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THE CAUCHY FUNCTION FOR nTH ORDER LINEAR DIFFERENCE EQUATIONS

ALLAN PETERSON AND JOHN SCHNEIDER

In this paper we will be concerned with the Cauchy function for the nth order linear difference equation

$$Ly(t) \equiv \sum_{i=0}^{n} p_i(t)y(t+i) = 0 \tag{1}$$

where t is an integer variable, and we assume that

$$p_0(t)p_n(t) \neq 0,$$

for all integers t. We assume the coefficients $p_i(t)$, $0 \leq i \leq n$, are real valued functions defined on the integers. The Cauchy function $K(t,s)$, for each fixed integer s, is defined to be the solution of the initial value problem (1),

$$K(s+k,s) = 0, \qquad 1 \leq k \leq n-1,$$
$$K(s+n,s) = \frac{1}{p_n(s)}.$$

Let a be an integer. Then it is well known that the solution for the initial value problem,

$$\sum_{i=0}^{n} p_i(t)y(t+i) = f(t), \qquad t \geq a$$
$$y(a+k) = 0, \qquad 0 \leq k \leq n-1$$

is given by

$$y(t) = \sum_{s=a}^{t-1} K(t,s)f(s).$$

Assume that we can factor (1) in the form

$$Ly(t) = MNy(t) = 0$$

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where

$$Ny(t) = \sum_{i=0}^{k} q_i(t)y(t+i)$$

and

$$Mu(t) = \sum_{i=0}^{n-k} r_i(t)u(t+i),$$

for $1 \leq k \leq n-1$ with $q_0(t)q_k(t) \neq 0$ and $r_0(t)r_{n-k}(t) \neq 0$ for $t \geq a$. Let $K_N(t,s)$ and $K_M(t,s)$ be the Cauchy functions for $Ny(t) = 0$ and $Mu(t) = 0$, respectively. Then it is easy to see that the Cauchy function for $Ly(t) = 0$ is given by

$$K(t,s) = \sum_{\tau=s+1}^{t-1} K_N(t,\tau)K_M(\tau,s). \tag{2}$$

This formula is given in a slightly different form in [**5**].

Consider the constant coefficient case of (1):

$$\sum_{i=0}^{n} p_i y(t+i) = 0, \tag{3}$$

where $p_0 p_n \neq 0$. If the characteristic roots $z_1, \dots, z_n$ of (3) are distinct, then the Cauchy function of (4) is given by

$$K(t,s) = \sum_{i=1}^{n} \frac{z_i^{t-s-1}}{p'(z_i)} \tag{4}$$

where $p(z)$ is the characteristic polynomial $p(z) = \sum_{i=0}^{n} p_i z^i$. In [**4**], this formula is derived from the variation of constants formula for the vector equation equivalent to (1). It is derived in a different way in [**5**]. Also, see Exercise 2.15.32 [**1**]. For the second order self-adjoint equation, see [**3**].

Similarly, it is well known that the Cauchy function for (3) in the case where there is a single characteristic value z_1 with multiplicity n, is given by

$$K(t,s) = \frac{(t-s-1)^{(n-1)}}{p_n(n-1)!} z_1^{t-s-n}. \tag{5}$$

Here $t^{(n)}$ denotes the factorial polynomial

$$t^{(n)} = t(t-1)\cdots(t-n+1).$$

Also we could write

$$K(t,s) = \frac{1}{p_n}\binom{t-s-1}{n-1} z_1^{t-s-n}$$

where $\binom{t-s-1}{n-1}$ is the binomial coefficient.

