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VALUE OF A BOEHMIAN AT A POINT AND AT INFINITY

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ABSTRACT. We define the notion of a value of a Boehmian at a point and study its properties. We prove that a Boehmian which has a value at a point is a Borel measure in a neighborhood of that point. We also define the notion of a value of a Boehmian at infinity.

0. Introduction. The name Boehmians is given to all objects defined by an algebraic construction described first in [**3**]. The construction applied to function spaces yields various spaces of generalized functions, see [**5, 6, 8, 10**]. Boehmians include all Schwartz distributions and all regular operators introduced by T.K. Boehme in [**2**]. It is interesting to investigate which properties of distributions extend onto Boehmians. The spaces of Boehmians have all basic properties we expect from a space of generalized functions as stated in [**13**, p. 135]. Some interesting results have been obtained in the area of the Fourier transform and the Fourier series (for periodic Boehmians), see [**6, 10, 11, 12**], as well as other integral transforms [**7, 8**] and [**9**]. In this note we investigate properties of the notion of a value of a Boehmian at a point and at infinity.

Generalized functions do not assign values to points. For example, the Dirac delta distribution does not have a value at the origin. On the other hand it is natural to say that it is equal to zero at any other point. A value of a distribution at a point can be defined in more than one way, see [**1**]. In this note we show that one of those definitions can be adopted for Boehmians and that the concept has desirable properties.

For convenience of the reader, the definition of Boehmians is given in Section 1. Section 2 is devoted to the notion of a value at a point and Section 3 to the notion of a value at infinity.

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1. Boehmians. Let $\mathcal{E}$ be a Banach space. By $\mathcal{C}(\mathbf{R}^N, \mathcal{E})$ we denote the space of continuous $\mathcal{E}$-valued functions on $\mathbf{R}^N$ and by $C^\infty(\mathbf{R}^N, \mathcal{E})$ the space of infinitely differentiable $\mathcal{E}$-valued functions on $\mathbf{R}^N$. Let $\mathcal{D}(\mathbf{R}^N)$ denote the space of infinitely differentiable real-valued functions with compact support in $\mathbf{R}^N$. If $f \in \mathcal{C}(\mathbf{R}^N, \mathcal{E})$ and $\phi \in \mathcal{D}(\mathbf{R}^N)$, then by $f * \phi$ we mean the convolution of f and ϕ, i.e.,

$$(f * \phi)(x) = \int_{\mathbf{R}^N} f(u)\phi(x-u)\,du,$$

where the righthand side is the Bochner integral. For $\phi \in \mathcal{D}(\mathbf{R}^N)$ define $\rho(\phi)$ to be the radius of the smallest closed ball centered at the origin which contains the support of ϕ, i.e.,

$$\rho(\phi) = \inf\{\varepsilon > 0 : \phi(x) = 0 \text{ for } ||x|| \geq \varepsilon\}$$

where, for $x = (x_1, \dots, x_N)$, $||x|| = \sqrt{x_1^2 + \cdots + x_N^2}$.

A sequence $\delta_1, \delta_2, \ldots \in \mathcal{D}(\mathbf{R}^N)$ is called a *delta sequence* if

$$\int_{\mathbf{R}^N} \delta_n(x)\,dx = 1 \quad \text{for every } n \in \mathbf{N},$$

$$\int_{\mathbf{R}^N} |\delta_n(x)|\,dx < M \quad \text{for some } M \in \mathbf{R} \text{ and all } n \in \mathbf{N},$$

$$\rho(\delta_n) \to 0 \quad \text{as } n \to \infty.$$

We will use the fact that, for any pair of delta sequences $\{\phi_n\}$ and $\{\psi_n\}$, the sequence of convolutions $\{\phi_n * \psi_n\}$ is a delta sequence. Moreover, for any $f \in \mathcal{C}(\mathbf{R}^N, \mathcal{E})$ and any delta sequence $\{\delta_n\}$, the sequence $\{f * \delta_n\}$ converges to f uniformly on compact subsets of $\mathbf{R}^N$.

A pair of sequences (f_n, ϕ_n), $n = 1, 2, \ldots$, is called a *quotient of sequences* and denoted by f_n/ϕ_n, if $f_n \in \mathcal{C}(\mathbf{R}^N, \mathcal{E})$, $\{\phi_n\}$ is a delta sequence, and $f_n * \phi_m = f_m * \phi_n$ for all $m, n \in \mathbf{N}$. Two quotients of sequences f_n/ϕ_n and g_n/ψ_n are equivalent if $f_n * \psi_m = g_m * \phi_n$ for every $m, n \in \mathbf{N}$. The equivalence class of a quotient of sequences f_n/ϕ_n will be denoted by $[f_n/\phi_n]$. The space of equivalence classes of quotients of sequences will be denoted by $\mathfrak{B}(\mathbf{R}^N, \mathcal{E})$, or $\mathfrak{B}$ for short, and its elements will be called *Boehmians*.

