CONFORMAL INVARIANTS FOR CURVES AND SURFACES IN THREE DIMENSIONAL SPACE FORMS

GRANT CAIRNS, RICHARD SHARPE AND LYNETTE WEBB

ABSTRACT. Conformal invariants of submanifolds have been known for a hundred years. And yet many recent papers seem to ignore the major works of yesteryear and the complete list of invariants that they defined. This present paper aims to provide a readable account of the existence of conformal invariants in their most natural setting, that of curves and surfaces in 3-space.

1. Introduction. In the same way that curves and surfaces possess invariants under the group of isometries (curvature, torsion, principal curvatures, etc.), it has been known since the beginning of this century that there are other functions which are invariant under the larger group of conformal transformations; hence there is a *conformal curvature*, a *conformal torsion*, etc. Now, conformal invariants have received a revival of interest in recent years [**8, 46, 22, 18, 19, 17**], in part because of Willmore's conjecture [**47**], and the associated conformal invariant known as the Willmore integrand (see [**31**]). Nevertheless, despite this interest, there have been very few references in the literature to the complete list of invariants and to the papers which established them (important exceptions are [**20, 33, 35, 36**]. See also [**45**]). The purpose of this present paper is to provide what we hope will be a timely review of these invariants.

To be specific, we are concerned in this paper with conformal geometry in the following sense. Let N denote a 3-dimensional Riemannian manifold with metric g of constant sectional curvature (which means that locally it is elliptic, hyperbolic or Euclidean space). We study those properties of N which remain invariant under conformal change of metric, that is under replacement of g by ρg, where ρ is a positive function on N and ρg still has constant sectional curvature, but possibly with a different constant. For example we can take N to be the

round sphere $\mathbf{S}^3$. Then the properties that interest us will not only be invariant under the group of Möbius transformations of $\mathbf{S}^3$ but they will also be invariant under stereographic projection from Euclidean space $\mathbf{E}^3$. In particular, we are interested in the invariants of the local extrinsic conformal geometry of "generic" curves and surfaces in N; that is to say, for "generic" locally defined curves and surfaces M in N, we study those functions on M which remain invariant under conformal changes as described above.

The key works in the development of conformal invariants were Tresse's 1892 paper on surfaces [40] and Liebmann's 1923 paper on curves [23] (see also [11, 7, 26, 21]). Complete sets of invariants were then obtained independently by Takasu [37, 38], Thomsen [39, 4] and Vessiot [42, 43, 44] (see also [25, 28]). These invariants were of such interest in the 1920's that Vessiot could write in 1927, concerning the invariant now known as the Willmore integrand [44 part II, p. 56]: "Cette integrale a déjà été introduite par divers auteurs, ainsi que les surfaces pour lesquelles elle est stationnaire (surfaces minima au point de vue conforme)."

In the 1940's the theory of conformal invariants was rediscovered and generalized by Fialkow [14, 15, 16], who also provided invariants in arbitrary dimension and for spaces of non-constant curvature. For the purposes of exposition, we have chosen in this paper to present the invariants only in the case of three dimensional space forms. Let us pause a moment to explain this choice. Whereas Fialkow describes his strategy in the following words: "In our development of conformal differential geometry, we have largely ignored the conformal transformations which define the geometry and have attempted to take maximum advantage of the methods of Riemannian geometry" ([16, p. 311]), it is clear that conversely, by restricting one's attention to space forms, one is fixing the group of conformal transformations. The necessary calculations then have a very concrete form and have natural geometric interpretations. In addition, in this special case, the formulas for the invariants are relatively simple, thus enabling one to gain a certain feel for their significance (the formulas in the variable curvature case are complicated by off-diagonal Ricci curvature terms). And finally, as we have kept in mind during the preparation of this paper, many of the known notions (such as the Banchoff-White invariant [2]) and open problems (such as Willmore's conjecture) pertain to space

forms.

Now let us summarize the situation. For curves, the algebra of conformal invariants is generated by an *infinitesimal conformal arc-length* and two invariant functions; a *conformal torsion* and a *conformal curvature* (of order 4 and 5 respectively). The infinitesimal conformal arc-length is a 1-form whose zeros are the vertices of the curve, where we define the vertices to be the stationary points of the curvature that are *simultaneously* zeros of the torsion or the curvature (see Section 4). Up to conformal transformation, the three invariants determine the curve away from the vertices. For surfaces there are two conformally invariant 1-forms and three conformally invariant functions (two of order 3 and one of order 4); they determine the surface away from the umbilics.

The derivation of these invariants by Takasu, Thomsen and Vessiot is somewhat difficult to follow, due to their use of "pentaspherical coordinates," which are a coordinatization of the set of spheres in $\mathbf{E}^3$. One chooses 5 mutually orthogonal spheres in $\mathbf{E}^3$ (of course, this is only possible if one allows spheres of null or imaginary radius); a given sphere is then coordinatized by its 5 "angles of intersection" with the chosen "pentasphere." Now at each point of a given curve or surface, one chooses a pentasphere, thus obtaining a moving "pentaspherical frame." Vessiot writes in 1927 [**42**, p. 101]: "La méthode du pentasphère mobile a été introduite par A. Demoulin [**11**]. Elle est aujourd'hui classique... ." It seems however that pentaspherical coordinates subsequently fell into disfavor, being surpassed by Cartan's more powerful formalism. Fortunately, it turns out that the method of pentaspherical coordinates has a simple modern expression in terms of Bryant's conformal Gauss map (see Section 3). In this way, the associated calculations for surfaces are extremely simple and the invariants are quickly obtained by hand. For curves the situation is less straightforward however, due to the fact that unlike the principal directions of a surface, the normal and bi-normal directions of a curve are not conformally invariant notions. So in Section 4, for curves, we prefer to use the direct method of reduced equations originally employed by Tresse [**41**].

It was known by the geometers of the 1920's that the conformal invariants described above actually form a generating set for all the conformal invariants of curves and surfaces in 3-space. Section 5 is devoted to formalizing this notion of generating set and clearly stating

the results.

Sections 6 and 7 contain some original contributions. In Section 6 we consider some of the applications of the invariants for curves; we give three results. We give a generalized 4-vertex theorem for "spherical" curves. We show that the curves with zero conformal torsion are precisely the spherical curves. And we interpret the integral of the conformal torsion along a closed curve as an $\mathbf{R}$ valued lift of Banchoff and White's $\mathbf{R}/\mathbf{Z}$ conformal invariant "total twist" [**2**]. In Section 7 we give a number of remarks concerning the invariants for surfaces and their connection with Willmore's conjecture.

Throughout this paper we have attempted to provide a readable account of the invariants and of the methods employed. For expediency, we have suppressed many of the necessary calculations, which are elementary but occasionally tedious and which in the main we have performed with the aid of the symbolic processor *Mathematica*.

