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ABELIAN GROUPS WITH SEMI-SIMPLE ARTINIAN QUASI-ENDOMORPHISM RINGS

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1. Introduction. In 1937, Baer published a paper which can still be considered as probably the most important contribution to the theory of torsion-free abelian groups [**8**]. In it, he gave a complete set of numerical invariants for the subgroups of the rational numbers, $\mathbf{Q}$, and their direct sums. Particular attention was given to homogeneous completely decomposable groups, which are groups of the form $\oplus_I A$ for some subgroup A of $\mathbf{Q}$. Baer's investigations resulted in a series of splitting properties for these groups, which have become important tools in the discussion of torsion-free abelian groups of finite rank [**10**, Section 86].

Because of this, the question arose whether it is possible to replace the subgroup A of $\mathbf{Q}$ in the definition of a homogeneous completely decomposable group by a more general group without losing the previously mentioned splitting properties. Although it soon became apparent that some restrictions on A are needed, it was not until 1975 that Arnold and Lady showed that the most natural way to introduce these restrictions is in terms of the endomorphism ring, $E(A)$ of A.

Following their approach, we consider a torsion-free abelian group A and call a group P *A-projective of finite A-rank* if P is a direct summand of A^n for some $n < \omega$. If G is an abelian group, then $S_A(G) = \langle f(A) \mid f \in \mathrm{Hom}\,(A, G)\rangle$ is the *A-socle of G*, while $R_A(G) = \cap\{\ker f \mid f \in \mathrm{Hom}\,(G, A)\}$ denotes the *A-radical of G*.

[**6,2**] and [**3**] were mainly concerned with the splitting and quasi-splitting of exact sequences of the form $0 \to P \overset{\alpha}{\to} G \overset{\beta}{\to} B \to 0$ and $0 \to B \overset{\alpha}{\to} G \overset{\beta}{\to} P \to 0$ in which P is a quasi-summand of an A-projective group of finite A-rank. We say that A has the *radical-splitting property* if every such sequence $0 \to P \overset{\alpha}{\to} G$ with $\alpha(P) \cap R_A(G) = 0$ quasi-splits, while A has the *finite quasi-Baer-*

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splitting property if every sequence of the form $0 \to B \overset{\alpha}{\to} G \overset{\beta}{\to} P \to 0$ with $\alpha(B) + S_A(G) \doteq G$ quasi-splits.

However, these efforts failed to capture one of the most important properties of homogeneous completely decomposable groups of finite rank, namely that their pure subgroups are direct summands [**10**, Lemma 86.8]. The author and Goeters attempted to recover this property in [**4**] but the obtained results were not completely satisfactory. Several of the difficulties encountered in [**4**] can be avoided if we follow the approach taken in [**2**] and [**3**], and consider the following quasi-version of [**10**, Lemma 86.8]: We say that a torsion-free group A has the *socle splitting property* if every exact sequence $0 \to B \overset{\alpha}{\to} A^n$ with $n < \omega$ and $S_A(B) \doteq B$ quasi-splits. A torsion-free group of finite rank with the socle splitting property clearly has the radical splitting property, too. However, the converse fails in general.

In Section 2, we investigate the socle splitting property in conjunction with the finite quasi-Baer-splitting property. Theorem 2.3 yields that the socle splitting- and the finite quasi-Baer-splitting property for A, together, imply that the quasi-endomorphism ring, $\mathbf{Q}E(A)$, of A is semi-simple Artinian, provided A is self-small, i.e., $\mathrm{Hom}\,(A, -)$ preserves direct sums of copies of A (for details on self-small groups, see [**7**]). Because of Corollary 2.4, a strongly indecomposable group A of finite rank has the socle-splitting property if and only if $\mathbf{Q}E(A)$ is a division algebra. In contrast, every strongly indecomposable group A has the radical- and the finite quasi-Baer splitting property.

The results of Section 2 raise the immediate question of whether there are groups with the socle splitting property which do not have the quasi-Baer splitting property. Such groups are constructed as a consequence of Theorem 3.1, which completely determines the structure of the torsion-free groups of finite rank which have the socle-splitting property. This structure result is used in Corollary 3.3 to construct a group A of rank 3 which has the socle-splitting property, but whose quasi-endomorphism ring is not semi-simple Artinian. In particular, this group A does not have the finite quasi-Baer-splitting property.

As an application of the results of this paper, we discuss the relation between the class of quasi-summands of A-projective groups of finite A-rank and the class of groups which are quasi-isomorphic to A-projective groups of finite A-rank. While the latter class is a subclass of the former, the converse fails, in general, as is shown in Section 4.

Another application of the results in the first two sections of this paper was found during the Oberwolfach conference on abelian groups in 1989, where Hausen and the author used them to determine the torsion-free abelian groups A of finite rank without proper, nonzero fully-invariant quasi-summands, such that A^n has the quasi-summand intersection property for all $0 < n < \omega$.

The notation of this paper is the standard one which was introduced in [**5**] and [**10**]. The symbol ω denotes the first infinite ordinal and all maps are written on the left. In particular, $\doteq$ and $\dot\sim$ denote quasi-equality and quasi-isomorphism, respectively.

