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STABILITY OF RANDOM MATRIX MODELS

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ABSTRACT. Random matrices have been widely studied as neutral models for the stability of large systems and, in particular, ecosystems. However, many ecologists interpret stability in terms of low variability and persistence, not Lyapunov stability studied in matrix theory. Following Harrison [**5**], Hastings [**6**] suggested a close relationship between Lyapunov stability and low variability for random matrix models with additional noise terms. We report on a simulation study confirming this conjecture and its extension to certain products of random matrices.

1. Introduction. "Will a large complex system be stable?" (May [**11**]). This important and intriguing problem has been extensively studied by Gardner and Ashby [**4**], May [**11, 12**], Hastings [**6**], Cohen and Newman [**3**], and Hastings, Juhasz and Schreiber [**7**] using linear models of the form

$$\underline{x}(t+1) = M\underline{x}(t) \tag{1}$$

in which M is a random matrix. Following Wigner [**14**], May [**11**] made the following conjecture.

The May-Wigner stability theorem. *Let M be an $n \times n$ matrix with connectance C (each entry of M is nonzero with probability C, independent of other entries) and root mean square (rms) interaction strength α (the nonzero entries are chosen independently from a symmetric distribution with expected square α^2). Then the spectral radius of M is given by*

$$\rho = \rho(M) = \alpha(nC)^{1/2} \tag{2}$$

asymptotically for large n.

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Bai and Yin [**1**] proved the May-Wigner upper bound,

$$\rho(M) \leq \alpha(nC)^{1/2},$$

under suitable technical hypothesis, and an extension to somewhat more general matrices. Let M be an $n \times n$ matrix with independent, identically distributed (iid) entries selected from a fixed distribution of expectation 0, expected square α^2, and bounded fourth moment. Then, almost surely,

$$\limsup_{n\to\infty}((1/n)\rho(M)) \leq \alpha^2. \tag{3}$$

In addition, under these hypotheses, extensive numerical evidence supports May's original conjecture (cf. McMurtrie [**13**]) in the form

$$\limsup_{n\to\infty}((1/n)\rho(M)) = \alpha^2. \tag{4}$$

The May-Wigner stability theorem implies asymptotic estimates on the growth of $\underline{x}(t)$ in equation (1), namely,

$$||\underline{x}(t)|| \approx \rho^t \tag{5}$$

in the sense that

$$\lim_{t\to\infty}(t^{-1}\log||\underline{x}(t)||) = \rho \tag{6}$$

whenever $\underline{x}(0) \neq 0$. Cohen and Newman [**3**] proved similar results for time-dependent systems

$$\underline{x}(t+1) = M(t)\underline{x}(t) \tag{7}$$

under suitable technical hypotheses. Here the random matrices $M(t)$ are chosen independently from a suitable statistical universe, and the growth rate ρ is given by the formula

$$\log\rho = E(\log||M(1)\underline{x}||/||\underline{x}||), \qquad \underline{x} \neq 0. \tag{8}$$

This estimate is similar but slightly less than the corresponding May-Wigner estimate

$$\rho = E(||M(1)\underline{x}||/||\underline{x}||), \qquad \underline{x} \neq 0$$

since the graph of the logarithm function is concave.

Lyapunov stability and low variability. Ecologists usually interpret stability in terms of low variability of population numbers and persistence of species rather than in terms of Lyapunov stability (cf. MacArthur [**10**], May [**12**], Harrison [**5**], Hastings [**6**]). How are these stability concepts related in the case of random matrix models?

Hastings [**6**] suggested an answer by considering stochastic models of the form

$$\underline{x}(t+1) = M\underline{x}(t) + \Delta\underline{w}(t) \tag{9}$$

in which the random walk process $\{\underline{w}(t)\}$ represents environmental noise. We call this noise *external*, in contrast to the internal noise of equation (4), since it does not affect the interspecific interactions. He then argued that, at large times

$$||\underline{x}(t)||^2 \approx E(\Delta\underline{w}^2)/(1-\rho^2), \tag{10}$$

provided that the system for the mean process (equation (1)) is Lyapunov stable, that is, provided that $|\rho| < 1$.

2. The models. We shall study stochastic models of the form

$$\underline{x}(t+1) = (M(t))\underline{x}(t) + \Delta\underline{w}(t) \tag{11}$$

where $M(t) = M + N(t)$. These models combine relevant features of both the Cohen-Newman models (equation (6)) and those previously studied by Hastings (equation (9)). Here the M is a time-independent, random $n \times n$ matrix with root mean square interaction strength α and connectance C. The community matrix M represents the mean value of the interspecific interactions in an ecological community. The $n \times n$ matrices $N(t)$ are independent samples from a fixed statistical universe and represent fluctuations in ecosystem dynamics (*internal noise*). The stochastic terms $\Delta\underline{w}(t)$ are the increments in a discrete-time random walk representing *external noise*. Of course, the increments $\Delta\underline{w}(t)$ are iid, Gaussian and of mean zero. Let σ^2 denote the expected square of $\Delta\underline{w}(t)$.

The spectral radius of $M(t) = M + N(t)$ is readily estimated with Bai and Yin's version of the May-Wigner theorem and numerical techniques. We obtain

$$\rho = \rho(M(t)) \approx (\alpha^2 C + \nu^2)^{1/2} n^{1/2} \tag{13}$$

in the sense of equations (4) and (5) above. As in formula (10), we conjecture that

$$E(||\underline{x}(t)||^2) \approx \frac{\sigma^2}{1-\rho^2} = \frac{\sigma^2}{1-(\alpha^2 C + \nu^2)n}, \tag{14}$$

provided that the expected spectral radii $|\rho|$ of the matrices $M(t)$ are strictly less than 1.