Addition and multiplication by a scalar are defined in $\mathfrak{B}$ as follows:

$$\lambda[f_n/\phi_n] = [\lambda f_n/\phi_n];$$
$$[f_n/\phi_n] + [g_n/\psi_n] = [(f_n * \psi_n + g_n * \phi_n)/(\phi_n * \psi_n)].$$

A function $f \in \mathcal{C}(\mathbf{R}^N, \mathcal{E})$ can be identified with the Boehmian $[(f * \delta_n)/\delta_n]$, where $\{\delta_n\}$ is any delta sequence. It can be shown that this identification is independent of the choice of the delta sequence $\{\delta_n\}$. Moreover, it is an isomorphism which is continuous with respect to uniform convergence on compact subsets of $\mathbf{R}^N$ in $\mathcal{C}(\mathbf{R}^N, \mathcal{E})$ and Δ-convergence in $\mathfrak{B}$.

If $F = [f_n/\delta_n] \in \mathfrak{B}$ and $\phi \in \mathcal{D}(\mathbf{R}^N)$, then the convolution $F * \phi$ is defined as $[(f_n * \phi)/\delta_n]$, which is a Boehmian. Note that $[f_n/\phi_n] * \phi_k = f_k$, or more precisely, $[f_n/\phi_n] * \phi_k$ is equal to the Boehmian identified with f_k. If $F \in \mathfrak{B}$ and, for some delta sequence $\{\phi_n\}$, all the convolutions $F * \phi_n$ represent continuous functions, then $F = [(F * \phi_n)/\phi_n]$.

We say that a sequence of Boehmians $\{F_n\}$ is Δ-*convergent* to a Boehmian F, and we write $\Delta\text{-}\lim_{n\to\infty} F_n = F$, if there exists a delta sequence $\{\delta_n\}$ such that $(F_n - F) * \delta_n \in \mathcal{C}(\mathbf{R}^N, \mathcal{E})$ for every $n \in \mathbf{N}$, and the sequence $\{(F_n - F) * \delta_n\}$ converges to zero uniformly on compact sets. It can be proved that $\mathfrak{B}$ equipped with Δ-convergence is a complete quasi-normed space. Addition, multiplication by scalars, and convolution with functions from $\mathcal{D}(\mathbf{R}^N)$ are continuous operations in $\mathfrak{B}$.

It is often more convenient to use another type of convergence in $\mathfrak{B}$, called δ-convergence. We say that a sequence of Boehmians $\{F_n\}$ is δ-*convergent* to a Boehmian F, and we write $\delta\text{-}\lim_{n\to\infty} F_n = F$, if there exists a delta sequence $\{\delta_n\}$ such that $(F_n - F) * \delta_k \in \mathcal{C}(\mathbf{R}^N, \mathcal{E})$, for every $n, k \in \mathbf{N}$, and for every $k \in \mathbf{N}$ the sequence $\{(F_n - F) * \delta_k\}$ converges to zero uniformly on compact sets as $n \to \infty$. This convergence is not topological. A sequence of Boehmians $\{F_n\}$ is Δ-convergent to F if and only if every subsequence of $\{F_n\}$ contains a subsequence δ-convergent to F.

It is easily seen that every Boehmian has a representation $F = [f_n/\phi_n]$ such that $f_n \in \mathcal{C}^\infty(\mathbf{R}^N, \mathcal{E})$ for all $n \in \mathbf{N}$. Indeed, if $F = [g_n/\psi_n]$ is any representation of F, then the representation $F = [(f_n *$

$\psi_n)/(\psi_n * \psi_n)]$ has the desired property. Using such a representation we can define derivatives of a Boehmian F by

$$\frac{\partial^{|k|}F}{\partial x^k} = \left[\frac{\partial^{|k|}f_n}{\partial x^k}\bigg/\phi_n\right],$$

where $k = (k_1, \dots, k_N)$, $k_1, \dots, k_N$ are nonnegative integers, $|k| = k_1 + \cdots + k_N$, and $x^k = x_1^{k_1} \cdots x_N^{k_N}$. The defined operation has all the usual properties and it is continuous with respect to Δ-convergence.

Although Boehmians are defined globally, they can be localized. Let U be an open subset of $\mathbf{R}^N$. Two Boehmians F and G are said to be equal on U if for every compact set $K \subset U$ there exists a delta sequence $\{\delta_n\}$ such that $F * \delta_n$, $G * \delta_n \in \mathcal{C}(\mathbf{R}^N, \mathcal{E})$ and $F * \delta_n = G * \delta_n$ on K for all $n \in \mathbf{N}$.

Proofs of most of the results mentioned above can be found in [**4**] and [**5**].

Let $F \in \mathfrak{B}(\mathbf{R}, \mathcal{E})$. A Boehmian $G \in \mathfrak{B}(\mathbf{R}, \mathcal{E})$ will be called an *antiderivative* of F if $G' = F$. Note that, if $[f_n/\phi_n]$ represents a Boehmian and all the functions f_n are constant functions, then they are identical and thus $[f_n/\phi_n]$ represents a constant function. Consequently, if the derivative F' of a Boehmian equals zero, then F is a constant function. Moreover, any two antiderivatives of a Boehmian differ by a constant. We will prove that every Boehmian has an antiderivative.