2. Preliminary Remarks. We begin by considering an umbilic-free surface M in Euclidean space $\mathbf{E}^3$. Let κ_1 and κ_2 denote the principal curvatures of M, with $\kappa_1 > \kappa_2$, and let X_1 and X_2 be corresponding orthonormal principal vector fields. Now let x be a point in M and consider the set of 2-spheres that are tangent to M at x and which have contact to second order at x with some geodesic in M. Of course the mean curvature (that is, the reciprocal of the radius) of these spheres varies with the direction of the associated geodesic at x. By definition, the maximum and minimum values obtained are κ_1 and κ_2 respectively. We are interested in how these spheres behave under conformal transformation.

Given any round 2-sphere σ in $\mathbf{E}^3$, let us denote the mean curvature of σ by H_σ. Recall that by Liouville's Theorem (see for instance [**13**]), if U is an open subset of $\mathbf{E}^3$, every conformal map $\phi \colon U \to \mathbf{E}^3$ can be written as the restriction to U of a composition of reflections in planes and inversions in round 2-spheres. We use the following elementary result from inversive geometry, which we leave to the reader to verify.

Lemma 2.1. *Let $\phi : \mathbf{E}^3 \cup \{\infty\} \to \mathbf{E}^3 \cup \{\infty\}$ be a conformal transformation, let x be some point in $\mathbf{E}^3$ and let P be some plane through x. Then there exist real numbers a and b such that for all*

spheres σ passing through x and tangent to P at x, one has

$$H_{\phi(\sigma)} = aH_\sigma + b.$$

Now apply this lemma to the case where P is the tangent plane to M at the point x. Let $\bar{\kappa}_1$ and $\bar{\kappa}_2$ denote the principal curvatures at $\phi(x)$ of the image $\phi(M)$ of M, under the transformation ϕ. It follows from the lemma that $\bar{\kappa}_1 - \bar{\kappa}_2 = a(\kappa_1 - \kappa_2)$. Moreover, this same argument also shows that the principal directions of $\phi(M)$ at $\phi(x)$ are just the images under ϕ of the principal directions of M at x. One can combine these two facts as follows. Notice that the map ϕ induces a homothetic transformation ϕ_* between the tangent space to M at x and the tangent space to $\phi(M)$ at $\phi(x)$. It is not difficult to see that the number a provided by Lemma 2.1 is just the reciprocal of the homothetic proportionality constant of ϕ_*. Consequently, writing $\mu = (\kappa_1 - \kappa_2)/2$, one has that the vector fields $\xi_1 = X_1/\mu$ and $\xi_2 = X_2/\mu$ are conformally invariant.

These invariant vector fields ξ_1 and ξ_2 can be used to immediately produce two conformally invariant functions. Indeed, the Lie bracket $[\xi_1, \xi_2]$ can be expressed as a function linear combination of ξ_1 and ξ_2:

$$[\xi_1, \xi_2] = -\frac{1}{2}\theta_2\xi_1 - \frac{1}{2}\theta_1\xi_2$$

(the factors $-1/2$ here are chosen for later ease of expression). Then the coefficients θ_1 are θ_2 are necessarily conformal invariants. (Note that the principal vector fields X_1 and X_2, and hence θ_1 and θ_2, are determined by the orientation of the surface only up to a simultaneous change in sign. 'Invariance' therefore is here only defined up to a sign. We return to this matter in Section 4.)

It is an elementary exercise in Riemannian geometry to show that $[X_1, X_2] = -\{X_2(\kappa_1)/2\mu\}X_1 - \{X_1(\kappa_2)/2\mu\}X_2$, whence

$$(2.1) \qquad \theta_1 = \frac{X_1(\kappa_1)}{\mu^2}, \quad \text{and} \quad \theta_2 = \frac{X_2(\kappa_2)}{\mu^2}.$$

These are the two invariants determined by Tresse [**40**]. They are third order invariants and may be regarded as "principal conformal curvatures."

Notice that from the above remarks, the 2-form $\mu^2 \mathrm{d}(area)$ is conformally invariant. This is precisely the Willmore integrand Ω. Indeed, $\mu^2 \mathrm{d}(area) = \{(\kappa_1 + \kappa_2)^2/4 - \kappa_1 \kappa_2\} \mathrm{d}(area) = \Omega$. If we let ω_1 and ω_2 denote the 1-forms dual to the vector fields ξ_1 and ξ_2 respectively, then ω_1 and ω_2 are also conformally invariant (up to a sign) and furthermore $\Omega = \omega_1 \wedge \omega_2$.

Now consider the mean curvature of the surface. It is clear that mean curvature is not in itself a conformal invariant. Nevertheless, it is associated with a certain invariance, as we will now explain. Let σ denote a sphere tangent at x to the surface M and suppose that the mean curvature of σ coincides with the mean curvature of M at x. Then it is clear from the above lemma that under the conformal transformation ϕ, the mean curvature of $\phi(\sigma)$ coincides with the mean curvature of $\phi(M)$ at $\phi(x)$. Thus, though mean curvature is itself not invariant, the osculating sphere that realizes the mean curvature is an invariant. We will return to this point in the next section.

Having derived the invariants $\theta_1, \theta_2, \xi_1, \xi_2$ and Ω in $\mathbf{E}^3$, one may wonder if they are invariants for surfaces in other 3-spaces. Now it is well known that the 3-spaces of constant curvature are all locally conformally equivalent (see for instance [**13**]). In particular, the natural embedding of the hyperbolic upper half space $\mathbf{H}^3$ into $\mathbf{E}^3$ is a conformal map, as is the stereographic projection of $\mathbf{E}^3$ into the round 3-sphere $\mathbf{S}^3$. Thus, one may imagine that the general conformal invariance (as formulated in the Introduction) follows directly from the invariance in $\mathbf{E}^3$. However, this appearance is deceptive, since one must verify that the formulas for the various invariants still hold when calculated with respect to the spherical or hyperbolic metric. In fact, rather more is true of the invariants presented above; not only are they invariant under conformal transformations of spaces of constant curvature, but they are invariant under conformal maps between spaces of arbitrary curvature. We omit the proof of this fact, which is a simple exercise in Riemannian geometry.

3. The Bryant map. In this section we continue our study of surfaces by now considering them as being embedded in $\mathbf{S}^3$. Here, the local conformal geometry reduces to the study of the group $M\ddot{o}b_3$ of general Möbius transformations of $\mathbf{S}^3$.

