tag{1.15}$$

where $\hat u(1) = \int_\Omega u\phi_1$. A check of the conditions in the hypothesis of Theorem 1 shows that each term on the left-hand side of the equal sign in (1.15) is nonnegative. Hence the integral on the right-hand side of the equal sign in (1.15) is nonnegative and the inequality in (1.11) is established.

We can prove a stronger result than the sufficiency condition given in Theorem 1. In particular we replace BVP (1.10) by the following

$$Qu = \lambda_1 u - \alpha u^- - f_1(x,u) + f_2(x,u) + f_3(x,u) + h \qquad u(0) = u(1) = 0 \tag{1.16}$$

where $f_2(x,s)$ and $f_3(x,-s)$ meet $(f-1)-(f-3)$.

The sufficiency condition in Theorem 1 is a corollary to the following result which we shall establish and which also generalizes [**4**].

Theorem 2. *Let Qu be given by* (1.2) *where a_1 and a_2 meet* $(a-1)$ *and* $(a-2)$ *and* (1.3)–(1.9) *hold. Suppose also that $f_1(x,s)$, $f_2(x,s)$ and $f_3(x,-s)$ meet $(f-1)-(f-3)$, that $h \in C(\Omega)\cap L^2(\Omega)$, that $\alpha > 0$ and that* (1.11) *holds. Then there exists $u \in C^1(\bar\Omega)\cap C^2(\Omega)\cap W^{2,2}(\Omega)$ which satisfies the boundary value problem* (1.16).

A candidate for $a_1(x)$ in (1.1) and (1.2) is

$$a_1(0) = 2, \qquad a_1(x) = 2 + x^2\sin(1/x), \qquad 0 < x \le 1.$$

It is easy to see that $a_1(x)$ meets $(a-1)$ and $(a-2)$ but $a_1(x) \notin C^1(\bar\Omega)$.

A candidate for $\sigma_{11}(u)$ is

$$\sigma_{11}(u) = 1 - \langle u,\phi_1\rangle^-/2[1 + ||u||_{L^2}||Du||^{\varepsilon}_{L^2}] \qquad \varepsilon > 0. \tag{1.17}$$

It is clear that σ_{11} meets (1.3) and (1.7).

There are many other possible candidates for $a_1(x)$ and σ_{11}. Likewise, in a similar vein, it is easy to find many candidates for $\sigma_{2i}(u)$ and $b_i(x,s)$ $i = 1,2$.

2. Fundamental lemmas. We take it as well known that associated with L given by (1.1) where a_1 and a_2 both meet $(a-1)$ and $(a-2)$ are sequences $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\phi_n\}_{n=1}^{\infty}$ such that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to +\infty; \tag{2.1}$$

$$\phi_n(x) \in C^1(\bar{\Omega}) \cap C^2(\Omega) \cap W^{2,\infty}(\Omega) \quad \text{with } \phi_n(0) = \phi_n(1) = 0 \tag{2.2}$$

for all n;

$$L\phi_n(x) = \lambda_n \phi_n(x) \qquad \forall\, x \in \Omega \tag{2.3}$$

and for all n;

$$\{\phi_n\}_{n=1}^{\infty} \quad \text{is a complete orthonormal system for } L^2(\Omega). \tag{2.4}$$

Also, ϕ_1 satisfies (1.6). We set

$$\hat{v}(n) = \langle v, \phi_n \rangle \qquad \forall\, v \in L^2(\Omega), \tag{2.5}$$

$$S_n = \{v \in C^1(\Omega) : v = \sum_{i=1}^{n} \gamma_i \phi_i,\ \gamma_i \in \mathbf{R},\ i = 1, \dots, n\}, \tag{2.6}$$

and

$$[g]^n(x) = \begin{cases} n & \text{if } g(x) \geqq n \\ g(x) & \text{if } |g(x)| \leqq n \\ -n & \text{if } g(x) \leqq -n \end{cases} \tag{2.7}$$

The first lemma we establish is

Lemma 1. *Let n be a fixed positive integer. Then under the conditions in the hypothesis of Theorem 2, there exists $u_n \in S_n$ such that*

$$\underset{\sim}{Q}(u_n, v) = \left\langle \lambda_1 u_n - \alpha[u_n^-]^n + \sum_{i=1}^{3} \delta_i [f_i]^n(\cdot, u_n) + h, v \right\rangle + \hat{u}_n(1)\hat{v}(1)/n \tag{2.8}$$

for all $v \in S_n$ where S_n is given by (2.6), $\hat{v}$ by (2.5), Q by (1.12) and δ_j is defined in (2.11) below.