Theorem 1. *Assume that $z_1, \dots, z_k$, $k \ge 2$, are the distinct characteristic roots of* (3)*, where z_1 has multiplicity $m \ge 1$ and $z_2, \dots, z_k$ have multiplicity one. Then the Cauchy function for* (3) *is given by*

$$K(t,s) = \sum_{j=2}^{k} \frac{z_j^{t-s-1}}{p'(z_j)} - \sum_{i=0}^{m-1} \left(\sum_{j=2}^{k} \frac{(z_j - z_1)^i}{p'(z_j)} \right) \frac{(t-s-1)^{(i)}}{i!} z_1^{t-s-1}.$$

Proof. First factor (3) in the form $MNy(t) = 0$ where $M = (E - z_2)\cdots(E - z_k)$, $N = p_n(E - z_1)^m$, where E is the shift operator defined by $Ey(k) = y(t+1)$. By (4) and (5) we get that the Cauchy functions for $Mu(t) = 0$ and $Ny(t) = 0$ are given, respectively, by

$$K_M(t,s) = \sum_{j=2}^{k} \frac{z_j^{t-s-1}}{q'(z_j)}, \qquad K_N(t,s) = \frac{(t-s-1)^{(m-1)}}{p_n(m-1)!} z_1^{t-s-m}$$

where $q(z) = \prod_{j=2}^{k}(z - z_j)$ is the characteristic polynomial for $Mu(t) = 0$.

Hence, by (2), the Cauchy function for (3) is given by

$$\begin{aligned}
K(t,s) &= \sum_{\tau=s+1}^{t-1} K_N(t,\tau) K_M(\tau,s) \\
&= \sum_{\tau=s+1}^{t-1} \frac{(t-\tau-1)^{(m-1)}}{p_n(m-1)!} z_1^{t-\tau-m} \sum_{j=2}^{k} \frac{z_j^{\tau-s-1}}{q'(z_j)} \\
&= \sum_{j=2}^{k} \frac{z_1^{t-m}}{p_n z_j^{s+1} q'(z_j)} \sum_{\tau=s+1}^{t-1} \frac{(t-\tau-1)^{(m-1)}}{(m-1)!} \left(\frac{z_j}{z_1}\right)^{\tau} \\
&= \sum_{j=2}^{k} \frac{z_1^{t-m}}{p_n z_j^{s+1} q'(z_j)} u(t),
\end{aligned}$$

where, for each fixed s, $u(t) = u(t,s)$, given by

$$u(t) = \sum_{\tau=s+1}^{t-1} \frac{(t-\tau-1)^{(m-1)}}{(m-1)!}\left(\frac{z_j}{z_1}\right)^{\tau}$$

is, by (5), the solution of

$$\Delta^m u(t) = \left(\frac{z_j}{z_1}\right)^t \tag{6}$$

$$u(s+i) = 0, \qquad 1 \le i \le m. \tag{7}$$

Here Δ is the difference operator defined by $\Delta y(t) = y(t+1) - y(t)$.

We now show by induction on l, $0 \le l \le m$, that

$$\begin{aligned} \Delta^{m-l} u(t) &= \left(\frac{z_1}{z_j - z_1}\right)^l \left(\frac{z_j}{z_1}\right)^t \\ &\quad - \left(\frac{z_j}{z_1}\right)^{s+1} \sum_{i=0}^{l-1} \frac{(t-s-1)^{(i)}}{i!}\left(\frac{z_1}{z_j - z_1}\right)^{l-i}. \end{aligned} \tag{8}$$

Summing both sides of (6) from $s+1$ to $t-1$ and using (7), we get that

$$\begin{aligned} \Delta^{m-1} u(t) &= \frac{1}{z_j/z_1 - 1}\left[\left(\frac{z_j}{z_1}\right)^t - \left(\frac{z_j}{z_1}\right)^{s+1}\right] \\ &= \frac{z_1}{z_j - z_1}\left(\frac{z_j}{z_1}\right)^t - \frac{z_1}{z_j - z_1}\left(\frac{z_j}{z_1}\right)^{s+1}. \end{aligned}$$