2. Semi-simple quasi-endomorphism rings. We consider an abelian group A and define an adjoint pair (H_A, T_A) of functors between the category $\mathcal{A}$ of abelian groups and the category $\mathcal{M}_{E(A)}$ of right $E(A)$-modules. If G is an abelian group, then the group $H_A(G) = \mathrm{Hom}\,(A, G)$ carries a natural right $E(A)$-module structure which is induced by composition of functions. Since A itself is a left $E(A)$-module, $T_A(M) = M \otimes_{E(A)} A$ defines an abelian group for all $M \in \mathcal{M}_{E(A)}$. There exist natural maps $\theta_G : T_A H_A(G) \to G$ and $\phi_M : M \to H_A T_A(M)$ for all $G \in \mathcal{A}$ and $M \in \mathcal{M}_{E(A)}$ which are defined by $\theta_G(\alpha \otimes a) = \alpha(a)$ and $[\phi_M(m)](a) = m \otimes a$ for all $\alpha \in H_A(G)$, $m \in M$, and $a \in A$. The maps θ_G and ϕ_M are isomorphisms, if G is A-projective of finite A-rank, and M is a finitely generated projective right $E(A)$-module [**6**].

Before we begin our discussion of the socle splitting property, we shortly summarize the results of [**2**] which are needed in the following.

Lemma 2.1 [**2**, Proposition 2.1]. *Let A be a torsion-free abelian group.*

(a) *The functors H_A and T_A preserve quasi-isomorphisms and quasi-splitting homomorphisms.*

(b) *If B and G are abelian groups, such that B is a quasi-summand of G, and θ_G is a quasi-isomorphism, then θ_B is a quasi-isomorphism.*

Furthermore, we obtained the following description of the self-small torsion-free abelian groups with the finite quasi-Baer-splitting property:

Proposition 2.2 [**2**, Corollary 2.5]. *The following conditions are equivalent for a self-small torsion-free abelian group A:*

(a) *A has the finite quasi-Baer-splitting property.*

(b) *If I is a right ideal of $E(A)$ such that A/IA is bounded, then $E(A)/I$ is bounded as an abelian group.*

Condition (b) is obviously satisfied if $\mathbf{Q}E(A)$ is semi-simple Artinian. Moreover, groups with semi-simple Artinian quasi-endomorphism rings arise immediately if we consider the socle splitting property in conjunction with the finite Baer-splitting property:

Theorem 2.3. *A torsion-free abelian group A has a semi-simple Artinian quasi-endomorphism ring, if and only if A is self-small and has the finite quasi-Baer- and the socle-splitting property.*

Proof. Suppose that $\mathbf{Q}E(A)$ is semi-simple Artinian. It remains to show that A has the socle-splitting property. We consider an exact sequence $0 \to B \overset{\alpha}{\to} A^n$ where $n < \omega$ and $S_A(B) \doteq B$. It induces the exact sequence $0 \to \mathbf{Q} \otimes_{\mathbf{Z}} H_A(B) \overset{id_{\mathbf{Q}} \otimes H_A(\alpha)}{\longrightarrow} \mathbf{Q} \otimes_{\mathbf{Z}} H_A(A^n)$ of right $\mathbf{Q}E(A)$-modules. Since $\mathbf{Q}E(A)$ is semi-simple Artinian, there is a $\mathbf{Q}E(A)$-module homomorphism $\psi : \mathbf{Q} \otimes_{\mathbf{Z}} H_A(A^n) \to \mathbf{Q} \otimes_{\mathbf{Z}} H_A(B)$ with $\psi(id_{\mathbf{Q}} \otimes H_A(\alpha)) = id_{\mathbf{Q} \otimes_{\mathbf{Z}} H_A(B)}$. Using the fact that $H_A(A^n)$ is finitely generated as an $E(A)$-module, we can find a nonzero integer m with $m\psi(H_A(A^n)) \subseteq H_A(B)$. Thus, $m\psi : H_A(A^n) \to H_A(B)$ satisfies $(m\psi)H_A(\alpha) = m \cdot id_{H_A(B)}$.

We obtain the following commutative diagram whose rows are exact:

$$\begin{array}{ccccc} & & T_AH_A(B) & \overset{T_A(m\psi)}{\underset{T_AH_A(\alpha)}{\rightleftarrows}} & T_AH_A(A^n) \\ & & \downarrow \theta_B & & l \downarrow \theta_{A^n} \\ 0 & \longrightarrow & B & \xrightarrow{\alpha} & A^n \end{array}$$

Choose a nonzero integer k with $kB \subseteq S_A(B) = im\theta_B \subseteq B$ and define a map $\beta : A^n \to B$ by $\beta = \theta_B T_A(m\psi)\theta_{A^n}^{-1}(k \cdot id_{A^n})$. If $x \in B$, then there is a $y \in T_A H_A(B)$ with $\theta_B(y) = kx$. Hence, $\beta\alpha(x) = \theta_B T_A(m\psi)\theta_{A^n}^{-1}\alpha\theta_B(y) = \theta_B T_A(m \cdot id_{H_A(B)})(y) = \theta_B(my) = (mk)x$. Thus, $\beta\alpha = (mk) \cdot id_B$ and α quasi-splits.