The group $M\ddot{o}b_3$ has a natural linear representation. Indeed, let $\mathbf{L}$ denote 5-dimensional Minkowski space; that is, $\mathbf{L}$ is the vector space $\mathbf{R}^5$ with basis vectors e_0, e_1, e_2, e_3, e_4 and pseudometric $\langle,\rangle$ defined by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_0 y_4 + x_4 y_0$$

where $x = (x_0, x_1, x_2, x_3, x_4)$ and $y = (y_0, y_1, y_2, y_3, y_4)$ are points in $\mathbf{L}$ expressed in coordinates with respect to the basis $\{e_0, e_1, e_2, e_3, e_4\}$. Now consider the 3-sphere $\mathbf{S}^3$ obtained by taking the intersection in $\mathbf{L}$ of the hyperplane $x_0 - x_4 = \sqrt{2}$ with the positive light cone

$$L^+ = \{x = (x_0, x_1, x_2, x_3, x_4) \in \mathbf{L} : \langle x, x \rangle = 0, \ x_0 - x_4 > 0\}.$$

There is clearly a bijection between $\mathbf{S}^3$ and the set $\mathbf{P}L^+$ of lines in L^+ emanating from the origin. Let $O(4,1)^+$ denote the group of linear transformations of $\mathbf{L}$ that respect $\langle,\rangle$ and preserve L^+. Thus every element of $O(4,1)^+$ defines a permutation of the elements of $\mathbf{P}L^+$. It is an elementary exercise to show that the induced bijections of $\mathbf{S}^3$ are conformal transformations and furthermore, every conformal transformation of $\mathbf{S}^3$ arises in this way. The group $O(4,1)^+$ can thus be identified with the group $M\ddot{o}b_3$ of conformal transformations of $\mathbf{S}^3$.

Now let Σ denote the set of round two spheres in $\mathbf{S}^3$. Clearly, every conformal transformation g of $\mathbf{S}^3$ induces a permutation $\bar{g}$ of the elements of Σ. In fact there is a natural map $\chi: \Sigma \to \mathbf{L}$ which is equivariant with respect to this action of $O(4,1)^+$; that is, $\chi \circ \bar{g} = g \circ \chi$ for all $g \in O(4,1)^+$. The map χ is constructed as follows. Let σ be an element of Σ and let Q denote the unique hyperplane through the origin of $\mathbf{L}$ whose intersection with $\mathbf{S}^3$ is σ. Then $\chi(\sigma)$ is defined to be one of the unit vectors $\langle,\rangle$-perpendicular to Q (see Figure 1). Of course, χ is only defined up to a sign. We will return to this problem shortly.

We now derive a formula for χ. Consider a round 2-sphere σ in Σ and let p be a point on σ. Let $\zeta = (e_0 - e_4)/\sqrt{2}$ and let $p = x + \zeta$ so that x lies in the Euclidean subspace $\mathbf{E}^4$ of $\mathbf{L}$ spanned by e_1, e_2, e_3 and $e_0 + e_4$. Now let n be a unit normal vector to σ in $\mathbf{S}^3$ and let H denote the curvature of σ as a submanifold of $\mathbf{S}^3$. Finally let θ denote the angle between the vector x and the line joining the center c of σ to the point ζ (see Figure 2). A simple calculation shows that $H = \cot\theta$. The center c of σ is clearly given by
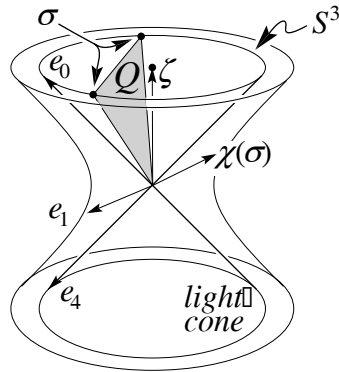
$$c = (x + n\tan\theta)\cos^2\theta + \zeta.$$

FIGURE 1.

Consequently

$$c = (Hx + n)\frac{H}{1 + H^2} + \zeta.$$

By definition, the vector $\chi(\sigma)$ is of unit length and is perpendicular to the vectors c and $p = x + \zeta$ and to the tangent space of σ at x in $\mathbf{S}^3$. It is clear that

(3.1) $\chi(\sigma) = n + H(x + \zeta)$

satisfies each of these properties and hence is the desired formula. The ambiguity in the sign of $\chi(\sigma)$ is clearly reflected in this formula in the choice of direction for n (which in turn determines the sign of H). Since we are interested in local geometry, we choose a local continuous choice of vector n, thus removing the ambiguity in the definition of χ.

The function χ can be regarded as a coordinatization of the set Σ of 2-spheres. This is the meaning of the 'pentaspherical coordinates' used by Takasu, Thomsen and Vessiot. Notice that the product $\langle,\rangle$ has a natural interpretation here. Indeed, by Equation (3.1), for two spheres σ_1 and σ_2, the number $\langle\chi(\sigma_1), \chi(\sigma_2)\rangle$ is just the cosine of the angle of intersection of the two spheres.

We now use χ to construct conformal invariants for the surface M embedded in $\mathbf{S}^3$. Let p be a point in M and let n be a continuous choice of unit vector field normal to M in $\mathbf{S}^3$, in some neighborhood of p. Now let σ be the unique 2-sphere which is tangent to M at p,
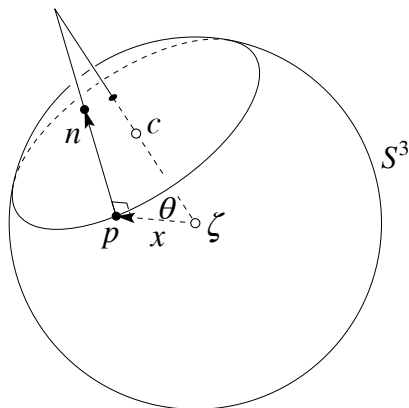
FIGURE 2.

which has the same mean curvature as the mean curvature H of M at p (taking n to be its normal vector at p). The map $\beta : M \ni p \mapsto \chi(\sigma)$ is Bryant's *conformal Gauss map* [**5**]. Since χ is equivariant, so too is β. Now suppose that M is umbilic-free and let X_1 and X_2 be orthonormal principal vector fields on M, corresponding to the principal curvatures κ_1 and κ_2 respectively. As we remarked in the previous section, the vector fields $\xi_1 = X_1/\mu$ and $\xi_2 = X_2/\mu$ are conformally invariant (up to a sign), where $\mu = (\kappa_1 - \kappa_2)/2$. It follows that the **L**-valued functions $\xi_1(\beta), \xi_2(\beta), \xi_1(\xi_1(\beta)), \xi_1(\xi_2(\beta))$, etc., are also equivariant (up to a sign), where by $\xi_1(\beta)$ for example, we mean the directional derivative of β with respect to ξ_1. Now by definition, one has

$$X_1(n) = -\kappa_1 X_1 \quad \text{and} \quad X_2(n) = -\kappa_2 X_2$$

and of course

$$X_1(x) = X_1 \quad \text{and} \quad X_2(x) = X_2.$$

Consequently, using Equation (3.1), one has

$$\xi_1(\beta) = \frac{X_1}{\mu}(\beta) = \frac{X_1}{\mu}(n + H(x + \zeta))$$

$$= -\frac{\kappa_1}{\mu}X_1 + \frac{X_1(H)}{\mu}(x + \zeta) + \frac{H}{\mu}X_1$$

$$= -X_1 + \frac{X_1(H)}{\mu}(x + \zeta)$$

$$= -\mu\xi_1 + \xi_1(H)(x + \zeta).$$

One can also use Equations (3.1) and (2.1) to calculate the higher derivatives. For instance,