Hence, (8) is true for $l = 1$. Now assume that $1 < l < m-1$, and that (8) is true for this l. Summing both sides of (8) from $s+1$ to $t-1$ and using (7), we get that

$$\begin{aligned} \Delta^{m-l-1} u(t) &= \left(\frac{z_1}{z_j - z_1}\right)^l \left\{\frac{z_1}{z_j - z_1}\left(\frac{z_j}{z_1}\right)^t - \frac{z_1}{z_j - z_1}\left(\frac{z_j}{z_1}\right)^{s+1}\right\} \\ &\quad - \left(\frac{z_j}{z_1}\right)^{s+1} \sum_{i=0}^{l-1} \binom{t-s-1}{i+1}\left(\frac{z_1}{z_j - z_1}\right)^{l-i} \\ &= \left(\frac{z_1}{z_j - z_1}\right)^{l+1}\left(\frac{z_j}{z_1}\right)^t - \left(\frac{z_1}{z_j - z_1}\right)^{l+1}\left(\frac{z_j}{z_1}\right)^{s+1} \\ &\quad - \left(\frac{z_j}{z_1}\right)^{s+1} \sum_{i=1}^{l} \binom{t-s-1}{i}\left(\frac{z_1}{z_j - z_1}\right)^{l-i+1}. \end{aligned}$$

Combining the last two terms we get that (8) holds for l replaced by $l+1$. Hence, (8) holds for $1 \leq l \leq m$. Letting $l = m$ in (8), we get that

$$u(t) = \left(\frac{z_1}{z_j - z_1}\right)^m \left(\frac{z_j}{z_1}\right)^t - \left(\frac{z_j}{z_1}\right)^{s+1} \sum_{i=0}^{m-1} \frac{(t-s-1)^{(i)}}{i!} \left(\frac{z_1}{z_j - z_1}\right)^{m-i}.$$

Hence, from earlier in the proof,

$$K(t,s) = \sum_{j=2}^{k} \frac{z_1^{t-m}}{p_n z_j^{s+1} q'(z_j)} \left(\frac{z_1}{z_j - z_1}\right)^m \left(\frac{z_j}{z_1}\right)^t - \sum_{j=2}^{k} \frac{z_1^{t-m}}{p_n z_j^{s+1} q'(z_j)} \left(\frac{z_j}{z_1}\right)^{s+1} \sum_{i=0}^{m-1} \frac{(t-s-1)^{(i)}}{i!} \left(\frac{z_1}{z_j - z_1}\right)^{m-i}.$$

Using $p_n q'(z_j)(z_j - z_1)^m = p'(z_j)$, we get the desired result:

$$K(t,s) = \sum_{j=2}^{k} \frac{z_j^{t-s-1}}{p'(z_j)} - \sum_{i=0}^{m-1} \left(\sum_{j=2}^{k} \frac{(z_j - z_1)^i}{p'(z_j)} \right) \frac{(t-s-1)^{(i)}}{i!} z_1^{t-s-1}.$$

Using Theorem 1, we get the following example.

Example 1. The Cauchy function for

$$(E-2)^3(E-1)(E-3)(E-4)y(t) = 0$$

is given by

$$K(t,s) = -\frac{1}{6} - \frac{1}{2} 3^{t-s-1} + \frac{1}{24} 4^{t-s-1} + \frac{15}{24} 2^{t-s-1} + \frac{1}{4}(t-s-1) 2^{t-s-2} + \frac{1}{4}(t-s-1)^{(2)} 2^{t-s-3}.$$

Theorem 2. *The Cauchy function for*

$$p_n(E-z_1)^{m_1}(E-z_2)^{m_2}y(t)=0$$

is given by

$$\begin{aligned}K(t,s)&=\frac{(-1)^{m_2}}{p_n(z_2-z_1)^{n-1}}\sum_{i=0}^{m_1-1}\frac{(n-i-2)^{(m_2-1)}}{(m_2-1)!}\\&\times(z_2-z_1)^i\frac{(t-s-1)^{(i)}}{i!}z_1^{t-s-i-1}+\frac{(-1)^{m_1}}{p_n(z_1-z_2)^{n-1}}\\&\times\sum_{i=0}^{m_2-1}\frac{(n-i-2)^{(m_1-1)}}{(m_1-1)!}(z_1-z_2)^i\frac{(t-s-1)^{(i)}}{i!}z_2^{t-s-i-1}\end{aligned}$$

where $n=m_1+m_2$.

