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STRICTLY CYCLIC VECTORS FOR INDUCED REPRESENTATIONS OF LOCALLY COMPACT GROUPS

ROBERT A. BEKES

ABSTRACT. An induced representation of a locally compact group has a strictly cyclic vector only if the coset space is finite. A nonzero subrepresentation of a representation induced from a compact or normal subgroup has a strictly cyclic vector only if the coset space is compact.

Preliminaries. Throughout G is a separable locally compact group and H is a closed subgroup. Let ν be left Haar measure on G and assume that there exists an invariant measure μ on G/H, the left cosets. Let π be a continuous unitary representation of H on a Hilbert space $\mathcal{H}(\pi)$ and ϕ a function from G to $\mathcal{H}(\pi)$ such that $\phi(xh) = \pi(h^{-1})\phi(x)$ for all $x \in G$ and all $h \in H$. Since π is unitary, the function $x \to ||\phi(x)||$ is constant on the left cosets of H. Therefore the space of weakly measurable functions $\phi : G \to \mathcal{H}(\pi)$ satisfying

i) $\phi(xh) = \pi(h^{-1})\phi(x)$ for $x \in G$ and $h \in H$, and

ii) $\int_{G/H} ||\phi(x)||^2 \, d\mu < \infty$

is a Hilbert space under the inner product $\langle \phi, \gamma \rangle = \int_{G/H} \langle \phi(x), \gamma(x) \rangle \, d\mu$. The induced representation π^G of G on this space, denoted by $\mathcal{H}(\pi^G)$, is defined by $\pi^G(s)\phi(x) = \phi(s^{-1}x)$. It follows that π^G is a continuous unitary representation of G, see [**4**].

Main results. For $f \in L_1(G)$ define the operator $\pi^G(f)$ on $\mathcal{H}(\pi^G)$ by $\int_G f(x)\pi^G(x) \, d\nu$, where this integral is taken in the weak sense. Then $||\pi^G(f)|| \le ||f||$. Therefore the map π^G defines a continuous representation of the Banach $*$-algebra $L_1(G)$ on $\mathcal{H}(\pi^G)$. Fix $\phi \in \mathcal{H}(\pi^G)$ and define the map T_ϕ from $L_1(G)$ to $\mathcal{H}(\pi^G)$ by $T_\phi f = \pi^G(f)\phi$. Then $||T_\phi|| \le ||\phi||$. Let T_ϕ^* denote the adjoint map. Then $T_\phi^* : \mathcal{H}(\pi^G) \to L_\infty(G)$ and $||T_\phi^*|| \le ||\phi||$.

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Lemma 1.

For $\gamma \in \mathcal{H}(\pi^G)$ *and* $s \in G$, $[T_\phi^*\gamma](s) = \int_{G/H} \langle \phi(s^{-1}x), \gamma(x)\rangle \, d\mu$.

Proof. Let $f \in L_1(G)$. Then

$$\begin{aligned}\int_G f(s)[T_\phi^*\gamma](s)\, d\mu &= \int_{G/H} \langle [T_\phi f](x), \gamma(x)\rangle \, d\nu \\ &= \int_{G/H} \left\{ \int_G f(s)\langle \phi(s^{-1}x), \gamma(x)\rangle \, d\mu \right\} d\nu \\ &= \int_G f(s) \int_{G/H} \langle \phi(s^{-1}x), \gamma(x)\rangle \, d\nu \, d\mu. \qquad \square\end{aligned}$$

Let $\mathcal{M}$ be a closed π^G invariant subspace of $\mathcal{H}(\pi^G)$. A vector $\phi \in \mathcal{M}$ is called strictly cyclic for $\mathcal{M}$ if $\pi^G(L_1(G))\phi = \mathcal{M}$. When this is the case, T_ϕ is an open map from $L_1(G)$ to $\mathcal{M}$. It follows from [**3**, II, 4.18b] that T_ϕ^* is a bicontinuous isomorphism from $\mathcal{M}^*$, the dual of $\mathcal{M}$, onto the polar of the kernel of T_ϕ in $L_\infty(G)$, see [**3**]. For $\gamma \in \mathcal{H}(\pi^G)$, let $||\gamma||_{\mathcal{M}^*}$ denote the norm of γ as a functional on $\mathcal{M}$.