(3.2)
$$\begin{cases} \xi_2(\beta) = \mu\xi_2 + \xi_2(H)(x + \zeta), \\ \xi_1(\xi_1(\beta)) = -\frac{\kappa_1}{\mu}n - \frac{\xi_2(\kappa_1)}{2}\xi_2 \\ \qquad + \frac{x}{\mu} + \xi_1(\xi_1(H))(x + \zeta) + \xi_1(H)\xi_1, \\ \xi_2(\xi_2(\beta)) = \frac{\kappa_2}{\mu}n - \frac{\xi_1(\kappa_2)}{2}\xi_1 - \frac{x}{\mu} \\ \qquad + \xi_2(\xi_2(H))(x + \zeta) + \xi_2(H)\xi_2. \end{cases}$$

One can now obtain invariant functions by evaluating these equivariant functions on the pseudometric $\langle,\rangle$. Of course, many of these functions will be uninteresting; for instance $\langle\xi_1(\beta), \xi_2(\beta)\rangle = 0$. Nevertheless, one does obtain nontrivial invariants. For example,

$$\langle\xi_1(\xi_1(\beta)), \xi_2(\beta)\rangle = \theta_2/2$$
$$\langle\xi_2(\xi_2(\beta)), \xi_1(\beta)\rangle = -\theta_1/2,$$

which are the third order invariants constructed in the previous section (see Equation (2.1)). In addition, one has a new invariant of order 4:

(3.3)
$$\frac{1}{2}\{\langle\xi_1(\xi_1(\beta)), \xi_1(\xi_1(\beta))\rangle - \langle\xi_2(\xi_2(\beta)), \xi_2(\xi_2(\beta))\rangle$$
$$- \langle\xi_1(\xi_1(\beta)), \xi_2(\beta)\rangle^2 + \langle\xi_2(\xi_2(\beta)), \xi_1(\beta)\rangle^2\}$$
$$= \frac{\Delta H + 2\mu^2 H}{\mu^3},$$

where Δ is the Laplacian in the metric induced on M from the ambient space. This invariant is not entirely new however. The equation $\Delta H + 2\mu^2 H = 0$ is just the Euler-Lagrange equation for the Willmore functional [5] and surfaces for which it is zero are known as *Willmore surfaces.* It is well known that the Willmore integral is conformally invariant and so it is not surprising that the function $(\Delta H + 2\mu^2 H)/\mu^3$ is a conformal invariant. It is perhaps more surprising that for arbitrary umbilic-free surfaces in 3-manifolds of constant curvature, the functions θ_1, θ_2 and $(\Delta H + 2\mu^2 H)/\mu^3$ are essentially the only invariants; that is, all other invariant functions can be derived from them and their derivatives. We will return to this matter in Section 5. Let us remark that the above calculation only establishes the invariance of the function $(\Delta H + 2\mu^2 H)/\mu^3$ for conformal transformations of $\mathbf{S}^3$. Its more general conformal invariance, in the sense described in the Introduction, can be explicitly verified by considering a conformal change of metric to another metric of constant curvature (cf. the discussion at the end of Section 4).

4. The method of reduced equations. Perhaps the most direct way of constructing conformal invariants is by means of the *method of reduced equations.* This was the method employed by Tresse in his derivation of the third order invariants for surfaces [41]. To apply it we do the following. First rescale the metric of the ambient manifold N so that it becomes locally Euclidean (flat). Since we only work locally, we may assume that $N = \mathbf{E}^3$. Now express the curve or surface as the zero set of a function $f : \mathbf{E}^3 \to \mathbf{E}^{3-m}$, where m equals 2 for surfaces and 1 for curves. If p is point in the submanifold under consideration, then acting on the ambient space $\mathbf{E}^3$ by a certain conformal transformation we may move the point p to the origin in such a way as to reduce some low order jet of f to a simple normal form. This is the reduced equation we seek and the coefficients in the Taylor series of the reduced equation may be interpreted as conformally invariant functions. Of course all this depends on the convention one chooses for the meaning of "simple normal form."

Let us describe the situation for a surface M. Here is one choice of normal form for a surface in $\mathbf{E}^3$ at a non-umbilic point.

$$z = \frac{1}{2}(x^2 - y^2) + O(x^3, y^3, x^2 y^2)$$

The stabilizer in the conformal group of this normal form is of order two and acts by changing the signs of x and y. Calculating some of the higher coefficients in this normal form, which may be regarded as invariant functions on the surface, we obtain

$$
\begin{aligned}
z = {} & \frac{1}{2}(x^2 - y^2) + \frac{1}{6}(\theta_1 x^3 + \theta_2 y^3) \\
& + \frac{1}{24}(ax^4 + 4bx^3y + 6\Psi x^2 y^2 + 4cxy^3 + dy^4) + O(5),
\end{aligned}
$$

where

$$
\begin{aligned}
a &= 3 + \theta_1^2 + \xi_1(\theta_1), & b &= -\theta_1\theta_2 + \xi_2(\theta_1), \\
c &= \theta_1\theta_2 + \xi_1(\theta_2), & d &= -3 - \theta_2^2 + \xi_2(\theta_2),
\end{aligned}
$$

and

$$
\Psi = \frac{\Delta H + 2\mu^2 H}{\mu^3} + \frac{1}{2}\{\theta_1^2 - \theta_2^2 + \xi_1(\theta_1) + \xi_2(\theta_2)\}.
$$

The coefficients, which depend upon p, are consequently just functions of the 3 invariants derived in the previous section and their derivatives with respect to the invariant vector fields ξ_1 and ξ_2.

Notice once again that the functions θ_1 and θ_2 are only determined up to a sign, because of the indetermination of normal form. To control this annoyance, we introduce the following notion (cf. [37]).

Definition. A double orientation of an umbilic-free surface is an orientation of each of the principal direction fields.

Of course, for an arbitrary umbilic-free surface there may not be a double orientation. Nevertheless, locally at least, one always exists. Given such a double orientation (in which case we say that the surface is *doubly oriented*), the indetermination in the normal form can now be removed by demanding that the x and y axes be positively aligned with the principal direction fields ξ_1 and ξ_2 respectively. The functions θ_1 and θ_2 are then uniquely defined. In the case of an oriented manifold M the double orientation may be taken compatibly with the given orientation of M. Then there is a canonical 2-fold cover of M (with group ±1), the *double orientation cover*, whose fiber at a point p

consists of the two possible double orientations at p. It is then possible to carry out our constructions of the invariants canonically on this double cover. We denote by L the *double orientation line bundle* which is the line bundle over M associated to the double orientation cover by the non-trivial representation of ± 1 on $\mathbf{R}$.