Proof. To establish (2.8), we take $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbf{R}^n$ and set

$$w = \sum_{k=1}^{n} \gamma_k \phi_k \tag{2.9}$$

and

(2.10)
$$F_j(\gamma) = \underset{\sim}{Q}(w, \delta_j \phi_j) - \langle w, \zeta_j \delta_j \phi_j \rangle / n + \langle -\lambda_1 w + \alpha [w^-]^n - \sum_{i=1}^{3} \delta_i [f_i]^n(\cdot, w) - h, \delta_j \phi_j \rangle$$

for $j = 1, \ldots, n$ where

$$\begin{aligned} &\delta_i = -1 \quad \text{and} \quad \delta_j = 1 \qquad j = 2, \ldots, n, \\ &\zeta_1 = 1 \quad\ \ \text{and} \quad \zeta_j = 0 \qquad j = 2, \ldots, n. \end{aligned} \tag{2.11}$$

It is clear under the conditions in the hypotheses of Theorem 2 that

$$F(\gamma) = [F_1(\gamma), \ldots, F_n(\gamma)] : \mathbf{R}^n \to \mathbf{R}^n \text{ is continuous.} \tag{2.12}$$

Also, with

$$\tilde{w} = \sum_{k=1}^{n} \delta_k \gamma_k \phi_k \quad \text{where } \delta_k \text{ is given in (2.11),} \tag{2.13}$$

we see from (2.10)–(2.13) that

$$F(\gamma) \cdot \gamma = \underset{\sim}{Q}(w, \tilde{w}) - \lambda_1 \langle w, \tilde{w} \rangle + \gamma_1^2 / n + \langle \alpha [w^-]^n - \sum_{i=1}^{3} \delta_i [f_i]^n(\cdot, w) - h, \tilde{w} \rangle \tag{2.14}$$

Now, using (2.3), (2.4) and (2.9), we see that

$$\langle Lw, \tilde{w} \rangle - \lambda_1 \langle w, \tilde{w} \rangle = \sum_{k=2}^{n} (\lambda_k - \lambda_1) \gamma_k^2. \tag{2.15}$$

Also, it follows from (2.7) that the absolute value of the fourth term on the right-hand side of the equal sign in (2.14) is majorized by $K_2 |\gamma|$ where K_2 is a positive constant. Furthermore, we see from (2.1) that

$$\exists \eta > 0 \quad \text{s.t. } \lambda_k - \lambda_1 \geq \eta \lambda_k \quad \text{for } k \geqq 2. \tag{2.16}$$

Hence, we conclude from (2.14) and (2.15) that

$$F(\gamma)\cdot\gamma \geqq \eta\sum_{k=2}^{n}\lambda_k\gamma_k^2+\gamma_1^2/n-K_2|\gamma|+\underset{\sim}{Q}(w,\tilde{w})-\langle Lw,\tilde{w}\rangle. \tag{2.17}$$

We claim

$$\lim_{|\gamma|\to\infty}|[\underset{\sim}{Q}(w,\tilde{w})-\langle Lw,\tilde{w}\rangle]/|\gamma|^2=0 \tag{2.18}$$

To establish (2.18), first observe from $(a-2)$ that

$$2\varepsilon_0\langle Dw,Dw\rangle \leqq \langle Lw,w\rangle=\sum_{j=1}^{n}\lambda_j\gamma_j^2. \tag{2.19}$$

Therefore, it follows from (1.12) and (1.8) (ii) that

$$\underset{\sim}{Q}(w,\tilde{w})-\langle Lw,\tilde{w}\rangle=\sum_{j=1}^{2}[\sigma_{1j}(w)-1]\langle a_jDw,D\tilde{w}\rangle+o(|\gamma|^2). \tag{2.20}$$