Conversely, suppose that A is self-small and satisfies the two quasi-splitting properties. Let I be a right ideal of $\mathbf{Q}E(A)$ and set $J = I \cap E(A)$. We have that J is a right ideal of $E(A)$ with $\mathbf{Q}J = I$. There exists an exact sequence $0 \to U \xrightarrow{\alpha} F \xrightarrow{\beta} J \to 0$ of right $E(A)$-modules in which F is free. Define a map $\delta : T_A(J) \to JA$ by $\delta(j \otimes a) = j(a)$ for all $j \in J$ and $a \in A$.

Since A has the socle splitting property J, A is a quasi-summand of A. Because A has the finite quasi-Baer-splitting property, the epimorphism $\delta T_A(\beta) : T_A(F) \to JA$ quasi-splits. Choose a nonzero integer m and a homomorphism $\psi : JA \to T_A(F)$ with $\delta T_A(B)\psi = m \cdot id_{JA}$. We obtain the diagram

$$\begin{array}{ccc} H_A T_A(F) & \underset{H_A(\delta T_A(\beta))}{\overset{H_A(\psi)}{\rightleftarrows}} & H_A(JA) \\ l \big\uparrow \phi_F & & \big\uparrow j_J \\ F & \xrightarrow{\quad\beta\quad} & J \to 0 \end{array}$$

in which j_J is the evaluation map. For all $x \in F$ and $a \in A$, we have $[H_A(\delta T_A(\beta))\phi_F(x)](a) = [\delta T_A(\beta)\phi_F(x)](a) = \delta T_A(\beta)(x \otimes a) = \delta(\beta(x) \otimes a) = [\beta(x)](a) = [j_J\beta(x)](a)$. Thus the diagram commutes.

Hence,

$$\begin{aligned} j_J\beta\phi_F^{-1}H_A(\psi)j_J &= H_A(\delta T_A(\beta))\phi_F\phi_F^{-1}H_A(\psi)j_J \\ &= H_A(\delta T_A(\beta)\psi)j_J = (m \cdot id_{H_A(JA)})j_J \\ &= j_J(m \cdot id_J). \end{aligned}$$

Since j_J is one-to-one as an evaluation map, $(\beta\phi_F^{-1}H_A(\psi))j_J = m \cdot id_J$. Moreover, $j_J\beta\phi_F^{-1}H_A(\psi) = H_A(\delta T_A(\beta))\phi_F\phi_F^{-1}H_A(\psi) = H_A(m \cdot id_{JA}) = m \cdot id_{H_A(JA)}$. Thus, j_J is a quasi-isomorphism.

Consider the inclusion $\lambda_J : J \to E(A)$. The inclusion map $\lambda_{JA} : JA \to A$ quasi-splits since A has the socle splitting property. For all $x \in J$ and all $a \in A$, we have $[H_A(\lambda_{JA})j_J(x)](a) = [\lambda_{JA}j(x)](a) = \lambda_{JA}(x(a)) = x(a)$. Thus, $H_A(\lambda_{JA})j_J = \lambda_J$. Since j_J is a quasi-isomorphism, and λ_{JA} quasi-splits, the map $\lambda_J : J \to E(A)$ quasi-splits by Lemma 2.1. Then, $I = \mathbf{Q}J$ is a direct summand of $\mathbf{Q}E(A)$. In particular, $\mathbf{Q}E(A)$ is semi-simple Artinian.

Corollary 2.4. *Let A be a strongly indecomposable group whose endomorphism ring has finite rank. Then, A has the socle splitting property if and only if $\mathbf{Q}E(A)$ is a division algebra.*

Proof. Every strongly indecomposable torsion-free group whose endomorphism ring has finite rank has the finite quasi-Baer-splitting property by [**2**]. If A has the socle splitting property, then $E(A)$ is semi-simple Artinian by Theorem 2.3. The fact that A is strongly indecomposable yields that $\mathbf{Q}E(A)$ does not have any nontrivial idempotents. However, a semi-simple Artinian ring with this property has to be a division algebra. Conversely, if $\mathbf{Q}E(A)$ is a division algebra, then A has the socle splitting property by Theorem 2.3. $\square$

Furthermore, the arguments used in the proof of Theorem 2.3 can be used to verify

Corollary 2.5. *Let A be a torsion-free abelian group whose quasi-endomorphism ring is semi-simple Artinian. The evaluation map $j_I : H_A(IA)$ is a quasi-isomorphism for all right ideals I of $E(A)$.*

3. The finite rank case. A family $\{A_1, \ldots, A_n\}$ of torsion-free abelian groups is *almost rigid* if the following conditions are satisfied for all indices $i, j \in \{1, \ldots, n\}$:

(i) If $R_{A_i}(A_j) = 0$, then $i = j$.