Theorem 1.1. *Let $F = [f_n/\delta_n] \in \mathfrak{B}(\mathbf{R}, \mathcal{E})$. Define, for $n = 1, 2, \dots$,*

$$g_n(x) = \int_0^x f_n(t)\,dt \quad \text{and} \quad h_n = g_n - g_n * \delta_k + g_k * \delta_n,$$

where k is a fixed natural number. Then $G = [h_n/\delta_n]$ is an antiderivative of F.

Proof. First we prove that h_n/δ_n is a quotient of sequences. For any $m, n \in \mathbf{N}$ we have

$$(g_n * \delta_m - g_m * \delta_n)' = g_n' * \delta_m - g_m' * \delta_n = f_n * \delta_m - f_m * \delta_n = 0.$$

Consequently, $g_n * \delta_m - g_m * \delta_n$ is a constant function. Now, for any $\phi \in \mathcal{D}(\mathbf{R})$, we have

$$\begin{aligned}(h_n * \delta_m - h_m * \delta_n) * \phi &= (g_n * \delta_m) * (\phi - \delta_k * \phi) - (g_m * \delta_n) * (\phi - \delta_k * \phi) \\ &= (g_n * \delta_m - g_m * \delta_n) * (\phi - \delta_k * \phi) = 0.\end{aligned}$$

The last equality follows from the fact that $g_n * \delta_m - g_m * \delta_n$ is a constant function. This proves that $h_n * \delta_m = h_m * \delta_n$, for all $m, n \in \mathbf{N}$, and hence $G = [h_n/\delta_n]$ is a Boehmian. Moreover,

$$G' = [h_n/\delta_n]' = [(f_n - f_n * \delta_k + f_k * \delta_n)/\delta_n] = [f_n/\delta_n],$$

which completes the proof. □

2. Value of a Boehmian at a point.

Definition 2.1. Let F be a *Boehmian*, and let $x \in \mathbf{R}^N$. If, for every representation $F = [f_n/\phi_n]$ we have $\lim_{n\to\infty} f_n(x) = a$, then we say that F has a *value* a *at* x, and denote this by $F(x) = a$.

If for every representation $[f_n/\phi_n]$ the sequence $\{f_n(x)\}$ converges, then the limit is the same for all representations. In fact, let $F \in \mathfrak{B}$, $F = [f_n/\phi_n] = [g_n/\psi_n]$, and $\lim_{n\to\infty} f_n(x) = a$, $\lim_{n\to\infty} g_n(x) = b$. Define $h_n = f_n$ if n is even, and $h_n = g_n$ if n is odd. Similarly, define $\delta_n = \phi_n$ if n is even, and $\delta_n = \psi_n$ if n is odd. Then $F = [h_n/\delta_n]$ and thus the sequence $\{h_n(x)\}$ converges. This can only happen if $a = b$.

A sequence $\delta_1, \delta_2, \ldots \in \mathcal{D}(\mathbf{R}^N)$ is called a *regular delta sequence* if

(A) $$\int_{\mathbf{R}^N} \delta_n(x)\, dx = 1 \quad \text{for every } n \in \mathbf{N}.$$

(B) For every multi-index $k = (k_1, \ldots, k_N)$, where $k_1, \ldots, k_N$ are nonnegative integers, there exists a positive constant M_k such that

$$(\rho(\delta_n))^{|k|} \int_{\mathbf{R}^N} \left| \frac{\partial^{|k|}}{\partial x^k} \delta_n(x) \right| dx \le M_k \quad \text{for all } n \in \mathbf{N},$$

(C) $$\rho(\delta_n) \to 0 \quad \text{as } n \to \infty.$$

Let $\psi \in \mathcal{D}(\mathbf{R}^N)$ be such that $\int_{\mathbf{R}^N} \psi(x)\,dx = 1$. For $0 < \alpha_n \to \infty$, define $\psi_n(x) = (\alpha_n)^N \psi(\alpha_n x)$. Then $\{\psi_n\}$ is a regular delta sequence.

Lemma 2.2. *If $\{\phi_n\}$ and $\{\psi_n\}$ are regular delta sequences, then the sequence $\{\delta_n\} = \{\phi_n * \psi_n\}$ is a regular delta sequence.*