Now consider an oriented curve γ in $\mathbf{E}^3$ and let s, κ and τ denote its arc-length, curvature and torsion respectively. In attempting to determine the normal form for γ, one finds that the following 1-form is conformally invariant:

$$\omega = \sqrt{\nu}\, ds, \quad \text{where } \nu = \sqrt{(\kappa')^2 + \kappa^2 \tau^2}, \quad \text{and } \kappa' = \frac{d\kappa}{ds}.$$

We call ω the *infinitesimal conformal arc-length* of γ. The zeros of ω are points of particular interest.

Definition. A vertex of γ is a point where the infinitesimal conformal arc-length ω of γ vanishes.

It should be remarked that there is another notion of vertex in the literature; namely, a point where the torsion vanishes (see for instance [**32**]). However, this is not an invariant notion, whereas the above definition of vertex is conformally invariant because of the invariance of ω. Moreover, the above definition is a clear generalization of the classical definition of vertex for planar curves.

Now, given a non-vertex point $p \in \gamma$ we may vary the curve by a conformal motion to obtain an equivalent curve in "standard position"; that is, the normal form of the curve γ at p. Here is the normal form for a curve in $\mathbf{E}^3$ at a non-vertex, the origin.

$$y = \frac{x^3}{3!} + O(x^5),$$
$$z = 0 + O(x^4),$$

where the x-axis is positively oriented with the direction of the curve. Calculating some of the higher coefficients in this normal form, we

obtain

$$y = \frac{x^3}{3!} + (2Q - T^2)\frac{x^5}{5!} + O(x^6),$$
$$z = T\frac{x^4}{4!} + \frac{1}{\sqrt{\nu}}\frac{dT}{ds}\frac{x^5}{5!} + O(x^6),$$

where

$$Q = \frac{4(\nu'' - (\kappa^2 + C)\nu)\nu - 5\nu'^2}{8\nu^3}$$

and

$$T = \frac{2\kappa'^2\tau + \kappa^2\tau^3 + \kappa\kappa'\tau' - \kappa\kappa''\tau}{\nu^{5/2}},$$

and where κ and τ are the curvature and torsion of the original curve γ and C is the sectional curvature of the ambient space (which, of course, for $\mathbf{E}^3$ is 0). We call Q the *conformal curvature* of γ and T the *conformal torsion*.

Note that although the expression for ω in terms of the Riemannian invariants breaks down at points where $\kappa = 0$, the conformal invariance of ω allows us to move the curve by a conformal transformation to a new curve for which $\kappa \neq 0$ so that ω is in fact defined at every point of γ. Similar remarks apply to Q and T showing that they are defined at every point where ν does not vanish. Thus a generic point on a curve is one for which $\nu \neq 0$. We note that by strong transversality [**1**] the curves in space for which every point is generic constitute an open dense subset of all smooth curves in the C^∞ topology and the analogous statements hold for closed curves. This contrasts with the situation for plane curves for which the "honest" vertices (cf. [**30, 27**]) are stable phenomena.

We now turn to the question of proving invariance. The above calculations show that for curves in $\mathbf{E}^3$, the 1-form ω and the functions Q and T are invariant under Möbius transformations, as are the vector fields ξ_1 and ξ_2 and the functions θ_1, θ_2 and Ψ for surfaces in $\mathbf{E}^3$. We must also show that the formulas for the various invariants still hold when calculated with respect to the spherical or hyperbolic metric. Let us indicate how one may proceed for the case of a curve γ, the case of surfaces being quite analogous.

The invariants ω, Q and T of γ are functions of the curvature and torsion of γ and their derivatives up to third order. We need to show that one obtains the same values for the invariants when one uses a constant curvature metric $\rho g = \rho(dx^2 + dy^2 + dz^2)$, where ρ is some positive function, as one does when one uses the flat metric $g = dx^2 + dy^2 + dz^2$. The curvature and torsion of γ with respect to the metric ρg can be easily calculated in terms of the Euclidean curvature and torsion of γ and the function ρ and its derivatives in the tangent, normal and bi-normal directions of γ (with respect to the Euclidean metric). One is thus required to establish three identities (one for each invariant) involving the function ρ, the Euclidean curvature and torsion of γ and various derivatives of these three functions. These identities can be established by calculating the curvature tensor for ρg in the moving frame basis (the Serret-Frenet frame) and applying the constant curvature hypothesis. We omit the details of these calculations which are somewhat involved.

5. Geometric forms and generating sets. In this section we see how the invariants introduced in the previous sections generate the entire algebra of invariants. In order to do this we must formalize the notion of 'generation.' Before doing this, however, it is perhaps instructive to see that the invariants actually determine the submanifold (up to conformal transformation). To see this, let us first fix some notation. Let M denote a generic 1 or 2 dimensional submanifold of $\mathbf{E}^3$; that is, a vertex-free curve or an umbilic-free surface. In either case, for each point in M there is a Möbius transformation g^{-1} that moves M into normal form, as described in Section 4. So, locally at least, we have a map $g\colon M \to M\ddot{o}b_3$, which we call the *canonical map* of M. (If we do the same construction on the double orientation cover we obtain a globally defined analogue satisfying $g(\sigma(p)) = g(p)\tau$ where σ is the covering transformation of the double orientation cover and $\tau = \operatorname{diag}(1, -1, -1, 1, 1) \in O(4, 1)^+$.) Notice that the canonical map completely describes M; indeed, for each point $x \in M$, the position of x in $\mathbf{E}^3$ is given by the image of the origin $O \in \mathbf{E}^3$ under the map $g(x)$. The 1-form $g^{-1}dg$ is a well defined 1-form on M with values in the Lie algebra of $M\ddot{o}b_3$. In order to present this 1-form we need a description of the group $M\ddot{o}b_3$ of Möbius transformations of $\mathbf{E}^3$. We saw in Section 3 how $M\ddot{o}b_3$ may be regarded as the group of positive orthogonal trans-

formations of Minkowski space $\mathbf{L}$. The relationship with $\mathbf{E}^3$ is obtained by identifying $\mathbf{E}^3 \cup \{\infty\}$ with the set $\mathbf{P}L^+$ of lines in the positive light cone L^+ of $\mathbf{L}$. To do this we choose the map

$$
\begin{aligned}
\mathbb{E}^3 \cup \{\infty\} &\longrightarrow \mathbf{P}L^+ \\
(x, y, z) &\longmapsto \text{span of } (1, x, y, z, -\frac{1}{2}(x^2 + y^2 + z^2)) \\
\infty &\longmapsto \text{span of } (0, 0, 0, 0, 1).
\end{aligned}
$$

This being understood, one has the following results.