(ii) If $U_1, \ldots, U_{i-1}, U_{i+1}, \ldots, U_n$ are subgroups of A_i with $R_{A_j}(A_i/U_j) = 0$ for $j \neq i$, then $A_i/(\cap_{j\neq i} U_j)$ is quasi-isomorphic to $\oplus_{j\neq i} A_j^{s_j}$ for some indices $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n < \omega$.

Theorem 3.1. *The following conditions are equivalent for a torsion-free abelian group A of finite rank*:

a) *A has the socle-splitting-property.*

b) *If $0 \to B \stackrel{\alpha}{\to} G$ is an exact sequence of torsion-free abelian groups, where B has finite rank, $S_A(B) \doteq B$ and $\alpha(B) \cap R_A(G) = 0$, then B is a quasi-summand of G.*

c) *$A \sim A_1^{m_1} \oplus \cdots \oplus A_n^{m_n}$ where $\{A_1, \ldots, A_n\}$ is an almost rigid family* *of strongly indecomposable groups whose quasi-endomorphism ring is a division algebra.*

Proof. a) $\Rightarrow$ b). There exists a homomorphism $\beta : G \to A^I$ for some index-set I such that $\operatorname{Ker} \beta = R_A(G)$. If J is a subset of I, then denote the canonical projection $A^I \to A^J$, whose kernel is $A^{I \setminus J}$ by π_J.

Suppose that $\pi_J \beta\alpha$ is not a monomorphismm if J is a finite subset of I. Since $\beta\alpha$ is one-to-one, there is a $j_0 \in I$ with $\pi_{j_0}\beta\alpha \neq 0$. If we have found indices $j_0, \dots, j_n \in I$ such that $U_n = \cap_{i=0}^n \ker(\pi_{j_i}\beta\alpha) \neq 0$, then choose a nonzero $x \in U_n$ and an index $j_{n+1} \in I \setminus \{j_0, \dots, j_n\}$ with $\pi_{j_{n+1}}\beta\alpha(x) \neq 0$. By our assumption, $\cap_{i=0}^{n+1} \ker(\pi_{j_i}\beta\alpha) \neq 0$. We obtain a properly descending chain $U_0 \supset \dots \supset U_n \supset \dots$ of infinite length of pure subgroups of B, which is not possible since B has finite rank.

Thus, there exists a finite subset J of I such that the map $\pi_J\beta\alpha$ is one-to-one. By a), there exist a nonzero integer s and a map $\delta : A^J \to B$ with $\delta\pi_J\beta\alpha = s \cdot id_B$.

b) $\Rightarrow$ c). Suppose $A \sim A_1^{m_1} \oplus \dots \oplus A_n^{m_n}$ where the A_i's are nonzero strongly indecomposable, and $A_i \sim A_j$ only if $i = j$. Since A_i is a quasi-summand of A, we have $S_A(A_i) \doteq A_i$. Let f be a nonzero element of $E(A_i)$. Then $S_A(f(A_i)) \doteq f(A_i) \neq 0$ and $f(A_i)$ is a quasi-summand of A by b) which is contained in A_i. Thus, $f(A_i)$ is a nonzero quasi-summand of A_i. Since A_i is strongly indecomposable, $f(A_i) \doteq A_i$. In particular, $\ker f = 0$ because $r_0(A_i) = r_0(f(A_i)) + r_0(\ker f) < \infty$. Thus, f is a quasi-isomorphism and $\mathbf{Q}E(A_i)$ is a division algebra.

Let $i, j \in \{1, \dots, n\}$ with $R_{A_i}(A_j) = 0$. Since $r_0(A_j) < \infty$, there exist $s < \omega$ and a monomorphism $\alpha : A_j \to A_i^s$. The map α quasi-splits by b) since $S_A(A_j) \doteq A_j$. Since A_i and A_j are strongly indecomposable, $A_i \simeq A_j$ by Jónsson's theorem [**11**]. Hence, $i = j$.

Finally, let $i \in \{1, \dots, n\}$ and suppose that $U_1, \dots, U_{i-1}, U_{i+1}, \dots, U_n$ are subgroups of A_i with $R_{A_j}(A_i/U_j) = 0$ for all $j \neq i$. For each of these j's, there exist $t_j < \omega$ and a homomorphism $\delta_j : A_i \to A_j^{t_j}$ whose kernel is U_j. We define $\delta : A_i \to \oplus_{j \neq i} A_j^{t_j}$ by $\delta(x) = (\delta_j(x))_{j \neq i}$ for all $x \in A_i$. Then $\cap_{j \neq i} U_j = \ker \delta$, and $A_i \doteq S_A(A_i)$ as a quasi-summand of A yields that $\delta(A_i)$ is a quasi-summand of $\oplus_{j \neq i} A_j^{t_j}$. By Jónsson's theorem, there exist $s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n < \omega$ with $A_i / \cap_{j \neq i} U_j \cong \delta(A_i) \sim \oplus_{j \neq i} A_j^{s_j}$.

c) $\Rightarrow$ a). We consider an exact sequence $0 \to B \overset{\alpha}{\to} A^k$ in which $S_A(B) \doteq B \neq 0$ and $k < \omega$. By Jónsson's theorem, and because of c), we have $A^k \sim A_1^{s_1} \oplus \cdots \oplus A_n^{s_n}$ for some $s_1, \ldots, s_n < \omega$. No generality is hence lost if we consider a sequence $0 \to B \overset{\alpha}{\to} F$, in which $F = A_1^{s_1} \oplus \cdots \oplus A_n^{s_n}$ and α is the inclusion map.