Lemma 2.3. *Let $\{\phi_n\}$ be a delta sequence. Then there exists a regular delta sequence $\{\delta_n\}$ such that*

$$\delta_n(-x) = \delta_n(x) \quad \textit{for all } x \in \mathbf{R}^N \textit{ and } n \in \mathbf{N}, \tag{2.4}$$

$$\operatorname{supp}\delta_n \textit{ contains a neighborhood of the origin for all } n \in \mathbf{N}, \tag{2.5}$$

$$\{\phi_n * \delta_n\} \textit{ is a regular delta sequence.} \tag{2.6}$$

Proof. Let $\{\phi_n\}$ be any delta sequence. Define

$$\delta_n(x) = (1/\rho(\phi_n))^N \psi(x/\rho(\phi_n))$$

where $\psi \in \mathcal{D}(\mathbf{R}^N)$ is such that $\psi(-x) = \psi(x)$, $\operatorname{supp}\psi = \{x \in \mathbf{R}^N : ||x|| \le 1\}$, and $\int_{\mathbf{R}^N} \psi(x)\,dx = 1$. Then $\{\delta_n\}$ is a regular delta sequence. Clearly $\phi_n * \delta_n \in \mathcal{D}(\mathbf{R}^N)$, $\int_{\mathbf{R}^N} (\phi_n * \delta_n)(x)\,dx = 1$ and $\rho(\phi_n * \delta_n) \to 0$. It remains to be shown that the sequence $\{\phi_n * \delta_n\}$ satisfies (B). Let k be a multi-index. Then there exists a positive number M_k such that

$$(\rho(\delta_n))^{|k|} \int_{\mathbf{R}^N} \left| \frac{\partial^{|k|}}{\partial x^k} \delta_n(x) \right| dx \le M_k.$$

Now, since $\rho(\delta_n) = \rho(\phi_n)$, we have

$$\begin{aligned}
(\rho(\phi_n * \delta_n))^{|k|} &\int_{\mathbf{R}^N} \left| \frac{\partial^{|k|}}{\partial x^k} (\phi_n * \delta_n)(x) \right| dx \\
&\le 2^{|k|} (\rho(\delta_n))^{|k|} \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} |\phi_n(t)| \left| \frac{\partial^{|k|}}{\partial x^k} \delta_n(x-t) \right| dt\,dx \\
&\le 2^{|k|} (\rho(\delta_n))^{|k|} \int_{\mathbf{R}^N} |\phi_n(x)|\,dx \int_{\mathbf{R}^N} \left| \frac{\partial^{|k|}}{\partial x^k} \delta_n(x) \right| dx \\
&\le 2^{|k|} M_k M,
\end{aligned}$$

where M is a positive number such that $\int_{\mathbf{R}^N} |\phi_n(x)|\,dx < M$. This shows that $\{\phi_n * \delta_n\}$ is a regular delta sequence. □

In [**1**] a distribution f is said to have the value a at a point x_0 if $\lim_{n\to\infty}(f * \delta_n)(x_0) = a$ for every regular sequence $\{\delta_n\}$. It may seem that our definition is more restrictive. Theorem 2.4 shows that it is not so.

Let $F \in \mathfrak{B}(\mathbf{R}, \mathcal{E})$, and let $\alpha \in \mathbf{R}$. By $T_\alpha F$ we denote the translation of F by α, i.e., $T_\alpha F = T_\alpha[f_n/\phi_n] = [(T_\alpha f_n)/\phi_n]$, where $T_\alpha f_n(x) = f_n(x-\alpha)$. It is easy to check that $T_\alpha F \in \mathfrak{B}(\mathbf{R}, \mathcal{E})$.

Theorems 2.4, 2.5, 2.7 and 2.8 are formulated for an arbitrary point x_0, but the proofs are written for $x_0 = 0$. This makes the presentation of the proofs simpler. Since $F(x_0) = a$ if and only if $T_{-x_0}F(0) = a$, the general case follows easily.

Theorem 2.4. *Let F be a Boehmian. Then the following are equivalent*:

(i) $F(x_0) = a$.

(ii) *For each representation $[f_n/\phi_n]$ of F, where $\{\phi_n\}$ is a regular delta sequence, we have* $\lim_{n\to\infty} f_n(x_0) = a$.

Proof. Let $x_0 = 0$. Clearly, (i) implies (ii). Assume (ii) and let $F = [f_n/\phi_n]$, where $\{\phi_n\}$ is a delta sequence. It must be shown that $\lim_{n\to\infty} f_n(0) = a$. Assume that $f_n(0)$ does not converge to a. Then there exist an $\varepsilon > 0$ and an increasing sequence of indices $\{p_n\}$ such that

$$||f_{p_n}(0) - a|| > 2\varepsilon \quad \text{for all } n \in \mathbf{N}.$$

Since the functions f_n are continuous, there exist positive numbers $\gamma_1, \gamma_2, \dots$ such that

$$||f_{p_n}(x) - f_{p_n}(0)|| < \varepsilon \quad \text{whenever } ||x|| < \gamma_n.$$

Let $\{\delta_n\}$ be a regular delta sequence such that

(1) $\delta_n(-x) = \delta_n(x)$ for all $x \in \mathbf{R}^N$ and $n \in \mathbf{N}$,

(2) $\operatorname{supp}\delta_n$ contains a neighborhood of the origin for all $n \in \mathbf{N}$,

(3) $\{\phi_n * \delta_n\}$ is a regular delta sequence.