Proof. Take $M=p_n(E-z_1)^{m_1}$, $N=(E-z_2)^{m_2}$. Then by (2),

(9)
$$\begin{aligned}K(t,s)&=\sum_{\tau=s+1}^{t-1}\frac{(t-\tau-1)^{(m_1-1)}}{p_n(m_1-1)!}z_1^{t-\tau-m_1}\frac{(\tau-s-1)^{(m_2-1)}}{(m_2-1)!}z_2^{\tau-s-m_2}\\&=\frac{z_1^{t-m_1}}{p_nz_2^{s+m_2}}u(t),\end{aligned}$$

where

$$u(t)=\sum_{\tau=s+1}^{t-1}\frac{(t-\tau-1)^{(m_1-1)}}{(m_1-1)!}\left[\frac{(\tau-s-1)^{(m_2-1)}}{(m_2-1)!}\left(\frac{z_2}{z_1}\right)^{\tau}\right].$$

Since $(t-\tau-1)^{(m_1-1)}/(m_1-1)!$ is the Cauchy function for $\Delta^{m_1}u(t)=0$, we have that, for each fixed s, $u(t)$ solves the initial value problem

$$\Delta^{m_1}u(t)=\frac{(t-s-1)^{(m_2-1)}}{(m_2-1)!}\left(\frac{z_2}{z_1}\right)^t \tag{10}$$

$$u(s+i)=0,\qquad 1\le i\le m_1. \tag{11}$$

We will now prove by induction on k, $1 \le k \le m_1$, that

$$\begin{aligned}\Delta^{m_1-k}u(t) = \left(\frac{z_1}{z_2-z_1}\right)^k \sum_{i=0}^{m_2-1} \frac{(m_2+k-i-2)^{(k-1)}}{(k-1)!} \\ \times \left(\frac{z_2}{z_1-z_2}\right)^{m_2-i-1} \frac{(t-s-1)^{(i)}}{i!}\left(\frac{z_2}{z_1}\right)^t \\ - \sum_{i=0}^{k-1}\left(\frac{z_1}{z_2-z_1}\right)^{k-i} \frac{(m_2+k-i-2)^{(k-i-1)}}{(k-i-1)!} \\ \times \left(\frac{z_2}{z_1-z_2}\right)^{m_2-1} \frac{(t-s-1)^{(i)}}{i!}\left(\frac{z_2}{z_1}\right)^{s+1}.\end{aligned} \tag{12}$$

We will use the formula

$$\begin{aligned}\sum_{\tau=s+1}^{t-1} \frac{(\tau-s-1)^{(i)}}{i!}\left(\frac{z_2}{z_1}\right)^{\tau} \\ = \frac{z_1}{z_2-z_1}\sum_{j=0}^{i}\left(\frac{z_2}{z_1-z_2}\right)^{i-j}\frac{(t-s-1)^{(j)}}{j!}\left(\frac{z_2}{z_1}\right)^t \\ - \left(\frac{z_2}{z_1-z_2}\right)^{i}\frac{z_1}{z_2-z_1}\left(\frac{z_2}{z_1}\right)^{s+1},\end{aligned} \tag{13}$$

which can be easily verified by repeated summation by parts.