Let $s \in G$, denote by $\bar{s}$ the projection of s onto G/H. For $B \subseteq G$ denote by $\overline{B}$ the projection of B onto G/H and by χ_B the characteristic function of the set B. Let $\tau : G/H \to G$ be a Borel cross-section.

Lemma 2. *Suppose* H *is open. If* $\mathcal{H}(\pi^G)$ *has a strictly cyclic vector, then* G/H *is finite.*

Proof. Let $\phi \in \mathcal{H}(\pi^G)$ be a strictly cyclic vector. By the open mapping theorem there exists $C > 0$ such that whenever $\gamma \in \mathcal{H}(\pi^G)$ with $||\gamma|| \le 1$ there exists $f \in L_1(G)$ with $||f|| \le C$ such that $T_\phi f = \gamma$. Choose $\phi_0 \in \mathcal{H}(\pi^G)$, with support KH, K compact, such that $||\phi_0 - \phi|| < 1/(2C)$. Then with γ and f as above we get $||T_{\phi_0} f - \gamma|| = ||T_{\phi_0} f - T_\phi f|| \le ||\phi_0 - \phi|| \, ||f||_1 < 1/2$. This shows, by [**1**, 1.2], that the map T_{ϕ_0} is surjective. Therefore, there exists $A > 0$ such that $||\gamma|| \le A||T_{\phi_0}^*\gamma||$ for all $\gamma \in \mathcal{H}(\pi^G)$.

Since H is open and closed, G/H is discrete. Normalize μ so that $\mu(\bar{s}) = 1$ for all $s \in G$. Now suppose that G/H is infinite. Let $\overline{K}$ be the projection of K on G/H. Then $\overline{K}$ is compact and hence finite. Let $M = \sup ||\phi_0(x)||$ and $\sqrt{N} > AM\mu(\overline{K})$. Choose $s_1, \ldots, s_N \in G$ such that the $\bar{s}_i$ are distinct. Let $\alpha \in \mathcal{H}(\pi)$, $||\alpha|| = 1$. Then the functions $\zeta_i(x) = \chi_{s_iH}(x)\pi(x^{-1}\tau(\bar{x}))\alpha$ belong to $\mathcal{H}(\pi^G)$ and are orthogonal. Therefore $||\sum_{i=1}^N \zeta_i|| = \sqrt{N}$.

For $s \in G$,

$$\begin{aligned}\left|\left\{T_{\phi_0}^* \sum_{i=1}^N \zeta_i\right\}(s)\right| &\le \int_{G/H} \sum_{i=1}^N |\langle \phi_0(s^{-1}x), \zeta_i(x)\rangle| \, d\mu \\ &\le \int_{G/H} \sum_{i=1}^N ||\phi_0(s^{-1}x)|| \, ||\zeta_i(x)|| \, d\mu \\ &= \sum_{i=1}^N ||\phi_0(s^{-1}s_i)|| \mu(\bar{s}_i) \\ &\le M\mu(\overline{K}).\end{aligned}$$

It follows that $||T_{\phi_0}^* \sum_{i=1}^N \zeta_i|| \le M\mu(\overline{K})$ contradicting $||\sum_{i=1}^N \zeta_i|| \le A||T_{\phi_0}^* \sum_{i=1}^N \zeta_i||$. Therefore, G/H must be finite. □

Theorem 3. *If $\mathcal{H}(\pi^G)$ has a strictly cyclic vector, then G/H is finite.*

Proof. Let ϕ_0 be as in Lemma 2. Suppose G/H is not discrete. Then there exists a Borel subset $B \subseteq G$ such that $0 < \mu(\overline{B}) < [AM]^{-2}$. Let $\alpha \in \mathcal{H}(\pi)$, $||\alpha|| = 1$, and let $\zeta(x) = \chi_{BH}(x)\pi(x^{-1}\tau(\bar{x}))\alpha$. It follows that $\zeta \in \mathcal{H}(\pi^G)$ and $||\zeta|| = \sqrt{\mu(\overline{B})}$.