Let $\{q_n\}$ be a subsequence of $\{p_n\}$ such that

$$\delta_{q_n}(x) = 0 \quad \text{whenever } ||x|| > \gamma_n.$$

Then

$$\begin{aligned}
||(f_{q_n} * \delta_{q_n})(0) - a|| &= \left\| \int_{\mathbf{R}^N} f_{q_n}(x)\delta_{q_n}(x)\,dx - a \right\| \\
&= \bigg\| \int_{\mathbf{R}^N} f_{q_n}(0)\delta_{q_n}(x)\,dx - a \\
&\qquad + \int_{\mathbf{R}^N} (f_{q_n}(x) - f_{q_n}(0))\delta_{q_n}(x)\,dx \bigg\| \\
&\geq \left\| \int_{\mathbf{R}^N} f_{q_n}(0)\delta_{q_n}(x)\,dx - a \right\| \\
&\quad - \left\| \int_{\mathbf{R}^N} (f_{q_n}(x) - f_{q_n}(0))\delta_{q_n}(x)\,dx \right\| \\
&\geq ||f_{q_n}(0) - a|| - \sup_{||x|| \leq \gamma_n} ||f_{q_n}(x) - f_{q_n}(0)|| \\
&> 2\varepsilon - \varepsilon = \varepsilon.
\end{aligned}$$

On the other hand, since $F = [(f_{q_n} * \delta_{q_n})/(\phi_{q_n} * \delta_{q_n})]$ and $\{\phi_{q_n} * \delta_{q_n}\}$ is a regular delta sequence, we have $\lim_{n\to\infty}(f_{q_n} * \delta_{q_n})(0) = a$. This contradiction proves the assertion. □

Theorem 2.5. *Let $F = [f_n/\phi_n] \in \mathfrak{B}$ such that $F(x_0) = a$. Then, for each $\varepsilon > 0$, there exist a positive number M and $n_0 \in \mathbf{N}$ such that*

$$\max_{||x-x_0|| \leq M} ||f_n(x) - a|| \leq \varepsilon \quad \textit{for each } n \geq n_0.$$

Proof. Let $x_0 = 0$. Assume, to the contrary, that there exist a positive number ε, a sequence of positive numbers $M_n \to 0$, and a sequence $p_n \to \infty$ such that

$$\max_{||x|| \leq M_n} ||f_{p_n}(x) - a|| > \varepsilon$$

for each $n \in \mathbf{N}$. Let $x_n \in \mathbf{R}^N$, $n \in \mathbf{N}$, be such that $||x_n|| \le M_n$ and

$$||f_{p_n}(x_n) - a|| = \max_{||x|| \le M_n} ||f_{p_n}(x) - a||.$$

Let $\{\gamma_n\}$ be a sequence of positive numbers such that, for all $n \in \mathbf{N}$,

$$\max_{||x-x_n|| \le \gamma_n} ||f_{p_n}(x) - a|| \ge \varepsilon \quad \text{and} \quad \max_{||x-x_n|| \le \gamma_n} ||f_{p_n}(x) - f_{p_n}(x_n)|| < \varepsilon/2.$$

Let $\{\delta_n\}$ be a delta sequence such that $\delta_n(x) = 0$ whenever $||x - x_n|| \ge \gamma_n$. Define $\tilde{\delta}_n(x) = \delta_n(-x)$. Then

$$\begin{aligned} ||(f_{p_n} * \tilde{\delta}_n)(0) - a|| &= \left\| \int_{\mathbf{R}^N} f_{p_n}(x)\delta_n(x)\,dx - a \right\| \\ &= \left\| \int_{\mathbf{R}^N} f_{p_n}(x_n)\delta_n(x)\,dx - a \right. \\ &\qquad \left. + \int_{\mathbf{R}^N} (f_{p_n}(x) - f_{p_n}(x_n))\delta_n(x)\,dx \right\| \\ &\ge \left\| \int_{\mathbf{R}^N} f_{p_n}(x_n)\delta_n(x)\,dx - a \right\| \\ &\quad - \left\| \int_{\mathbf{R}^N} (f_{p_n}(x) - f_{p_n}(x_n))\delta_n(x)\,dx \right\| \\ &\ge \varepsilon - \varepsilon/2 = \varepsilon/2. \end{aligned}$$

On the other hand, since $F = [(f_{p_n} * \tilde{\delta}_n)/(\phi_{p_n} * \tilde{\delta}_n)]$, we have $\lim_{n\to\infty}(f_{p_n} * \tilde{\delta}_n)(0) = a$. This contradiction proves the lemma. □

Corollary 2.6. *If $F = [f_n/\phi_n] \in \mathfrak{B}$ and F has a value at a point, then there exists a constant $C > 0$ such that $||f_n(x)|| \le C$ in some neighborhood of that point for all $n \in \mathbf{N}$.*

Theorem 2.7. *If a Boehmian $F \in \mathfrak{B}(\mathbf{R}, \mathcal{E})$ has a value at a point, then every anti-derivative of F has a value at that point.*

Proof. Assume that $F, G \in \mathfrak{B}(\mathbf{R}, \mathcal{E})$, $F(0) = a$, and $G' = F$. Let $G = [g_n/\phi_n]$, and let ε be an arbitrary positive number. Since $F = [g_n'/\phi_n]$ and $F(0) = a$, there exist $M > 0$ and $n_0 \in \mathbf{N}$ such that

$$||g_n'(x) - a|| \le \varepsilon \quad \text{for } |x| < M \text{ and } n \ge n_0.$$