Summing both sides of (10) from $s+1$ to $t-1$ and using (11), we get that

$$\Delta^{m_1-1}u(t) = \sum_{\tau=s+1}^{t-1} \frac{(\tau-s-1)^{(m_2-1)}}{(m_2-1)!}\left(\frac{z_2}{z_1}\right)^{\tau}.$$

Using (13) with $i = m_2 - 1$, we get that

$$\begin{aligned}\Delta^{m_1-1}u(t) = \frac{z_1}{z_2-z_1}\sum_{i=0}^{m_2-1}\left(\frac{z_2}{z_1-z_2}\right)^{m_2-i-1}\frac{(t-s-1)^{(i)}}{i!}\left(\frac{z_2}{z_1}\right)^t \\ - \left(\frac{z_2}{z_1-z_2}\right)^{m_2-1}\frac{z_1}{z_2-z_1}\left(\frac{z_2}{z_1}\right)^{s+1}.\end{aligned}$$

Hence, (12) holds for $k = 1$.

Now assume that (12) holds for a fixed k, $1 \le k \le m_1$. Summing both sides of (12) from $s+1$ to $t-1$ and using (11) and (13), we get that

$$\begin{aligned}
\Delta^{m_1-k-1}u(t) &= \left(\frac{z_1}{z_2-z_1}\right)^k \sum_{i=0}^{m_2-1} \frac{(m_2+k-i-2)^{(k-1)}}{(k-1)!}\left(\frac{z_2}{z_1-z_2}\right)^{m_2-i-1} \\
&\quad \times \left\{\frac{z_1}{z_2-z_1}\sum_{j=0}^{i}\left(\frac{z_1}{z_1-z_2}\right)^{i-j}\binom{t-s-1}{j}\left(\frac{z_2}{z_1}\right)^t \right. \\
&\qquad \left. - \left(\frac{z_2}{z_1-z_2}\right)^i \frac{z_1}{z_2-z_1}\left(\frac{z_2}{z_1}\right)^{s+1}\right\} \\
&\quad - \sum_{i=0}^{k-1}\left(\frac{z_1}{z_2-z_1}\right)^{k-i}\frac{(m_2+k-i-2)^{(k-i-1)}}{(k-i-1)!} \\
&\qquad \times \left(\frac{z_2}{z_1-z_2}\right)^{m_2-1}\binom{t-s-1}{i+1}\left(\frac{z_2}{z_1}\right)^{s+1}.
\end{aligned}$$

Interchanging order of summations in the first term and changing the index of summation in the last term, we get that

$$\begin{aligned}
\Delta^{m_1-k-1}u(t) &= \left(\frac{z_1}{z_2-z_1}\right)^{k+1}\sum_{j=0}^{m_2-1}\left(\frac{z_2}{z_1-z_2}\right)^{m_2-j-1}\frac{(t-s-1)^{(j)}}{j!} \\
&\quad \times \left(\frac{z_2}{z_1}\right)^t \sum_{i=j}^{m_2-1}\frac{(m_2+k-i-2)^{(k-1)}}{(k-1)!} - \left(\frac{z_1}{z_2-z_1}\right)^{k+1} \\
&\quad \times \left(\sum_{i=0}^{m_2-1}\frac{(m_2+k-i-2)^{(k-1)}}{(k-1)!}\right)\left(\frac{z_2}{z_1-z_2}\right)^{m_2-1}\left(\frac{z_2}{z_1}\right)^{s+1} \\
&\quad - \sum_{i=1}^{k}\left(\frac{z_1}{z_2-z_1}\right)^{k-i+1}\frac{(m_2+k-i-1)^{(k-i)}}{(k-i)!} \\
&\quad \times \left(\frac{z_2}{z_1-z_2}\right)^{m_2-1}\frac{(t-s-1)^{(i)}}{i!}\left(\frac{z_2}{z_1}\right)^{s+1}.
\end{aligned}$$

Note that

$$\sum_{i=j}^{m_2-1} \frac{(m_2+k-i-2)^{(k-1)}}{(k-1)!} = \sum_{l=0}^{m_2-j+1} \frac{(l+k-1)^{(k-1)}}{(k-1)!} = \frac{(m_2+k-j-1)^{(k)}}{k!}.$$