Theorem 5.1. *Let γ be an oriented connected vertex-free curve in $\mathbf{E}^3$. Then the canonical map $g \colon \gamma \to M\ddot{o}b_3$ satisfies*

$$
g^{-1}dg = \begin{pmatrix} 0 & Q & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -Q \\ 0 & 0 & 0 & -T & -1 \\ 0 & 0 & T & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} \omega
$$

where Q, T and ω are as in Section 4.

Corollary 5.2   (cf. [**15**, Theorem 7.2]).   *An oriented connected, vertex-free curve γ is determined up to conformal motion by its invariants Q, T and ω; that is, if $\gamma_1, \gamma_2 : (\alpha, \beta) \to \mathbf{E}^3$ are two oriented vertex-free curves with the same invariants Q, T and ω, then there exists a conformal transformation ϕ of $\mathbf{E}^3 \cup \{\infty\}$ such that $\gamma_1 = \phi \circ \gamma_2$. Conversely, given any pair of smooth functions $Q, T \colon (\alpha, \beta) \to \mathbf{R}$ and given any nowhere zero 1-form ω on (α, β), they are the conformal curvature, conformal torsion and infinitesimal conformal arclength of a curve $\gamma \colon (\alpha, \beta) \to \mathbf{E}^3$.*

Theorem 5.3. *Let M be a doubly oriented umbilic-free surface in*

$\mathbf{E}^3$. *Then the canonical map* $g\colon M \to M\ddot{o}b_3$ *satisfies*

$$g^{-1}dg = \begin{pmatrix} \frac{1}{2}\theta_1 & -\frac{1}{2}(1+\Psi) & \frac{1}{2}b & \frac{1}{2}\theta_1 & 0 \\ 1 & 0 & 0 & -1 & \frac{1}{2}(\Psi+1) \\ 0 & 0 & 0 & 0 & -\frac{1}{2}b \\ 0 & 1 & 0 & 0 & -\frac{1}{2}\theta_1 \\ 0 & -1 & 0 & 0 & -\frac{1}{2}\theta_1 \end{pmatrix} \omega_1$$

$$+ \begin{pmatrix} -\frac{1}{2}\theta_2 & -\frac{1}{2}c & -\frac{1}{2}(1-\Psi) & \frac{1}{2}\theta_2 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}c \\ 1 & 0 & 0 & 1 & \frac{1}{2}(1-\Psi) \\ 0 & 0 & -1 & 0 & -\frac{1}{2}\theta_2 \\ 0 & 0 & -1 & 0 & \frac{1}{2}\theta_2 \end{pmatrix} \omega_2$$

where $\theta_1, \theta_2, \Psi, b$ *and* c *are as in Section 4 and* ω_1 *and* ω_2 *are the invariant 1-forms dual to* ξ_1 *and* ξ_2 *respectively.*

It is not difficult to verify that the 1-form $\omega = g^{-1}dg$, given by Theorem 5.3, satisfies the equation $d\omega + 1/2[\omega,\omega] = 0$. This may be regarded as the conformal Gauss-Codazzi equations for the surface M. (The same is true for the analogous form $\bar{\omega}$ on the double orientation cover. Moreover $\bar{\omega}$ satisfies the identity $\sigma^*\bar{\omega} = ad(\tau^{-1})\bar{\omega}$, and thus induces a 1-form ω on M with values in the flat bundle $L \otimes o(4,1)^+$ over M (with group ± 1) whose fiber is the Lie algebra $o(4,1)^+$. This version of ω is the global incarnation of the previous ω in the sense that they will agree on any connected open set where the local version is defined, provided they agree at one point. It is clear that if we integrate $\bar{\omega}$ along a loop based at the origin which goes once around an umbilic then we must obtain τ or the identity).

Corollary 5.4 (cf. [**16**, Theorem 31.1]). *A doubly oriented, connected, umbilic-free surface M is determined up to conformal motion by its invariants* $\omega_1, \omega_2, \theta_1, \theta_2, \Psi$*; that is, if M_1 and M_2 are submanifolds of $\mathbf{E}^3$ and $f\colon M_1 \to M_2$ is a diffeomorphism identifying the corresponding invariants, then f is the restriction of a conformal transformation of $\mathbf{E}^3 \cup \{\infty\}$. Conversely, if one is given a 1-connected domain $U \subset \mathbf{R}^2$, two independent 1-forms ω_1 and ω_2 on U and any triple of smooth functions*

$$\theta_1, \theta_2, \Psi \colon U \to \mathbf{R},$$

such that the corresponding form ω given by the formula in Theorem 5.3 satisfies the structural equation $d\omega + (1/2)[\omega, \omega] = 0$, then there is an immersed surface in $\mathbf{E}^3$ realizing these invariants. (More generally, let M be an oriented surface and let L be a real line bundle over M with group ± 1. Suppose that Ψ is a smooth function on M, θ_1, θ_2 are smooth sections of L, and ω_1, ω_2 are smooth 1-forms on M with values in L, such that the corresponding form ω on M with values in $L \otimes o(4,1)^+$ satisfies the structural equation $d\omega + (1/2)[\omega, \omega] = 0$ and the integral around any loop on M (see below) lies in $\{1, \tau\}$, then there is an immersed surface in $\mathbf{E}^3$ realizing these invariants).

Theorems 5.1 and 5.3 are just calculations. Corollaries 5.2 and 5.4 are consequences of the following result of Lie's: if G is a Lie group with Lie algebra $\mathbf{g}$ and if ω is a $\mathbf{g}$-valued 1-form on some 1-connected manifold U such that $d\omega + (1/2)[\omega, \omega] = 0$, then there exists a map $\phi \colon U \to G$, unique up to left translation by an element of G, such that ω is the pull-back by ϕ of the Maurer-Cartan form on G (see [**24**]). This also explains the meaning of "integration of an $L \otimes o(4,1)^+$ valued 1-form along a path in M."

Having seen that the invariants of Section 4 determine the curves and surfaces, we now turn to the existence of other invariants. Much of the following discussion is valid in arbitrary dimension and so for the moment we let N be an n-dimensional Riemannian manifold of constant sectional curvature and we let M be an m-dimensional embedded submanifold of N. As described in the introduction, a conformally invariant form on M is a differential form on M (determined by the embedding $M \hookrightarrow N$ and the Riemannian geometry of N) that is left unaltered by conformal change of metric on N to any other metric of constant curvature. In the remaining part of this section we require the following definition.

Definition. A conformal geometric p-form (or geometric form, for short) of dimension m $(0 \leq p \leq m)$ is an assignment ω which associates to each *pair* (N, M), where M is an m-dimensional submanifold of N, a p-form $\omega(N, M)$, defined at "generic" points of M, which is smooth, invariant and local in the sense that it satisfies the following three properties:

(i) $\omega(N, M)$ *varies smoothly under smooth variation of* M,

(ii) $\omega(N, M)$ *is invariant under conformal change of the metric on* N *as described above,*

(iii) *The value of* $\omega(N, M)$ *at* $x \in M$ *depends only on the* r*-jet of* M *in* N *at* x *for some* $r < \infty$.