Without loss of generality we can assume that $M < 1$. Then, for $n \geq n_0$,

$$\left\| \int_0^x (g_n'(t) - a)\, dt \right\| \leq \varepsilon \quad \text{for } |x| < M$$

and thus

$$||g_n(x) - ax - g_n(0)|| \leq \varepsilon \quad \text{for } |x| < M.$$

Let $k \in \mathbf{N}$ be such that $\rho(\phi_k) < M$. Then

$$\left\| (g_n * \phi_k)(0) - a \int_{\mathbf{R}} x\phi_k(-x)\, dx - g_n(0) \right\| = \left\| \int_{\mathbf{R}} (g_n(x) - ax - g_n(0))\phi_k(-x)\, dx \right\| \leq A\varepsilon$$

for all $n \geq n_0$, where $A = \int_{\mathbf{R}} |\phi_k(x)|\, dx$. Since $(g_n * \phi_k)(0) = (g_k * \phi_n)(0) \to g_k(0)$ as $n \to \infty$ and $a \int_{\mathbf{R}} x\phi_k(-x)\, dx$ is independent of n, we have

$$||g_n(0) - g_m(0)|| < 4A\varepsilon$$

for all m, $n > n_1$, where n_1 is some positive integer. This proves that $\{g_n(0)\}$ is a convergent sequence. Since $G = [g_n/\phi_n]$ is an arbitrary representation of G, G has a value at 0. □

Let μ be an $\mathcal{E}$-valued Borel measure on $\mathbf{R}^N$, and let $\{\delta_n\}$ be a delta sequence. Define $f_n(x) = \int_{\mathbf{R}^N} \delta_n(x - u)\, d\mu(u)$. Then $[f_n/\delta_n]$ is a Boehmian. The following theorem has been suggested by J. Burzyk (private correspondence).

Theorem 2.8. *Let $F = [f_n/\phi_n] \in \mathfrak{B}$. If there exists a constant $C > 0$ such that $||f_n|| \leq C$ in some neighborhood of $x_0 \in \mathbf{R}^N$, then F is a Borel vector measure in some neighborhood of x_0.*

Proof. Let $x_0 = 0$. Let U be an open neighborhood of 0 such that $||f_n|| < C$ in U, and let V be a bounded open neighborhood of 0 whose closure is in U. Let $\Omega \subseteq V$ be a Borel set. Then $\{\int_\Omega f_n\}$ is a bounded sequence. Suppose, for some increasing sequences of indices $\{p_n\}$ and $\{r_n\}$, we have

$$\lim_{n\to\infty} \int_\Omega f_{p_n} = a \quad \text{and} \quad \lim_{n\to\infty} \int_\Omega f_{r_n} = b.$$

Let ω be the characteristic function of Ω, and let $\tilde{\omega}(x) = \omega(-x)$. Since the sequence $\{f_{p_n} * \tilde{\omega}\}$ is equicontinuous in some neighborhood of 0, say U_0, there exists a subsequence $\{f_{q_n}\}$ of $\{f_{p_n}\}$ such that $\{f_{q_n} * \tilde{\omega}\}$ converges uniformly on U_0, by Ascoli's theorem. Then

$$\lim_{n\to\infty} (f_{q_n} * \tilde{\omega} * \phi_{r_n})(0) = \lim_{n\to\infty} (f_{q_n} * \tilde{\omega})(0) = a,$$

because $\{\phi_{r_n}\}$ is a delta sequence. On the other hand, since $[f_n/\phi_n] \in \mathfrak{B}$,

$$\lim_{n\to\infty} (f_{q_n} * \tilde{\omega} * \phi_{r_n})(0) = \lim_{n\to\infty} (f_{r_n} * \tilde{\omega} * \phi_{q_n})(0) = b.$$

Consequently, $a = b$. This allows us to define a set function on Borel subsets of V:

$$\mu(\Omega) = \lim_{n\to\infty} \int_\Omega f_n \, d\lambda,$$

where λ denotes the Lebesgue measure. We will show that μ is a measure.

Clearly $\mu(\varnothing) = \lim_{n\to\infty} \int_\varnothing f_n = 0$. Now let $\{A_i\}$ be a sequence of disjoint Borel subsets of V. We must show that $\mu(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$, or equivalently,

$$\lim_{n\to\infty} \int_{\cup_{i=1}^\infty A_i} f_n = \lim_{m\to\infty} \left(\lim_{n\to\infty} \int_{B_m} f_n \right),$$

where $B_m = \cup_{i=1}^m A_i$. Let $\varepsilon > 0$. Since V is bounded, there exist $n_0 \in \mathbf{N}$ such that $\lambda(\cup_{i=m+1}^\infty A_i) < \varepsilon/C$ for each $m > n_0$. Consequently,