From (1.7) (iii) and (2.9) we see that $[\sigma_{1j}(w)-1]\to 0$ as $|\gamma|\to\infty$. Hence we conclude from (2.19) that each term in the summation of (2.20) is also $o(|\gamma|^2)$ as $|\gamma|\to\infty$, and thus claim (2.18) is established. But then it follows from (2.17)–(2.18) that there exists $s_0>0$ such that

$$F(\gamma)\cdot\gamma>0 \quad \text{for } |\gamma|\geqq s_0.$$

We conclude from (2.12) (See [**6**, p. 219] or [**8**, p. 18]) that there exists $\gamma^{\#}=(\gamma_1^{\#},\dots,\gamma_n^{\#})$ with $|\gamma^{\#}|<s_0$ such that $F_k(\gamma^{\#})=0$ for $k=1,\dots,n$. In particular $-F_1(\gamma^{\#})=0$. So taking $u_n=\sum_{k=1}^{n}\gamma_k^{\#}\phi_k$, we have from (2.10) that

$$\underset{\sim}{Q}(u_n,\phi_1)=\langle\lambda_1u_n-\alpha[u_n]^-+\sum_{i=1}^{3}\delta_i[f_i]^n(\cdot,u_n)+h,\phi_1\rangle+\hat{u}_n(1)/n \tag{2.21}$$

This fact joined with $F_k(\gamma^{\#})=0$ for $k=2,\dots,n$, when used with the definition of S_n establishes (2.8). □

The next lemma we establish is the following.

Lemma 2. *Given* $v \in W_0^{1,2}(\Omega)$, *set* $v_n = \sum_{k=1}^n \hat{v}(k)\phi_k$. *Then*

$$\lim_{n\to\infty} ||Dv_n - Dv||_{L^2} = 0. \tag{2.22}$$

Proof. To establish the lemma, for u, $v \in W_0^{1,2}(\Omega)$ set

$$\mathcal{L}(u,v) = \langle (a_1 + a_2)Du, Dv\rangle. \tag{2.23}$$

Then it follows from $(a-2)$ that

$$2\varepsilon_0 ||Du||_{L^2}^2 \leqq \mathcal{L}(u,u) \quad \text{where } \varepsilon_0 > 0. \tag{2.24}$$

Hence, it follows from (2.23) and the Poincare inequality joined with (2.24) that $\mathcal{L}(u,v)$ is a real-inner product on $W_0^{1,2}(\Omega)$. Also, it is easy to see from (2.3) and (2.4) that $\{\phi_n/\lambda_n^{1/2}\}_{n=1}^\infty$ is a complete orthonormal system for $W_0^{1,2}(\Omega)$ with respect to this inner product. Now $\mathcal{L}(v, \phi_k/\lambda_k^{1/2}) = \langle v, \phi_k\rangle \lambda_k^{1/2}$. Therefore $v_n = \sum_{k=1}^n \hat{v}(k)\phi_k = \sum_{k=1}^n \mathcal{L}(v, \phi_k/\lambda_k^{1/2})\phi_k \lambda_k^{-1/2}$ and we conclude from well-known Hilbert space theory that

$$\lim_{n\to\infty} \mathcal{L}(v - v_n,\, v - v_n) = 0$$

Setting $u = v - v_n$ in (2.24), we see from this last limit that (2.22) holds. □

Next, we establish a regularity lemma motivated by the technique to be found on [**5**, p. 202].

Lemma 3. *Suppose the conditions in the hypothesis of Theorem 2 hold and suppose furthermore that*

$$u \in C(\bar{\Omega}) \cap W_0^{1,2}(\Omega), \tag{2.25}$$

and

(2.26)

$$\underset{\sim}{Q}(u,v) = \left\langle \lambda_1 u - \alpha u^- + \sum_{j=1}^3 \delta_j f_j(\cdot, u) + h, v \right\rangle \qquad \forall\, v \in W_0^{1,2}(\Omega).$$

Then $u \in C^1(\bar{\Omega}) \cap C^2(\Omega) \cap W^{2,2}(\Omega)$.