In the first step, we consider the case that B satisfies $S_{A_1}(B) \doteq B$ but fails to be a quasi-summand of F. We choose B to be of minimal rank with respect to these properties. For each $j \in \{1, \ldots, n\}$, $\pi_j : F \to A_j^{s_j}$ denotes the projection whose kernel is $\oplus_{i \neq j} A_i^{s_i}$. Let ϕ be a nonzero element of $\operatorname{Hom}(A_1, B)$ and set $U_j = \ker \pi_j \phi$ for all j. Since A_1/U_j is isomorphic to a subgroup of $A_j^{s_j}$, the subgroup $V = U_2 \cap \cdots \cap U_n$ of A_1 satisfies $A_1/V \sim A_2^{t_2} \oplus \cdots \oplus A_n^{t_n}$ by c). If $V = 0$, then $A_1 \sim A_2^{t_2} \oplus \cdots \oplus A_n^{t_n}$ which is not possible by c) and Jónsson's theorem since A_i is strongly indecomposable for $i = 1, \ldots, n$. Thus, $V \neq 0$. By Theorem 2.3, A_1 has the finite quasi-Baer- and the socle-splitting property. Thus, $\pi_1\phi(A_1)$ is a quasi-summand of $A_1^{s_1}$. The finite quasi-Baer-splitting property for A_1 yields the quasi-splitting of $0 \to U_1 \to A_1 \to \pi_1\phi(A_1) \to 0$. Since A_1 is strongly indecomposable, we have either $U_1 = 0$ or $U_1 \doteq A_1$.

If $U_1 = 0$, then $\pi_1\phi : A_1 \to A_1^{s_1}$ a monomorphism which quasi-splits by Theorem 2.3. Choose a map $\tau : A_1^{s_1} \to A_1$ and a nonzero integer m with $\tau\pi_1\phi = m \cdot id_{A_1}$. Then $\phi : A_1 \to F$ quasi-splits. We write $F \doteq \phi(A_1) \oplus E$ for some subgroup E of F. This yields $B \doteq \phi(A_1) \oplus (E \cap B)$. Moreover, $B \doteq S_{A_1}(B)$ implies $S_{A_1}(E \cap B) \doteq E \cap B$. Since $r_0(E \cap B) < r_0(B) < \infty$, the group $E \cap B$ is a quasi-summand of F, say $F \doteq (E \cap B) \oplus D$ for some subgroup D of F. Then, $E \doteq (E \cap B) \oplus (E \cap D)$ and $F \doteq \phi(A_1) \oplus (E \cap B) \oplus (E \cap D) \doteq B \cap (E \cap D)$, because of $B \doteq \phi(A_1) \oplus (E \cap B)$, and F is torsion-free. This contradicts the choice of B.

On the other hand, $U_1 \doteq A_1$ implies $U_1 = A_1$ since U_1 is a pure subgroup of A_1. Then $\pi_1\phi = 0$ and $\phi(A_1) \subseteq A_2^{s_2} \oplus \cdots \oplus A_n^{s_n}$. In particular, $\ker \phi = \cap_{j=2}^{n} \ker \pi_j \phi = V$. This yields $\phi(A_1) \cong A/V \sim A_2^{t_2} \oplus \cdots \oplus A_n^{t_n}$. Once we have shown that A has the radical-splitting property, then $\phi(A_1) \subseteq A_2^{s_2} \oplus \cdots \oplus A_n^{s_n}$ implies that $\phi(A_1)$ is a quasi-summand of $A_2^{s_2} \oplus \cdots \oplus A_n^{s_n}$. This yields a quasi-decomposition $F \doteq \phi(A_1) \oplus E$ for some subgroup E of F. Now we proceed to obtain a contradiction as in the case $U_1 = 0$.

To see that A has the radical-splitting property, we consider nonzero subgroups $W_1, \dots, W_n$ of A_i for some $i \in \{1, \dots, n\}$ such that $R_{A_j}(A_i/W_j) = 0$ for all j. Since $R_{A_i}(A_j) \neq 0$ for $i \neq j$ by c), it suffices to show $\cap_{j=1}^n W_j \neq 0$ to guarantee that A has the radical splitting property by [**3**, Theorem 3.1]. There is an $r_i < \omega$ such that A_i/W_i is isomorphic to a subgroup of $A_i^{r_i}$. By Theorem 2.3, A_i/W_i is a quasi-summand of $A_i^{r_i}$, and the sequence $0 \to W_i \to A_i \to A_i/W_i \to 0$ quasi-splits. Since A_i is strongly indecomposable, and $0 \neq W_i$ is pure in A_i, we have $W_i = A_i$. As in the case of V, we show $\cap_{j \neq i} W_j \neq 0$. Consequently, $\cap_{j=1}^n W_j = \cap_{j \neq i} W_j \neq 0$.