$$\begin{aligned}\left\| \left(\lim_{n\to\infty} \int_{\cup_{i=1}^\infty A_i} f_n - \left(\lim_{n\to\infty} \int_{B_m} f_n \right) \right\| &= \left\| \lim_{n\to\infty} \int_{\cup_{i=m+1}^\infty A_i} f_n \right\| \\ &\le C\lambda\left(\cup_{i=m+1}^\infty A_i \right) < \varepsilon\end{aligned}$$

for each $m > n_0$. Finally, if $\{A_i\}$ is a sequence of disjoint Borel subsets of V, then

$$\begin{aligned}\sum_{i=1}^\infty ||\mu(A_i)|| &= \sum_{i=1}^\infty || \lim_{n\to\infty} \int_{A_i} f_n || \\ &\le \sum_{i=1}^\infty C\lambda(A_i) \\ &= C\lambda\left(\cup_{i=1}^\infty A_i \right) < \infty.\end{aligned}$$

This shows that μ has finite variation. $\square$

Corollary 2.9. *If a Boehmian F has a value at a point, then F is a Borel vector measure in some neighborhood of that point.*

Corollary 2.10. *If $F \in \mathfrak{B}(\mathbf{R}^N, \mathbf{C})$ has a value at a point, then F is a Schwartz distribution in some neighborhood of that point.*

3. Value of a Boehmian at infinity. In this section we assume that the dimension $N = 1$.

Definition 3.1. A Boehmian $F \in \mathfrak{B}(\mathbf{R}, \mathcal{E})$ is said to have a *value at infinity*, if the sequence of Boehmians $\{T_{-\alpha_n} F\}$ is Δ-convergent for every $\alpha_n \to \infty$.

Note that if F has a value at infinity, then there exists a unique $G \in \mathfrak{B}(\mathbf{R}, \mathcal{E})$ such that $\Delta\text{-}\lim_{n\to\infty} T_{-\alpha_n} F = G$ for every $\alpha_n \to \infty$. We will prove that G is a constant function, or more precisely, $G = [(C * \phi_n)/\phi_n]$ for some $C \in \mathcal{E}$. That constant C will be called the value of F at infinity and we will write $\lim_{x\to\infty} F(x) = C$.

Let $F = [f_n/\phi_n]$ be a Boehmian. Note that for every $\alpha \in \mathbf{R}$ and $n \in \mathbf{N}$, we have $(T_\alpha F) * \phi_n = T_\alpha (F * \phi_n)$.

Lemma 3.2. *Let $F = [f_n/\phi_n] \in \mathfrak{B}(\mathbf{R}, \mathcal{E})$, $f_n \in \mathcal{C}^\infty(\mathbf{R}, \mathcal{E})$. Then, for every $\varepsilon_n \to 0$, $\varepsilon_n \neq 0$, we have*

$$\delta\text{-}\lim_{n\to\infty} \frac{T_{-\varepsilon_n} F - F}{\varepsilon_n} = F'.$$

Proof. Let $0 \neq \varepsilon_n \to 0$, and let $k \in \mathbf{N}$. Then

$$\begin{aligned}\left(\frac{T_{-\varepsilon_n} F - F}{\varepsilon_n} - F'\right) * \phi_k &= \frac{T_{-\varepsilon_n}(F * \phi_k) - F * \phi_k}{\varepsilon_n} - F' * \phi_k \\ &= \frac{T_{-\varepsilon_n} f_k - f_k}{\varepsilon_n} - f_k' \to 0. \qquad \square\end{aligned}$$

Note that Lemma 3.2 implies that, for every $\varepsilon_n \to 0$, $\varepsilon_n \neq 0$, we have

$$\Delta\text{-}\lim_{n\to\infty} \frac{T_{-\varepsilon_n}F - F}{\varepsilon_n} = F'.$$

Theorem 3.3. *If a Boehmian F has a value at infinity, then, for every $\alpha_n \to \infty$, the sequence $\{T_{-\alpha_n}F\}$ is Δ-convergent to a constant function.*

Proof. Let F have a value at infinity, and let $\alpha_n \to \infty$. Then Δ-$\lim_{n\to\infty} T_{-\alpha_n}F = G$, for some $G \in \mathfrak{B}(\mathbf{R}, \mathcal{E})$. For any $\beta \in \mathbf{R}$, we have

$$T_\beta G = \Delta\text{-}\lim_{n\to\infty} T_{\beta-\alpha_n}F = G.$$

Consequently, for every $\varepsilon_n \to 0$, $\varepsilon_n \neq 0$, we have

$$G' = \delta\text{-}\lim_{n\to\infty} \frac{T_{-\varepsilon_n}G - G}{\varepsilon_n} = \delta\text{-}\lim_{n\to\infty} \frac{G - G}{\varepsilon_n} = 0.$$

Thus, G is a constant function. □

Using the fact that differentiation is continuous with respect to Δ-convergence and Theorem 3.3, we obtain:

Corollary 3.4. *If F has a value at infinity, then F' has the value* 0 *at infinity.*