Proof. For the proof of this lemma, we take

$$w \in C^2(\bar{\Omega}) \quad \text{with } w(0) = w'(0) = w(1) = w'(1) = 0. \tag{2.27}$$

Also, we take

$$C_{1j}(x) = \sigma_{1j}(u)a_j(x) \quad \text{and} \quad C_{2j}(x) = \sigma_{2j}(u)b_j(x,u), \qquad j = 1,2 \tag{2.28}$$

and observe from $(a-2)$ and (1.7) that

$$C_{1j}(x) \geqq \varepsilon_0^2 \qquad \forall\, x \in \Omega,\ j = 1,2 \tag{2.29}$$

Then it follows from (1.12), (2.26)–(2.28) and integration by parts twice that

$$\begin{aligned} -\int_0^1 u(C_{11}+C_{12})D^2w + \int_0^1 \left[\int_0^x u(t)(DC_{11}+DC_{12})\,dt\right] D^2 w \\ = \int_0^1 \left\{ \int_0^x \int_0^t \left[\lambda_1 u(s) - \alpha u^-(s) + \sum_{j=1}^3 \delta_j f_j(s,u) + h(s) \right.\right. \\ \left.\left. - C_{22}[Du]^- - C_{21}[Du]^+ \right] ds dt \right\} D^2 w \end{aligned}$$

We conclude from $(a-1)$, $(a-2)$, (2.29) and [**5**, p. 10] first that $u \in C^1(\Omega)$ and next that

$$\begin{aligned} -[C_{11}(x)+C_{12}(x)]Du(x)+k_1 = \int_0^x \Big\{ \lambda_1 u(t) - \alpha(t)u^-(t) + \sum_{j=1}^3 \delta_j f_j(t,u) + h(t) \\ - C_{22}[Du]^-(t) - C_{21}[Du]^+(t) \Big\}\, dt, \end{aligned}$$

where k_1 is a constant. From this last equality we obtain that $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ and finally from (2.29) that $D^2u \in L^2(\Omega)$. □

3. Proof of Theorem 2. To prove Theorem 2, we invoke Lemma 1 and obtain a sequence of functions $\{u_n\}_{n=1}^{\infty}$ such that

$$u_n \in S_n \quad \text{and} \quad u_n \text{ satisfies (2.8) for } n = 1, 2, \ldots. \tag{3.1}$$

We claim

$$\exists K_3 > 0 \quad \text{such that } ||u_n||_{W^{1,2}} \leqq K_3 \qquad \forall\, n. \tag{3.2}$$

Suppose (3.2) is false. Then there exists a subsequence which for ease of notation we take to be the full sequence such that

$$\lim_{n\to\infty} ||u_n||_{W^{1,2}} = \infty. \tag{3.3}$$

We first claim that (3.3) implies that

$$\lim_{n\to\infty} ||u_n||_{L^2} = \infty \tag{3.4}$$

Suppose this fact is false. Then (once again using the full sequence for ease of notation) we would have that

$$\exists K_4 > 0 \quad \text{such that } ||u_n||_{L^2} \leqq K_4 \qquad \forall\, n. \tag{3.5}$$

Now it follows from (1.12) and (3.1) that

$$\begin{aligned}\sum_{j=1}^{2}\sigma_{1j}(u_n)\langle a_j Du_n, Du_n\rangle = &-\sigma_{21}(u_n)\langle b_1(\cdot,u_n)[Du]^+, u_n\rangle \\ &-\sigma_{22}(u_n)\langle b_2(\cdot,u_n)[Du_n]^-, u_n\rangle + [\hat{u}_n(1)]^2/n \\ &+\langle \lambda_1 u_n - \alpha[u_n^-]^n + \sum_{j=1}^{3}\delta_j[f_j]^n(\cdot,u_n) + h, u_n\rangle.\end{aligned} \tag{3.6}$$