We may similarly define conformal geometric p-vector fields or, more generally, geometric tensors of mixed type. The meaning of "generic" in the above definition depends on the case at hand; for curves it means vertex-free, while for surfaces it means umbilic-free.

Now we discuss the notion of a *generating set* of conformal geometric forms of dimension m. Suppose we have a collection of geometric 1-forms $\omega_1, \omega_2, \ldots, \omega_m$ such that for each pair (N, M) the 1-forms $\omega_k(N, M)$, where $1 \leq k \leq m$, form a basis for the space of 1-forms at generic points of M. We call such a collection *a basis of geometric 1-forms*. By multiplication these forms yield geometric p-forms at generic points of M, in terms of which an arbitrary geometric p-form can be expressed as a linear combination with coefficients which are geometric functions. Thus if we are given a basis of geometric 1-forms (or dually, a basis of geometric vector fields) in order to describe all geometric forms and fields it suffices to describe all geometric functions.

Let f be a geometric function. Then df is a geometric 1-form and can be expressed as

$$df = \sum_{j=1}^{m} f_j \omega_j$$

where the f_j's are also geometric functions, the partial derivatives of f. Now let S be any collection of geometric functions. The collection consisting of S and all the successive derivatives of elements of S generate an algebra $A \supset S$. Finally we enlarge A to obtain the algebra $\langle S \rangle \supset A \supset S$ consisting of all the composites of the form $F(a_1, a_2, \ldots, a_r)$ where $F \colon \mathbf{R}^r \to \mathbf{R}$ is a smooth function and $a_1, a_2, \ldots, a_r \in A$. We call $\langle S \rangle$ the algebra of geometric functions generated by S. It is clear that $\langle S \rangle$ is independent of the particular basis of geometric 1-forms used to define the derivatives (always providing that such a basis exists).

Definition.   A generating set for the geometric forms and fields consists of

(i) A basis ω_k, $1 \leq k \leq M$ of geometric 1-forms or vector fields,

and

(ii)   A set S of geometric functions such that $\langle S \rangle$ consists of all geometric functions.

Now we are in a position to state the following results.

Theorem 5.5   (cf. [**15**, Theorem 9.1]).   *The geometric invariants* ω, Q *and* T *constitute a generating set for the geometric forms and fields for oriented vertex-free curves in* 3-*dimensional space forms.*

Theorem 5.6   (cf. [**16**, Theorem 32.1]).   *The geometric invariants* ξ_1, ξ_2, θ_1, θ_2, Ψ *constitute a generating set for the geometric forms and fields for doubly oriented umbilic-free surfaces in* 3-*dimensional space forms.*

We omit the proofs of these theorems. They are easily established by induction, using the results of Section 4 (cf. [**6**]).

6. Some remarks on the invariants for curves.   Consider a curve γ in some 3-space of constant curvature. It follows from the definition in Section 4 of *vertex* that the vertices of γ are invariant under conformal transformation. It is also obvious that if γ lies in the Euclidean plane, then the definition of vertex coincides with the classical one (that is, a stationary point of the curvature). Now by the classical 4-vertex theorem, every simple closed planar curve has at least four vertices (see [**6**] for a simple proof). It follows immediately from the above remarks that every simple closed spherical curve γ in $\mathbf{E}^3$ has at least four vertices; indeed, if γ lies in a sphere S, then by inversion in a sphere centered at some point in $S \backslash \gamma$, the curve γ becomes planar and one can apply the classical theorem. More generally, if γ is a curve in a 3-manifold N of constant curvature, then we will say that γ is *spherical* if γ lies in a totally umbilic surface in N. By passage to the universal covering space of N and from there to Euclidean space, we

obtain the following result (cf. **6**]).

Theorem 6.1. *Let γ be a simple closed null-homotopic curve in a 3-manifold N of constant curvature. If γ is spherical, then γ has at least 4 vertices.*

Let us now turn to the conformal torsion,

$$T = \frac{2\kappa'^2\tau + \kappa^2\tau^3 + \kappa\kappa'\tau' - \kappa\kappa''\tau}{\nu^{5/2}},$$

of a curve γ in $\mathbf{E}^3$. Let us suppose for the moment that γ has non-zero curvature κ and non-zero torsion τ. Then one has

$$\frac{\nu^{5/2}}{\tau^2\kappa^3}T = \left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]' + \frac{\tau}{\kappa}.$$

This expression is no doubt more familiar to the reader; as is well-known, it equals zero precisely when γ is spherical (see [**49**]). Consequently T is zero if and only if γ is spherical. Furthermore, our assumption that κ and τ are non-zero can be weakened. Indeed, as long as γ is vertex-free, one may always move γ by a conformal transformation so that the curvature and torsion of the image curve are both locally non-zero. So one has the following result.

Theorem 6.2. *Let γ be a vertex-free curve in a 3-manifold N of constant curvature. Then γ is spherical if and only if its conformal torsion T vanishes identically.*

Now consider a closed curve γ in $\mathbf{E}^3$ and suppose that γ is regular; that is, its curvature κ is nowhere zero (and hence its torsion τ is defined at all points). Then, according to Scherrer's Theorem (see [**49**]), $\int \tau \mathrm{d}s = 0$ if γ is spherical. In fact, Banchoff and White [**2**] have shown that the residue $(1/2\pi)\int \tau ds$ mod 1, the "total twist," is a conformal invariant of the curve γ (see also [**9**]). We will show that for generic γ the total twist is actually the reduction mod 1 of the $\mathbf{R}$ valued conformal invariant $(1/2\pi)\int_\gamma T\omega$.

Theorem 6.3. *Let γ be a regular closed vertex-free curve in a 3-manifold N of constant curvature. Then*

$$\frac{1}{2\pi} \int_\gamma T\omega \equiv \frac{1}{2\pi} \int_\gamma \tau \, ds \qquad (\mathrm{mod}\ 1)$$

where $\omega \ (= \sqrt{\nu}\, ds = \sqrt[4]{(\kappa')^2 + \kappa^2 \tau^2}\, ds)$ is the conformal arc-length.

Proof. First note that from the definition of T one has

$$T\omega = \tau \, ds + \frac{\kappa'^2 \tau + \kappa \kappa' \tau' - \kappa \kappa'' \tau}{\nu^2} \, ds.$$

Now let $\mathbf{n}$ denote the unit normal vector field to γ, let $\mathbf{b}$ denote its unit binormal vector field and let α denote the angle between $\mathbf{n}$ and $\kappa' \mathbf{n} + \kappa \tau \mathbf{b}$ (this latter vector field is nowhere zero because γ is vertex-free). Then a simple calculation shows that

$$T\,\omega = \tau \, ds + d\alpha.$$

This gives the required result by integration around γ. $\quad\square$

7. Some remarks on the invariants for surfaces. Consider a doubly oriented umbilic-free surface M in a 3-space N of constant curvature. As in the previous paragraphs, let κ_1 and κ_2 denote the principal curvatures of M, with $\kappa_1 > \kappa_2$, let X_1 and X_2 be the corresponding orthonormal principal vector fields, let $\mu = (\kappa_1 - \kappa_2)/2$ and let $\xi_1 = X_1/\mu$ and $\xi_2 = X_2/\mu$. In this section we make 5 remarks.