Now let $s \in G$. Then

$$\begin{aligned}|\{T_{\phi_0}^*\zeta\}(s)| &\le \int_{G/H} |\langle \phi_0(s^{-1}x), \zeta(x)\rangle| \, d\mu \\ &\le \int_{G/H} ||\phi_0(s^{-1}x)|| \, ||\zeta(x)|| \, d\mu \\ &= \int_{G/H} ||\phi_0(s^{-1}x)|| \, ||\chi_{BH}(x)|| \, d\mu \\ &\le M\mu(\overline{B}).\end{aligned}$$

Therefore $||T^*_{\phi_0}\zeta|| \leq M\mu(\overline{B})$. But $||\zeta|| \leq A||T^*_{\phi_0}\zeta||$ implies $\sqrt{\mu(\overline{B})} \leq AM\mu(\overline{B})$, contradicting the choice of B. Therefore, G/H is discrete and so, by Lemma 2, is finite. $\square$

Now we consider subrepresentations of induced representations from compact or normal subgroups. In all that follows H will always be a compact or normal subgroup. Let 1 denote the identity of G.

Lemma 4. *Let K_1 and K_2 be compact subsets of G and suppose that G/H is not compact. Then for any positive integer N there exists $s_1, \dots, s_N \in G$ such that*

i) $s_1K_1H, \dots, s_NK_1H$ *are disjoint*

ii) $s_1K_2, H, \dots, s_NK_2H$ *are disjoint*

iii) $s_iK_1H \cap s_jK_2H = \varnothing$ *for* $i \neq j$

and

iv) $s_1K_1HK_2^{-1}, \dots, s_NK_1HK_2^{-1}$ *are disjoint.*

Proof. Let $K = \{1\} \cup K_1 \cup K_2$. Then K is compact. If H is compact, then so is $KHK^{-1}KHK^{-1}$. If H is normal, then $KHK^{-1}KHK^{-1} = KK^{-1}KK^{-1}H$ whose projection on G/H is compact. Therefore in both cases the projection of $KHK^{-1}KHK^{-1}$ on G/H is compact. We choose the s_i inductively: having chosen $s_1, \dots, s_j$ so that $s_1KHK^{-1}, \dots, s_jKHK^{-1}$ are disjoint pick $s_{j+1} \in G\backslash \cup_{i=1}^{j} s_iKHK^{-1}KHK^{-1}$, which is nonempty since the projection of $\cup_{i=1}^{j} s_iKHK^{-1}KHK^{-1}$ onto G/H is compact. If $s_{j+1}KHK^{-1} \cap s_mKHK^{-1} \neq \varnothing$ for some $m \leq j$, then $s_{j+1} \in s_mKHK^{-1}KHK^{-1}$, violating the choice of s_{j+1}. Therefore, $s_1KHK^{-1}, \dots, s_{j+1}KHK^{-1}$ are disjoint. Since $1 \in K$, the lemma follows. $\square$

Theorem 5. *If a nonzero subrepresentation of π^G has a strictly cyclic vector, then G/H must be compact.*

Proof. Let $\mathcal{M}$ be a nonzero closed G invariant subspace of $\mathcal{H}(\pi^G)$ and $\phi \in \mathcal{M}$ such that $\pi^G(L_1(G))\phi = \mathcal{M}$. Then there exists $A > 0$ such that $||\gamma||_{\mathcal{M}^*} \leq A||T_\phi^*\gamma||$ for all $\gamma \in \mathcal{M}^*$, the dual space of $\mathcal{M}$.