Theorem 3.5. *Let $F = [f_n/\phi_n] \in \mathfrak{B}(\mathbf{R}, \mathcal{E})$. If $\lim_{x\to\infty} f_n(x) = C$ for each $n \in \mathbf{N}$, then $\lim_{x\to\infty} F(x) = C$.*

Proof. Assume that, for each $n \in \mathbf{N}$, $\lim_{x\to\infty} f_n(x) = C$. Let $k \in \mathbf{N}$ and $0 < \alpha_n \to \infty$. Then $(T_{-\alpha_n}F) * \phi_k = T_{-\alpha_n}f_k \to C$ as $n \to \infty$. Hence, δ-$\lim_{n\to\infty} T_{-\alpha_n}F = C$, which implies that $\lim_{x\to\infty} F(x) = C$. □

Theorem 3.6. *Let $f \in \mathcal{C}(\mathbf{R}, \mathbf{R})$ and $F = [f * \phi_n/\phi_n]$. If $\lim_{x\to\infty} f(x) = C$, then $\lim_{x\to\infty} F(x) = C$.*

Proof. Let $0 < \alpha_n \to \infty$, and let $k \in \mathbf{N}$. Then $(T_{-\alpha_n}F) * \phi_k = (T_{-\alpha_n}f) * \phi_k$. Since the sequence $\{T_{-\alpha_n}f\}$ converges to C uniformly on compact subsets of $\mathbf{R}$, $(T_{-\alpha_n}F) * \phi_k$ converges to $C * \phi_k = C$ uniformly on compact subsets of $\mathbf{R}$. Hence, $\delta\text{-}\lim_{n\to\infty} T_{-\alpha_n}F = C$, and thus $\Delta\text{-}\lim_{n\to\infty} T_{-\alpha_n}F = C$, i.e., $\lim_{x\to\infty} F(x) = C$. □

The converse is not true as can be seen in the following example.

Example 3.7. Take $\phi \in \mathcal{C}(\mathbf{R}, \mathbf{R})$, $0 \le \phi \le 1$, $\rho(\phi) \le 1$, $\phi(0) = 1$, and $\int \phi(x)\, dx = 1$. Consider the function

$$f(x) = \sum_{n=1}^{\infty} \phi(2^n x - 4^n).$$

Clearly, f is a continuous function and $\lim_{x\to\infty} f(x)$ does not exist. On the other hand, the Boehmian $F = [f * \phi_n/\phi_n]$, where $\phi_n(x) = n\phi(nx)$, has the value 0 at infinity. Indeed, it is easy to check that $\lim_{x\to\infty}(f * \phi_n)(x) = 0$, for each $n \in \mathbf{N}$. Thus, F has the value 0 at infinity, by Theorem 3.5.

A Boehmian $F = [f_n/\phi_n] \in \mathfrak{B}(\mathbf{R}, \mathcal{E})$ is called *integrable* if f_n is Bochner integrable for every $n \in \mathbf{N}$. Note that if $F = [f_n/\phi_n]$ is integrable, then

$$\int_{\mathbf{R}} f_m = \int_{\mathbf{R}} f_m * \phi_1 = \int_{\mathbf{R}} f_1 * \phi_m = \int_{\mathbf{R}} f_1$$

for every $m \in \mathbf{N}$. The integral of an integrable Boehmian $F = [f_n/\phi_n]$ is defined as $\int_{\mathbf{R}} F = \int_{\mathbf{R}} f_1$.

Theorem 3.8. *If $F \in \mathfrak{B}(\mathbf{R}, \mathcal{E})$ is an integrable Boehmian, then F has the value 0 at infinity.*

Proof. Let $F = [f_n/\phi_n]$. Define, for $n \in \mathbf{N}$,

$$g_n(x) = \int_{-\infty}^{x} f_n(t)\, dt, \qquad h_n = g_n - g_n * \phi_1 + g_1 * \phi_n, \quad \text{and} \quad G = [h_n/\phi_n].$$

Let $C = \int_{\mathbf{R}} F$. Then $\lim_{x\to\infty} g_n(x) = C$ for every $n \in \mathbf{N}$. Hence, $\lim_{x\to\infty} h_n(x) = C$ for every $n \in \mathbf{N}$. Thus, by Theorem 3.5, G has the value C at infinity. Since $F = G'$, F has the value 0 at infinity, by Corollary 3.4. □

4. Concluding remarks. It can be proved that every Boehmian has a representation $[f_n/\phi_n]$ such that $\{\phi_n\}$ is a delta sequence "made of one function," i.e., $\{\phi_n\}$ has the form $\phi_n(x) = (\alpha_n)^N \phi(\alpha_n x)$. In the definition of the value of a Boehmian at a point, we can require that $\lim_{n\to\infty} f_n(x) = a$ for every representation $[f_n/\phi_n]$ such that $\{\phi_n\}$ is a delta sequence made of one function. Is this equivalent to Definition 2.1?

It can be proved that if a distribution f satisfies the equation $f(\alpha x) = f(x)$ for every nonzero α, then it is a constant function, see [**1**]. Is the same true for Boehmians?

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