(1) Since the vector fields ξ_1 and ξ_2 are conformally invariant, they may be regarded as orthonormal vector fields for a *conformally invariant Riemannian metric* on M (which is conformally equivalent to the original metric inherited from N). The Gaussian curvature of M with respect to this conformally invariant metric is consequently a natural conformal invariant of M; we denote it K_c and call it the *conformal curvature* of M. This invariant was first introduced in [**12**]. In view of the results of Section 5, we must be able to express K_c in terms of the elementary invariants θ_1, θ_2, Ψ, ξ_1 and ξ_2. Indeed, it is a simple exercise to show that

(7.1) $$4K_c = 2\xi_2(\theta_2) - 2\xi_1(\theta_1) - \theta_1^2 - \theta_2^2.$$

It can be re-expressed as

(7.2)
$$K_c = \frac{K - \Delta \log(\mu)}{\mu^2}$$

where K is the Gaussian curvature of M with respect to the metric induced on M from N and Δ is the Laplacian on M with respect to the same metric. When N is the round sphere $\mathbf{S}^3$, one can also express K_c in terms of the $\mathbf{L}$-valued functions of Section 3. Indeed, one calculates easily that

$$\frac{1}{2}\{\langle \xi_1(\xi_1(\beta)), \xi_1(\xi_1(\beta)) \rangle + \langle \xi_2(\xi_2(\beta)), \xi_2(\xi_2(\beta)) \rangle$$
$$-\langle \xi_1(\xi_1(\beta)), \xi_2(\beta) \rangle^2 - \langle \xi_2(\xi_2(\beta)), \xi_1(\beta) \rangle^2\}$$
$$= 2 - K_c,$$

which should be compared to Equation (3.3). One may equally verify that

$$K_c = \langle \xi_1(\xi_1(\beta)), \xi_2(\xi_2(\beta)) \rangle.$$

(2) Instead of using the Laplacian Δ on M, one may equally well use the Laplacian Δ_c determined by the conformally invariant Riemannian metric on M. For any function f on M one has $\Delta f = \mu^2 \Delta_c f$. Thus for instance, the Willmore surfaces, which are the surfaces for which $\Delta H + 2\mu^2 H = 0$, are equally characterized by the equation

$$\Delta_c H = -2H.$$

It is also of interest to calculate the Laplacian of the conformal Gauss map β of M (see Section 3). A straightforward calculation, using Equations (3.2), shows that M is a Willmore surface if and only if one has (cf. [**29**]):
$$\Delta_c \beta = -2\beta.$$

(3) Now consider the case where M is an umbilic-free embedded torus in $\mathbf{S}^3$. The Willmore conjecture states that $\int_M \mu^2 \mathrm{d}(area) \geq 2\pi^2$. The value of $2\pi^2$ is attained by the *Clifford torus*: this is the minimally embedded torus in $\mathbf{S}^3$ obtained by taking the pre-image under the Hopf map of the equator of $\mathbf{S}^2$. It is known that the infimum of the Willmore

integral $\int_M \mu^2 \mathrm{d}(area)$, over all embedded tori in $\mathbf{S}^3$, is attained by some torus and that this torus is a Willmore surface [**34**]. A strong version of the Willmore conjecture is therefore: *Willmore surfaces with minimal Willmore integral can be mapped, by conformal transformation of $\mathbf{S}^3$, to the Clifford torus.*

(4) It is well known that up to an isometric motion of $\mathbf{S}^3$, an embedded torus in $\mathbf{S}^3$ is the Clifford torus if and only if its principal curvatures κ_1 and κ_2 are ± 1 respectively (see [**10**]). This is equivalent to the condition $\mu \equiv 1$. Indeed, one implication is trivial, whilst if $\mu \equiv 1$, then one has

$$
\begin{aligned}
\int_M H^2 \mathrm{d}(area) &= \int_M (\mu^2 + \kappa_1\kappa_2)\mathrm{d}(area) \\
&= \int_M (\mu^2 + K - 1)\mathrm{d}(area) \\
&= \int_M K\mathrm{d}(area) = 0,
\end{aligned}
$$

using the Gauss-Bonnet Theorem. Hence $H \equiv 0$ and thus $\kappa_1 = -\kappa_2 = 1$.

One would however like to have a condition for a surface to be the Clifford torus, up to *conformal* transformation of $\mathbf{S}^3$. To this end, consider a torus of revolution M in $\mathbf{E}^3$; that is, an anchor ring. Let the two radii of M be denoted r and R. It was known by Vessiot ([**44**, see also **36**]) that up to a conformal transformation, the anchor rings are precisely the tori for which the two fundamental invariants θ_1 and θ_2 are zero. Moreover, it is well known that the Clifford torus is the image under the stereographic projection of the anchor ring with $R = \sqrt{2}$ and $r = 1$ (see [**46**]). On the other hand, a direct calculation shows that for a general anchor ring,

$$
\frac{\Delta H + 2\mu^2 H}{\mu^3} = \frac{2 - (R/r)^2}{4}.
$$

Combining these observations, one has the following result:

Proposition 7.1. *Let M be an embedded umbilic-free torus in $\mathbf{S}^3$. Then M is a Willmore torus if the invariant $(\Delta H + 2\mu^2 H)/\mu^3$*

is zero and M can be mapped by a conformal transformation of $\mathbf{S}^3$ to the Clifford torus if and only if all three invariants θ_1, θ_2 and $(\Delta H + 2\mu^2 H)/\mu^3$ are zero.

In particular, the Clifford torus and its conformal images are the simplest conformal surfaces after the round spheres (the latter are regarded as completely degenerate in conformal geometry since every point is umbilic).

(5) In order to establish the Willmore conjecture for umbilic-free tori, it suffices, in view of the above Proposition, to show that an embedded torus, with $(\Delta H + 2\mu^2 H)/\mu^3 = 0$ and minimal Willmore integral, necessarily has $\theta_1 = \theta_2 = 0$. Notice that in fact it is only necessary to show that θ_1 and θ_2 are constant. Indeed, by equation (7.1), if θ_1 and θ_2 are constant, then integrating over M and applying the Gauss-Bonnet Theorem, one has

$$0 = \int_M (\theta_1^2 + \theta_2^2)$$

and so $\theta_1 = \theta_2 = 0$.

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DEPARTMENT OF MATHEMATICS, LA TROBE UNIVERSITY, MELBOURNE, AUSTRALIA 3083

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, CANADA M5S 1A1

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MELBOURNE, MELBOURNE, AUSTRALIA 3052