Suppose G/H is not compact. Choose $\gamma \in \mathcal{H}(\pi^G)$, $||\gamma|| = 1$, such that $\langle \phi, \gamma \rangle > 0$ and γ is supported in K_1H where K_1 is compact in G. By taking a scalar multiple of ϕ, if necessary, we may assume that $\langle \phi, \gamma \rangle = 1$.

Let N be a positive integer. Choose $\phi_0 \in \mathcal{H}(\pi^G)$ such that $||\phi - \phi_0|| < 1/(N\sqrt{N})$ and whose support is contained in K_2H where K_2 is compact in G. Let K_1 and K_2 be as above and choose $s_1, \dots, s_N$ by Lemma 4. Then

1. $\pi^G(s_i)\gamma$ are orthogonal in $\mathcal{H}(\pi^G)$
2. $\pi^G(s_i)\phi_0$ are orthogonal in $\mathcal{H}(\pi^G)$
3. $\langle \pi^G(s_j)\phi_0, \pi^G(s_i)\gamma \rangle = 0$ for $j \neq i$
4. for any $s \in G$, $\langle \pi^G(s)\phi_0, \pi^G(s_i)\gamma \rangle \neq 0$ implies $\langle \pi^G(s)\phi_0, \pi^G(s_j)\gamma \rangle = 0$ for all $j \neq i$.

The first three assertions follow directly from i), ii) and iii) of Lemma 4. Now suppose that $\langle \pi^G(s)\phi_0, \pi^G(s_i)\gamma \rangle \neq 0$ and $\langle \pi^G(s)\phi_0, \pi^G(s_j)\gamma \rangle \neq 0$ where $i \neq j$. Then $sK_2H \cap s_iK_1H \neq \varnothing$ and $sK_2H \cap s_jK_1H \neq \varnothing$. And so $s_iK_1HK_2^{-1} \cap s_jK_1HK_2^{-1} \neq \varnothing$, contradicting the choice of $s_1, \dots, s_N$. Therefore, assertion 4 holds.

By assertion 1, $||\sum_{i=1}^N \pi^G(s_i)\gamma|| = \sqrt{N}$. Now fix $s \in G$. By assertion 4 above, there exists k such that $|\langle \pi^G(s)\phi_0, \pi^G(s_k)\gamma \rangle| = \max_{1 \leq i \leq N} |\langle \pi^G(s)\phi_0, \pi^G(s_i)\gamma \rangle|$. Therefore,

$$\begin{aligned}
\left[T_\phi^* \sum_{i=1}^N \pi^G(s_i)\gamma\right](s) &\leq \left|\left\langle \pi^G(s)\phi - \pi^G(s)\phi_0, \sum_{i=1}^N \pi^G(s_i)\gamma \right\rangle\right| \\
&\quad + \left|\left\langle \pi^G(s)\phi_0, \sum_{i=1}^N \pi^G(s_i)\gamma \right\rangle\right| \\
&\leq ||\phi - \phi_0||\sqrt{N} + \left|\left\langle \pi^G(s)\phi_0, \sum_{i=1}^N \pi^G(s_i)\gamma \right\rangle\right| \\
&\leq \frac{1}{N} + |\langle \pi^G(s)\phi_0, \pi^G(s_k)\gamma \rangle| \\
&\leq \frac{1}{N} + |\langle \pi^G(s)\phi_0 - \pi^G(s)\phi, \pi^G(s_k)\gamma \rangle| \\
&\quad + |\langle \pi^G(s)\phi, \pi^G(s_k)\gamma \rangle|
\end{aligned}$$

$$\begin{aligned}&= \frac{1}{N} + |\langle \pi^G(s)\phi_0 - \pi^G(s)\phi, \pi^G(s_k)\gamma\rangle| \\ &\quad + |\langle \pi^G(s_k^{-1}s)\phi, \gamma\rangle| \\ &\leq \frac{1}{N} + ||\phi - \phi_0|| + ||T_\phi^*\gamma|| \\ &\leq \frac{1}{N} + \frac{1}{N\sqrt{N}} + ||T_\phi^*\gamma||.\end{aligned}$$