But then it follows from $(a-2)$ and (1.7) (i) that

$$\sum_{j=1}^{2}\sigma_{1j}(u_n)\langle a_j Du_n, Du_n\rangle \geqq 2\varepsilon_0^2||Du_n||_{L^2}^2 \qquad \forall\, n. \tag{3.7}$$

On the other hand, we have from (3.5) and the conditions in the hypothesis of the theorem that the right-hand side of the inequality in (3.6) is majorized by $K_5||Du_n||_{L^2} + K_5$ where K_5 is a positive constant. We conclude from (3.7) that $||Du_n||_{L^2} \leqq K_5\varepsilon_0^{-2}(1+||Du_n||_{L^2}^{-1})/2$. This along with (3.5) gives a contradiction to (3.3). Hence (3.4) is indeed true under assumption (3.3). Also, it follows from (3.3), (3.6), (3.7) and Schwarz's inequality that

$$||Du_n||_{L^2} \leqq K_6||u_n||_{L^2} \qquad \forall\, n \quad \text{where } K_6 \text{ is a positive constant.} \tag{3.8}$$

Next, we set

$$W_n = u_n/||u_n||_{L^2} \tag{3.9}$$

and observe from (3.8) that

$$||W_n||_{W^{1,2}} \leqq K_7 \qquad \forall\, n \quad \text{where } K_7 \text{ is a positive constant.} \tag{3.10}$$

Hence, it follows from Sobolev's compact imbedding theorem [**1**, p. 144] that

$$\begin{aligned} &(i) \quad \exists W \in C(\bar{\Omega}) \cap W^{1,2}(\Omega) \quad \text{s.t.} \lim_{n\to\infty} W_n(x) = W(x) \\ &\qquad \text{uniformly } \forall\, x \in \bar{\Omega}, \\ &(ii) \quad \lim_{n\to\infty} \int_\Omega DW_n v = \int_\Omega DWv \qquad \forall\, v \in L^2(\Omega) \end{aligned} \tag{3.11}$$

where we have used the full sequence for ease of notation.

Using (3.1), we next take $v = \phi_1$ in (2.8) and obtain from (1.12) and the fact that $\langle LW_n, \phi_1\rangle = \lambda_1\langle W_n, \phi_1\rangle$ that

$$\begin{aligned} &\sum_{j=1}^{2}[\sigma_{1j}(u_n) - 1]\langle a_j DW_n, \phi_1\rangle + \sigma_{21}(u_n)\langle b_1(\cdot, u_n)[DW_n]^+, \phi_1\rangle \\ &\quad + \sigma_{22}(u_n)\langle b_2(\cdot, u_n)[DW_n]^-, \phi_1\rangle \\ &= -\alpha\langle [W_n^-]^n, \phi_1\rangle + ||u_n||_{L^2}^{-1}\left\langle \sum_{j=1}^{3}\delta_j[f_j]^n(\cdot, u_n) + h, \phi_1\right\rangle + \widehat{W}_n(1)/n \end{aligned} \tag{3.12}$$

Since $\sigma_{1j}(u_n) \to 1$ and $\sigma_{2j}(u_n) \to 0$ as $n \to \infty$ for $j = 1, 2$ by (3.4), (1.7) and (1.8), we conclude from (3.10) that the left-hand side of (3.12) tends to zero. Likewise it follows from $(f-3)$ that that second term on the right-hand side of (3.12) tends to zero. We consequently obtain from (3.12) coupled with (3.11) (i) that

$$\alpha\langle W^-, \phi_1\rangle = 0.$$

But $\phi_1(x) > 0$ for all $x \in \Omega$. Furthermore, $\alpha > 0$. Hence $[W(x)]^- \phi_1(x) = 0$ for all $x \in \Omega$. We conclude $[W]^-(x) = 0$ for $x \in \Omega$. Therefore

$$W(x) \geqq 0 \qquad \forall\, x \in \Omega. \tag{3.13}$$

Next, we replace ϕ_1 by ϕ_k in the left-hand side of (3.12) and observe that this expression is equal to

$$[\underset{\sim}{Q}(u_n, \phi_k) - \langle Lu_n, \phi_k\rangle]/||u_n||_{L^2} \tag{3.14}$$