We shall now derive formula (14) in the case of no noise, $N(t) = 0$. However, error estimates appear difficult to estimate analytically, especially for the small systems of ecological interest. The general case, $N(t) \neq 0$, is even harder. We shall therefore turn to simulation methods in Section 4.

3. Derivation of formula (14) in the case of no noise. In the case of no noise, the state of the system described by equations (11) and (12) above is given by

$$\underline{x}(t) = M^t \underline{x}(0) + \Sigma M^{t-s+1} \Delta \underline{w}(s). \tag{15}$$

Here and below, all sums range over $0 \le s \le t-1$. Let $\rho = \rho(M)$ denote the spectral radius of M.

We now make precise the assumption that the matrices $M(t)$ have expected spectral radius strictly less than 1. Assume that, for some $\varepsilon > 0$,

$$\alpha(nC)^{1/2} \le 1 - 2\varepsilon.$$

By Bai and Yin [**1**], we may also assume that n is sufficiently large that

$$E(\rho) \le 1 - \varepsilon.$$

We are now ready to compute the right-hand side of equation (15). Our assumptions imply that

$$\lim_{t\to\infty} M^t \underline{x}(0) = 0,$$

and thus

$$\lim_{t\to\infty} E(||\underline{x}(t)||^2) = \lim_{t\to\infty} E(||\Sigma M^{t-s+1}\Delta\underline{w}(s)||^2).$$

Since the terms $\Delta\underline{w}(s)$ are independent by assumption, for $s \neq s'$,

$$E([M^{t-s+1}\Delta\underline{w}(s)]\cdot[M^{t-s'+1}\Delta\underline{w}(s')]) = 0.$$

Therefore,

$$\lim_{t\to\infty} E(||\underline{x}(t)||^2) = \lim_{t\to\infty} \Sigma E(||M^{t-s+1}\Delta\underline{w}(s)||^2). \tag{16}$$

The required asymptotic estimate (formula (14)) now follows easily from formula (5).

4. Simulation analysis. We studied the dynamics of both the autonomous stochastic model (equation (8), above)

$$\underline{x}(t+1) = M\underline{x}(t) + \Delta\underline{w}(t)$$

(the *May-Wigner case*) and an analogous time-dependent model (equations (11) and (12), above):

$$\underline{x}(t+1) = (M + N(t))\underline{x}(t) + \Delta\underline{w}(t). \tag{17}$$

In both models, M is an $n \times n$ random matrix of connectance C. We chose the rms interaction strength $\alpha = (2nC)^{-1/2}$ in order to the expected spectral radius $\rho(M) = 1/2$. As above, the terms $\Delta\underline{w}(t)$ are the increments in a random walk. In addition, M is either fully connected ($C = 1$) or $nC \gg \log(n)$, making the graph of M connected with high probability (cf. Bollobas [**2**]). Here is the detailed methodology.

The May-Wigner case. First generate a random matrix M as above. Next, compute the actual spectral radius of M by studying the growth rate of M^t. Then let $\underline{x}(0) = \underline{0}$, and use equation (18) to generate 50 successive values of $\underline{x}(t)$. Finally, compare the theoretical value of the asymptotic mean square norm of the vector $\underline{x}(t)$, namely $1/(1-\rho^2)$ by Section 2, with the average of the last 25 experimental values of

$||\underline{x}(t)||^2$; that is, $||\underline{x}(26)||^2, ||\underline{x}(27)||^2, \dots, ||\underline{x}(50)||^2$. (As it is customary with stochastic models, we let the model stabilize before gathering statistics.)

The case with internal noise included. We followed the same simulation methodology as above, using the time-dependent model (17). The noise matrices $N(t)$ are fully connected ($C = 1$), have rms interaction strength $\nu = 0.01$ or 0.1, and were chosen independently at each time t. In order to compare theoretical and experimental results, we computed the expected spectral radius of each of the matrices $M + N(t)$ in equation (17) using equation (13) above. We then proceeded as in the May-Wigner case.

We used the same values of n and C used previously in [**6**]; these values seem typical of small ecosystems. We ran 25 replicates for each choice of parameters. The results are given in Table 1. As we have shown, experimental values tended to agree closely with theoretical values of equation (14); however, we do not understand why large amounts of internal noise reduced experimental values below theoretical values.

TABLE 1. Ratio of experimental to theoretical values of $||\underline{x}(t)||^2$.

Size	Connectance	Internal Noise	(mean ± standard error)
8	1	0.0	1.017 ± 0.018
		0.01	1.015 ± 0.017
		0.1	0.979 ± 0.019
16	1	0.0	1.008 ± 0.014
		0.01	1.030 ± 0.014
		0.1	0.987 ± 0.017
16	0.5	0.0	1.001 ± 0.013
		0.01	1.008 ± 0.018
		0.1	0.986 ± 0.017

5. Discussion. We have found a simple relationship between Lyapunov stability and persistence and, in particular, the following equation relating ecosystem parameters (size, connectance, interaction strength), the maximum acceptable amount of environmental noise (internal noise $\sqrt{E(\Delta \underline{w}^2)}$ and external noise ν) to the size of population fluctuations $\underline{x}(t)$:

$$E(||\underline{x}(t)||^2) = \frac{E(\Delta \underline{w}^2)}{1 - (\alpha^2 C + \nu^2)n}.$$

This equation relates Lyapunov stability of model ecosystems to variability, and thus indirectly to persistence. Thus, random matrix models can be used to study persistence in the presence of environmental noise.

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