It follows that $||[T_\phi^* \sum_{i=1}^N \pi^G(s_i)\gamma||$ is uniformly bounded as a function of N. Now

$$\begin{aligned}\left\| \sum_{i=1}^N \pi^G(s_i)\phi \right\| &\leq \left\| \sum_{i=1}^N \pi^G(s_i)\phi - \sum_{i=1}^N \pi^G(s_i)\phi_0 \right\| + \left\| \sum_{i=1}^N \pi^G(s_i)\phi_0 \right\| \\ &\leq \frac{1}{\sqrt{N}} + \sqrt{N}||\phi_0|| \\ &\leq \frac{1}{\sqrt{N}} + \sqrt{N}\left\{ ||\phi|| + \frac{1}{N\sqrt{N}} \right\} \\ &\leq 2\sqrt{N}||\phi|| \qquad \text{for } N \text{ sufficiently large.}\end{aligned}$$

Therefore,

$$\begin{aligned}&\left\| \sum_{i=1}^N \pi^G(s_i)\gamma \right\|_{\mathcal{M}^*} \\ &\quad \geq \frac{1}{|| \sum_{i=1}^N \pi^G(s_i)\phi||} \left| \left\langle \sum_{i=1}^N \pi^G(s_i)\phi, \sum_{i=1}^N \pi^G(s_i)\gamma \right\rangle \right| \\ &\quad \geq \frac{1}{2\sqrt{N}||\phi||} \left| \left\langle \sum_{i=1}^N \pi^G(s_i)\phi_0, \sum_{i=1}^N \pi^G(s_i)\gamma \right\rangle \right| \\ &\qquad - \frac{1}{2\sqrt{N}||\phi||} \left| \left\langle \sum_{i=1}^N \pi^G(s_i)\phi - \sum_{i=1}^N \pi^G(s_i)\phi_0, \sum_{i=1}^N \pi^G(s_i)\gamma \right\rangle \right| \\ &\quad = \frac{N}{2\sqrt{N}||\phi||} |\langle \phi_0, \gamma\rangle| \\ &\qquad - \frac{1}{2\sqrt{N}||\phi||} \left| \left\langle \sum_{i=1}^N \pi^G(s_i)\phi - \sum_{i=1}^N \pi^G(s_i)\phi_0, \sum_{i=1}^N \pi^G(s_i)\gamma \right\rangle \right|\end{aligned}$$

$$\geq \frac{\sqrt{N}}{2||\phi||}\left\{|\langle\phi,\gamma\rangle| - \frac{1}{N\sqrt{N}}\right\} - \frac{1}{2\sqrt{N}||\phi||}N||\phi-\phi_0||\sqrt{N}$$
$$\geq \frac{\sqrt{N}}{4||\phi||} - \frac{1}{2\sqrt{N}||\phi||}.$$

And so $||\sum_{i=1}^N \pi^G(s_i)\gamma||_{\mathcal{M}^*} \to \infty$ as $N \to \infty$ while $||[T_\phi^* \sum_{i=1}^N \pi^G(s_i)\gamma||$ remains bounded. This contradicts the choice of A. Therefore G/H must be compact. □

Let $\mathcal{H}(\pi)$ be one dimensional and $H = \{1\}$ in Theorem 5. Then the following is Corollary 2.1 of [**2**].

Corollary 6. *If a nonzero subrepresentation of the left regular representation has a strictly cyclic vector, then G is compact.*

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Santa Clara University, Department of Mathematics, Santa Clara, CA 95053