But taking the limit of the left-hand side of (3.12) with ϕ_1 replaced by ϕ_k as $n \to \infty$ and using (3.4), (3.10), (1.7) and (1.8), we see that this limit is zero. Hence the limit of the expression in (3.14) is zero. However $\langle LW_n, \phi_k\rangle = \lambda_k \widehat{W}_n(k)$. We consequently conclude from (3.11) (i) and (3.14) that

$$\lim_{n\to\infty} \underset{\sim}{Q}(u_n, \phi_k)/||u_n||_{L^2} = \lambda_k \widehat{W}(k) \tag{3.15}$$

Also, we observe from (3.11) and (3.13) that

$$\lim_{n\to\infty} \langle |u_n^-|^n, \phi_k\rangle/||u_n||_{L^2} = \langle W^-, \phi_k\rangle = 0. \tag{3.16}$$

Consequently, we see from (3.1) that if we set $v = \phi_k$ in (2.8), divide both sides of (2.8) by $||u_n||_{L^2}$ and pass to the limit as $n \to \infty$ using (3.15), (3.16) and $(f-3)$ that

$$\lambda_k \widehat{W}(k) = \lambda_1 \widehat{W}(k)$$

But from (2.17), we have that $\lambda_k > \lambda_1$ for $k \geqq 2$. Therefore

$$\widehat{W}(k) = 0, \qquad k \geqq 2. \tag{3.17}$$

From (3.9) and (3.11), we have that $||W||_{L^2} = 1$. We consequently conclude from (3.13) and (3.17) that

$$W(x) = \phi_1(x) \qquad \forall\, x \in \Omega. \tag{3.18}$$

Now $u_n(x) = \sum_{k=1}^n \hat{u}_n(k)\phi_k(x)$. Therefore from (2.3), we have

$$Lu_n = \sum_{k=1}^n \lambda_k \hat{u}_n(k)\phi_k \tag{3.19}$$

So from (2.6) we see that $Lu_n \in S_n$. Hence using (3.1) once again, we take $v = Lu_n$ in (2.8) and obtain

$$-\sum_{j=1}^2 \sigma_{1j}(u_n)\langle Da_j Du_n, Lu_n\rangle = A_n - B_n \tag{3.20}$$

where

$$A_n = \quad \text{the right-hand side of (2.8) with } v = Lu_n, \tag{3.21}$$

and

$$\begin{aligned} B_n = {} & \sigma_{21}(u_n)\langle b_1(\cdot, u_n)[Du_n]^+, Lu_n\rangle \\ & + \sigma_{22}(u_n)\langle b_2(\cdot, u_n)[Du_n]^-, Lu_n\rangle \end{aligned} \tag{3.22}$$

Now it follows from (1.1) and (3.20) that

$$\langle Lu_n, Lu_n\rangle = A_n - B_n + C_n \tag{3.23}$$

where

$$C_n = \sum_{j=1}^2 [\sigma_{1j}(u_n) - 1]\langle Da_j Du_n, Lu_n\rangle \tag{3.24}$$

Also, we see from (1.1) and the fact that a_1 and a_2 meet $(a-1)$ and $(a-2)$ that

$$\exists K_8 > 0 \quad \text{s.t. } |D^2 u_n| \leqq K_8[|Lu_n| + |Du_n|] \qquad \forall\, n. \tag{3.25}$$

Also, it follows from (2.24) and Poincare's inequality that there exists a constant K_8' such that $||Du_n||_{L^2} \leqq K_8'||Lu_n||_{L^2}$ for all n. Hence, it follows from (1.7), (3.4), (3.24) and (3.25) that

$$\lim_{n\to\infty} |C_n|/\langle Lu_n, Lu_n\rangle = 0 \tag{3.26}$$

In a similar manner, it follows from (1.8) and (3.22) that

$$\lim_{n\to\infty} |B_n|/\langle Lu_n, Lu_n\rangle = 0 \tag{3.27}$$

On the other hand, we see from (3.21), (3.4) and $(f-3)$ that

$$\exists K_9 > 0 \quad \text{s.t. } |A_n| \leqq K_9 ||u_n||_{L^2} ||Lu_n||_{L^2} \qquad \forall\, n. \tag{3.28}$$