Thus, we have shown that a subgroup B of F with $S_{A_i}(B) \doteq B$ for some $i \in \{1, \dots, n\}$ is a quasi-summand of F. In the general case we assume that B is a nonzero subgroup of F with $S_A(B) \doteq B$ which is not a quasi-summand of F, but of minimal rank with this property. Because of $S_A(B) \doteq B \neq 0$, there is a nonzero map $\phi \in \mathrm{Hom}(A, B)$. Since $A \sim A_1^{m_1} \oplus \dots \oplus A_n^{m_n}$, there is an $i \in \{1, \dots, n\}$ with $\phi(A_i) \neq 0$. Thus, $B_i = S_{A_i}(B)$ is nonzero. The results of the first step yield $F \doteq B_i \oplus E$ for some subgroup E of F. Hence, $B \doteq B_i \oplus (E \cap B)$. We have $r_0(E \cap B) < r_0(B) < \infty$ and $S_A(E \cap B) \doteq E \cap B$. Consequently, $F \doteq (E \cap B) \oplus D$ for some subgroup D of F. Then $E \doteq (E \cap B) \oplus (E \cap D)$ yields $F \doteq B_i \oplus E \doteq B_i \oplus (E \cap B) \oplus (E \cap D) \doteq B \oplus (E \cap D)$ since F is torsion-free and $B \doteq B_i \oplus (E \cap B)$. Thus, B is a quasi-summand of F, a contradiction. □

For the sake of comparison, we mention a related result for semi-simple Artinian quasi-endomorphism ring from [**12**]:

Proposition 3.2. *A torsion-free abelian group A of finite rank has a semi-simple Artinian quasi-endomorphism ring if and only if $A \sim A_1^{m_1} \oplus \dots \oplus A_n^{m_n}$, where the A_i's are strongly indecomposable groups whose quasi-endomorphism ring is a division algebra and* $\mathrm{Hom}(A_i, A_j) = 0$ *for $i \neq j$.*

However, there exists a group A which has the socle-splitting property, but whose quasi-endomorphism ring is not semi-simple:

Corollary 3.3. a) *There exists a torsion-free abelian group A of* *rank* 3 *with*

$$\mathbf{Q}E(A) \cong \begin{pmatrix} \mathbf{Q} & 0 \\ \mathbf{Q} & \mathbf{Q} \end{pmatrix}$$

which has the socle-splitting property.

b) *There exists a torsion-free group of rank* 3 *which does not have the socle-splitting property with*

$$\mathbf{Q}E(A) \cong \begin{pmatrix} \mathbf{Q} & 0 \\ \mathbf{Q} & \mathbf{Q} \end{pmatrix}.$$

Proof. a) For primes $p_1, \dots, p_n$ of $\mathbf{Z}$, let $\mathbf{Z}_{p_1,\dots,p_n}$ be the subgroup of $\mathbf{Q}$ whose elements can be represented by fractions of the form r/s such that p_i does not divide s for $i = 1, \dots, n$.

Choose distinct primes p_1, p_2, p_3 and p_4 of $\mathbf{Z}$ and a two-dimensional $\mathbf{Q}$-vector space $V = \mathbf{Q}e_1 \oplus \mathbf{Q}e_2$. Set $G = \mathbf{Z}_{p_1,p_2,p_3}e_1$, $G_2 = \mathbf{Z}_{p_1,p_2,p_4}e_2$, and $G = \langle G_1, G_2, p_1^{-n}(e_1 + e_2) \mid n < \omega \rangle$. By [**5**, Example 2.4], G is a strongly indecomposable group with $\mathbf{Q}E(G) = \mathbf{Q}$ and G_1 and G_2 are pure, fully invariant subgroups of G.

If $p_3(G/G_2) = G/G_2$, then $p_3G = G$ since $p_3G_2 = G_2$. Then $p_3G_1 = G_1 \cap p_3G = G_1$, which is not possible. Now assume that $p_2(G/G_2) = G/G_2$. For each $n < \omega$, we can find $a_n \in \mathbf{Z}_{p_1,p_2,p_3}$, $b_n \in \mathbf{Z}_{p_1,p_2,p_4}$, $m_n \in \mathbf{Z}$ and $k_n < \omega$ such that $e_1 + b_ne_2 = p_2^n[a_ne_1 + m_np_1^{-k_n}(e_1 + e_2)]$. Comparing the coefficients of e_1 and e_2, we obtain $p_1^{k_n} = p_2^n(p_1^{k_n}a_n + m_n)$ and $p_1^{k_n}b_n = p_2^n m_n$. Hence, 1 has infinite p_2-height in $\mathbf{Z}_{p_1,p_2,p_3}$ since $p_1^{k_n}a_n + m_n$ is an element of the latter group. This contradiction shows that $p_2(G/G_2) \neq G/G_2$. On the other hand, G/G_2 is q-divisible for all primes q of $\mathbf{Z}$ different from p_2 and p_3, since $\langle e_1 + e_2 \rangle_* \cap G_2 = 0$ yields $p_1(G/G_2) = G/G_2$. Thus, $G/G_2 \cong \mathbf{Z}_{p_2,p_3}$.

Let $B = \mathbf{Z}_{p_2,p_3}$ and $A = G \oplus B$. Since $p_3G_2 = G_2$, we have $\phi(G_2) = 0$ for all $\phi \in \operatorname{Hom}(G, B)$. In particular, $R_B(G) = G_2 \neq 0$. Since B is divisible by p_1 and p_4, any image of B in G is contained in $p_1^\omega G = \langle e_1 + e_2 \rangle_*$, which is not divisible by p_4. Thus, $\operatorname{Hom}(B, G) = 0$. Since B has rank 1, there is no subgroup U of B with $R_G(B/U) = 0$. On the other hand, suppose that there is a subgroup V of G with $R_B(G/V) = 0$. Then $G_2 = R_B(G) \subseteq V$ implies $V/G_2 \subseteq G/G_2$. If $V \neq G_2$, then V has rank at least 2. Since V is pure in G, we have

$G = V$. Thus, $G/V \cong B$ or $G/V = 0$. Thus $\{B, G\}$ is almost rigid. Since $\mathbf{Q}E(G) \cong \mathbf{Q}E(B) \cong \mathbf{Q}$, A has the socle splitting property by Theorem 3.1. Moreover, $\mathrm{Hom}\,(G, B) \cong \mathrm{Hom}\,(G/G_2, B)$ has rank 1 and $\mathrm{Hom}\,(B, G) = 0$. Hence,

$$\mathbf{Q}E(A) \cong \begin{pmatrix} \mathbf{Q}E(B) & \mathbf{Q}\mathrm{Hom}\,(B,G) \\ \mathbf{Q}\mathrm{Hom}\,(G,B) & \mathbf{Q}E(G) \end{pmatrix} \cong \begin{pmatrix} \mathbf{Q} & 0 \\ \mathbf{Q} & \mathbf{Q} \end{pmatrix}.$$

b) We use the notation of part a) and let $A = G \oplus \mathbf{Z}_{p_3}$. As before

$$\mathbf{Q}E(A) \cong \begin{pmatrix} \mathbf{Q} & 0 \\ \mathbf{Q} & \mathbf{Q} \end{pmatrix}.$$

Since $G/G_2 \cong \mathbf{Z}_{p_2,p_3}$ is not quasi-isomorphic to a direct sum of copies of $\mathbf{Z}_{p_3}$, although $R_{\mathbf{Z}_{p_3}}(G/G_2)$ is zero, the group A cannot have the socle-splitting property by Theorem 3.1. □

If A has rank 2, then the situation improves in comparison to Corollary 3.3:

Corollary 3.4. *The following conditions are equivalent for a torsion-free group of rank* 2.

a) $\mathbf{Q}E(A)$ *is semi-simple Artinian.*

b) A *has the socle splitting property.*

Proof. It remains to show b) $\Rightarrow$ a). Suppose A has the socle condition, but $\mathbf{Q}E(A)$ is not semi-simple Artinian. By Corollary 2.4, A cannot be strongly indecomposable. Thus, $A \dot{\sim} A_1 \oplus A_2$ for rank 1 groups A_1 and A_2. If $A_1 \cong A_2$ or if A_1 and A_2 have incomparable type, then $\mathbf{Q}E(A)$ is semi-simple. Hence $\mathrm{Hom}\,(A_1, A_2) \neq 0$ and $\mathrm{Hom}\,(A_2, A_1) = 0$ or vice-versa. No generality is lost if we assume $\mathrm{Hom}\,(A_1, A_2) \neq 0$ and $\mathrm{Hom}\,(A_2, A_1) = 0$. If $\phi : A_1 \to A_2$ is nonzero, then $\phi(A_1) \cong A_1$, and $\phi(A_1)$ is not quasi-equal to A_2. By b), $\phi(A_1)$ is a quasi-summand of A_2. Hence, $\phi(A_1) \doteq A_2$, a contradiction. □

4. Applications. In the discussion of the various splitting properties in [**6, 2**] and [**3**], quasi-summands of A-projective groups of finite A-rank play a central role. It is the goal of this section to show that such a quasi-summand is not always quasi-isomorphic to an A-projective group of finite A-rank as the next two results show, although the converse always holds:

Theorem 4.1. *The following conditions are equivalent for a torsion-free abelian group A, whose quasi-endomorphism ring is semi-simple Artinian*:

a) *Every nonzero right ideal I of $E(A)$ contains a nonzero projective right ideal.*

b) *If B is a quasi-summand of an A-projective group of finite A-rank, then B is quasi-isomorphic to an A-projective group of finite A-rank.*