We conclude from (3.23) and (3.26)–(3.28) that

$$\exists n_0 > 0 \quad \text{s.t. } ||LW_n||_{L^2} \leqq 2K_9 \quad \text{for } n \geqq n_0.$$

This fact joined with (3.10) and (3.25) gives

$$||D^2 W_n||_{L^2} \leqq K_8[2K_9 + K_7] \quad \text{for } n \geqq n_0 \tag{3.29}$$

Hence, it follows from (3.10), (3.11), (3.18), (3.29) and the Sobolev compact imbedding theorem [**1**, p. 144] that

$$\lim_{n\to\infty} DW_n(x) = D\phi_1(x) \quad \text{uniformly for } x \in \bar{\Omega}. \tag{3.30}$$

Now from [**9**, p. 4] we have that

$$D\phi_1(0) > 0 \quad \text{and} \quad D\phi_1(1) < 0. \tag{3.31}$$

Since $\phi_1(x) > 0$ for all $x \in \Omega$, we conclude from (3.11) (i), (3.18), (3.30) and (3.31) that there exists $n_1 > 0$ such that

$$W_n(x) > 0 \qquad \forall\, x \in \Omega \quad \text{and} \quad n \geqq n_1.$$

But $u_n(x) = W_n(x)||u_n||_{L^2}$. Therefore we have from this last fact that

$$u_n(x) > 0 \qquad \forall\, x \in \Omega \quad \text{and} \quad n \geqq n_1. \tag{3.32}$$

We invoke (3.1) once again and take $v = \phi_1$ in (2.8). It follows from (1.7) (ii), (1.9) (i), (1.12) and (3.32) that

$$\underset{\sim}{Q}(u_n, \phi_1) = \langle Lu_n, \phi_1\rangle = \lambda_1 \hat{u}_n(1) \quad \text{for } n \geqq n_1 \tag{3.33}$$

Likewise, it follows from (3.32) and $(f-2)$ that the right-hand side of (2.8) with $v=\phi_1$ becomes

$$(3.34)\quad \tilde{A}_n=\lambda_1\hat{u}_n(1)+\hat{u}_n(1)/n+\langle [f_3]^n(\cdot,u_n)+h,\phi_1\rangle \quad \text{for } n\geqq n_1$$

Since $\underset{\sim}{Q}(u_n,\phi_1)=\tilde{A}_n$; we conclude from (3.33) and (3.34) that

$$(3.35)\qquad \hat{u}_n(1)/n=-\langle [f_3]^n(\cdot,u_n)+h,\phi_1\rangle \quad \text{for } n\geqq n_1$$

Now from $(f-2)$, we see that $f_3(x,s)\geqq 0$ for $s\geqq 0$. Consequently, it follows from (1.11) and (3.35) that $\hat{u}_n(1)\leqq 0$. Therefore $\widehat{W}_n(1)\leqq 0$ for $n\geqq n_1$. But from (3.11) (i) and (3.18), we have that $\widehat{W}_n(1)\to 1$ as $n\to\infty$. We consequently obtain that $1\leqq 0$, a manifest contradiction. Hence (3.3) is false, and (3.2) is indeed true.

Next, we claim

$$(3.36)\qquad \exists K_3'>0 \quad \text{s.t. } ||D^2u_n||_{L^2}\leqq K_3' \qquad \forall\, n$$

To establish this fact, we recall from (3.19) that $Lu_n\in S_n$. Also we have from (1.1) that

$$(3.37)\qquad Lu_n=-(a_1+a_2)D^2u_n-(Da_1+Da_2)Du_n.$$

We consequently obtain from (3.20)–(3.22) in conjunction with (3.2), $(a-1)$, (1.7) (i) and (3.37) that

$$(3.38)\quad \exists K_{10}>0 \quad \text{s.t. } \sum_{j=1}^{2}\sigma_{1j}(u_n)\langle a_jD^2u_n,(a_1+a_2)D^2u_n\rangle \leqq K_{10}||D^2u_n||_{L^2}+K_{10} \qquad \forall\, n.$$

But then it follows from $(a-2)$ and (1.7) (i) applied to the inequality in (3.38) that

$$4\varepsilon_0^3||D^2u_n||_{L^2}^2\leqq K_{10}||D^2u_n||_{L^2}+K_{10} \qquad \forall\, n$$

where $\varepsilon_0>0$. We conclude from this last inequality that (3.36) does indeed hold.