Proof. a) $\Rightarrow$ b). Let B be a quasi-summand of A^n for some $n < \omega$. Choose a subgroup W of A^n and a nonzero integer m with $mA^n \subset B \oplus W \subset A^n$. Then $mH_A(A^n) = H_A(mA^n) \subset H_A(B) \oplus H_A(W) \subset H_A(A^n)$. Since $E(A)$ has finite Goldie-dimension, we can find an essential submodule U of $H_A(B)$ which is the direct sum of nonzero, uniform cyclic submodules $U_1, \dots, U_r$. Because $H_A(A^n)$ is a nonsingular right $E(A)$-module, each U_i contains a nonzero submodule V_i which is isomorphic to a right ideal of $E(A)$. By a), we may assume that each V_i is projective. Consequently, $V = V_1 \oplus \cdots \oplus V_r$ is an essential projective submodule of $H_A(B)$. Since $\mathbf{Q}E(A)$ is semi-simple Artinian, $H_A(B)/V$ is torsion as an abelian group.

Let $\pi : H_A(B) \oplus H_A(W) \to H_A(B)$ be the projection whose kernel is $H_A(W)$. Then $N = \pi(mH_A(A^n))$ is a finitely generated submodule of $H_A(B)$ with $mH_A(B) = \pi(m(H_A(B) \oplus H_A(W))) \subset \pi(mH_A(A^n)) = N$. Since $\langle N, V\rangle/V \cong N/N \cap V$ is finitely generated as an $E(A)$-module and $\langle N, V\rangle/V \subset H_A(B)/V$ is torsion as an abelian group, there exists a nonzero integer k with $kN \subset V$. Thus, $(km)H_A(B) \subset kN \subset V \subset H_A(B)$. Thus, $H_A(B)$ and V are quasi-equal. By Lemma 2.1, $T_A H_A(B)$ is quasi-isomorphic to $T_A(V)$, and the latter is A-projective of finite A-rank. Since B is a quasi-summand of A^n, the map θ_B is a quasi-isomorphism by Lemma 2.1. Consequently, B is quasi-isomorphic to $T_A(V)$ which is A-projective of finite A-rank.

b) $\Rightarrow$ a). Let I be a nonzero right ideal of $E(A)$. No generality is lost if we assume that I is cyclic, say $I = \phi E(A)$. By Theorem 2.3, $IA = \phi(A)$ is a quasi-summand of A. Because of b), there exists a nonzero integer n

and an A-projective group V of finite A-rank with $n\phi(A) \subset V \subset \phi(A)$. Then $nH_A(\phi(A)) = H_A(n\phi(A)) \subset H_A(V) \subset H_A(\phi(A))$, and we obtain that $H_A(\phi(A))$ is quasi-equal to the projective module $H_A(V)$ since A is self-small. Since $\mathbf{Q}E(A)$ is semi-simple Artinian, the evaluation map $j_J : J \to H_A(JA)$ is a quasi-isomorphism for every right ideal J of $E(A)$ by Corollary 2.5. Thus, $I = \phi E(A)$ is quasi-isomorphic to $H_A(IA) = H_A(\phi(A))$. This shows $I \dot{\sim} H_A(V)$, and hence I contains a nonzero projective submodule. □

Corollary 4.2. *Let A be a torsion-free abelian group such that $\mathbf{Q}E(A)$ is a division algebra. Then quasi-summands of A-projective groups of finite A-rank are quasi-isomorphic to A-projective groups of finite A-rank.*

Proof. Let I be a nonzero right ideal of $E(A)$. Then I is essential in $E(A)$ and $nE(A) \subset I$ for some nonzero integer n. Now apply Theorem 4.1. □

Example 4.3. *There exists a torsion-free abelian group A of finite rank for which there is a quasi-summand of an A-projective group of finite A-rank which is not quasi-isomorphic to an A-projective group.*

Proof. Let $R = \{(n + 2k, n + 2m) \mid n, k, m \in \mathbf{Z}\}$, a subring of $\mathbf{Z} \times \mathbf{Z}$. The ideal $I = (2, 0) \cdot R = \{(2n, 0) \mid n \in \mathbf{Z}\}$ of R is nonzero, and every ideal J of R, which is contained in I, is cyclic. If $(2m, 0)R$ is a projective ideal for some nonzero integer m, then the annihilator of $(2m, 0)$ is generated by some idempotent e of R. Choose integers r, s, and t with $e = (r + 2s, r + 2t)$. Then the equations $2m(r + 2s) = 0$ and $(r + 2t)^2 = r + 2t$ yield $r = -2s$ and $r(t - s)^2 = (t - s)$. Hence, $t = s$. Thus, $r + 2t = 0$ and $e = 0$. But then $(2m, 0) \cong R$ which is not possible. Therefore, I does not contain a nonzero projective ideal of R.

If A is an abelian group with $E(A) \cong R$ (such an A exists by either Corner's or Zassenhaus's realization theorem [**8**] or [**12**]), then there exists a quasi-summand of an A-projective group of finite A-rank which is not quasi-isomorphic to an A-projective group. □

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