It then follows from the Sobolev compact imbedding theorem [**1**, p. 144] in conjunction with (3.2) and (3.36) that there exists $u \in C^1(\bar{\Omega}) \cap W^{2,2}(\Omega)$ such that

$$\lim_{n\to\infty} |u_n(x) - u(x)| + |Du_n(x) - Du(x)| = 0 \quad \text{uniformly for } x \in \bar{\Omega}, \tag{3.39}$$

where we have used the full sequence for ease of notation.

Next, we observe from (1.4), (1.9) (ii), (3.39) and the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty} \int_\Omega |b_j(x,u_n) - b_j(x,u)|^2 |v|^2 = 0 \qquad \forall\, v \in L^2(\Omega),\ j = 1,2.$$

Hence, it follows from (3.39) and (3.2) that

$$\lim_{n\to\infty} \int_\Omega b_1(x,u_n)[Du_n]^+ v = \int_\Omega b_1(x,u)[Du]^+ v \qquad \forall\, v \in L^2(\Omega) \tag{3.40}$$

with a similar situation prevailing for $b_2(x,u)[Du]^-$.

From (3.39) we see that

$$\lim_{n\to\infty} u_n = u \quad \text{strongly in } W_0^{1,2}(\Omega).$$

Hence, it follows from (1.3) that $\sigma_{ij}(u_n) \to \sigma_{ij}(u)$, $i,j = 1,2$. We conclude from (1.12), (3.39) and (3.40) that

$$\lim_{n\to\infty} \underset{\sim}{Q}(u_n, v) = \underset{\sim}{Q}(u,v) \qquad \forall\, v \in W_0^{1,2}(\Omega). \tag{3.41}$$

It is clear from $(f-3)$ and (3.39) that $\{[f_j]^n(x,u_n)\}_{n=1}^\infty$ is absolutely equiintegrable for $j = 1,2,3$. Hence, it follows from Egoroff's theorem, $(f-1)$, and (3.39) that

$$\lim_{n\to\infty} \langle [f_j]^n(\cdot, u_n), v\rangle = \langle f_j(\cdot,u), v\rangle \qquad \forall\, v \in C(\bar{\Omega}) \tag{3.42}$$

for $j = 1,2,3$.

Next, we let $v \in \bigcup_{n=1}^\infty S_n$. Then, it follows from (3.1) that (2.8) holds for this v. We take the limit as $n \to \infty$ on both sides of (2.8) and obtain from (3.39), (3.41) and (3.42) that

$$\underset{\sim}{Q}(u,v) = \langle \lambda_1 u - \alpha u^- + \sum_{j=1}^3 \delta_j f_j(\cdot,u) + h, v\rangle \qquad \forall\, v \in \bigcup_{n=1}^\infty S_n. \tag{3.43}$$

Now, it follows from (2.4) and Lemma 2 that if $v \in W_0^{1,2}(\Omega)$ there exists a sequence $\{v_n\}_{n=1}^{\infty}$ with $v_n \in S_n$ such that

$$\lim_{n\to\infty} ||v_n - v||_{W^{1,2}} = 0.$$

It is then clear from (1.12) and this last fact that $\lim_{n\to\infty} \underset{\sim}{Q}(u, v_n) = \underset{\sim}{Q}(u, v)$. It is clear that a similar situation prevails for the right-hand side of (3.43). Hence we see that (3.43) is indeed true for all $v \in W_0^{1,2}(\Omega)$. But then it follows from Lemma 3 that $u \in C^1(\bar{\Omega}) \cap C^2(\Omega) \cap W^{2,2}(\Omega)$. □

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