Using this, we get that

$$\begin{aligned}\Delta^{m_1-k-1}u(t) = &\left(\frac{z_1}{z_2-z_1}\right)^{k+1} \sum_{j=0}^{m_2-1} \frac{(m_2+k-j-1)^{(k)}}{k!} \\ &\times \left(\frac{z_2}{z_1-z_2}\right)^{m_2-j+1} \frac{(t-s-1)^{(j)}}{j!}\left(\frac{z_2}{z_1}\right)^{t} \\ &- \left(\frac{z_1}{z_2-z_1}\right)^{k+1} \frac{(m_2+k-1)^{(k)}}{k!}\left(\frac{z_2}{z_1-z_2}\right)^{m_2-1}\left(\frac{z_2}{z_1}\right)^{s+1} \\ &- \sum_{i=1}^{k}\left(\frac{z_1}{z_2-z_1}\right)^{k-i+1} \frac{(m_2+k-i-1)^{(k-i)}}{(k-i)!} \\ &\times \left(\frac{z_2}{z_1-z_2}\right)^{m_2-1} \frac{(t-s-1)^{(i)}}{i!}\left(\frac{z_2}{z_1}\right)^{s+1}.\end{aligned}$$

This last equality implies that (12) holds.

Letting $k = m_1$ in (12), using $m_1 + m_2 = n$ and

$$\frac{(m_2+k-i-2)^{(k-i-1)}}{(k-i-1)!} = \frac{(m_2+k-i-2)^{(m_2-1)}}{(m_2-1)!}$$

we get that

$$\begin{aligned}u(t) = &\left(\frac{z_1}{z_2-z_1}\right)^{m_1} \sum_{i=0}^{m_2-1} \frac{(n-i-2)^{(m_1-1)}}{(m_1-1)!} \\ &\times \left(\frac{z_2}{z_1-z_2}\right)^{m_2-i+1} \frac{(t-s-1)^{(i)}}{i!}\left(\frac{z_2}{z_1}\right)^{t} \\ &- \sum_{i=0}^{m_1-1}\left(\frac{z_1}{z_2-z_1}\right)^{m_1-i} \frac{(n-i-2)^{(m_2-1)}}{(m_2-1)!} \\ &\times \left(\frac{z_2}{z_1-z_2}\right)^{m_2-1} \frac{(t-s-1)^{(i)}}{i!}\left(\frac{z_2}{z_1}\right)^{s+1}.\end{aligned}$$

Hence, by (9),

$$K(t,s) = \frac{(-1)^{m_2}}{p_n(z_2-z_1)^{n-1}} \sum_{i=0}^{m_1-1} \frac{(n-i-2)^{(m_2-1)}}{(m_2-1)!} \times (z_2-z_1)^i \frac{(t-s-1)^{(i)}}{i!} z_1^{t-s-i-1} + \frac{(-1)^{m_1}}{p_n(z_1-z_2)^{n-1}} \times \sum_{i=0}^{m_2-1} \frac{(n-i-2)^{(m_1-1)}}{(m_1-1)!} (z_1-z_2)^i \frac{(t-s-1)^{(i)}}{i!} z_2^{t-s-i-1}.$$

□

Example 2. Using Theorem 2 and some trigonometric identities, the Cauchy function for

$$(E^2+1)^m y(t) = 0$$

is given by

$$K(t,s) = \frac{\sin[\frac{\pi}{2}(t-s-1)]}{4^{m-1}(m-1)!} \sum_{j=0}^{m-1} (-1)^j \frac{(2m-j-2)^{(m-1)} 2^j (t-s-1)^{(j)}}{j!}.$$

Example 3. From Theorem 2, the Cauchy function for

$$(E+1)^3(E-2)^2 y(t) = 0$$

is given by

$$K(t,s) = \left(-\frac{1}{27} + \frac{t-s-1}{54}\right) 2^{t-s-1} + \left(\frac{1}{27} - \frac{2}{27}(t-s-1) + \frac{1}{18}(t-s-1)^{(2)}\right)(-1)^{t-s-1}.$$

Theorem 3. *The Cauchy function for* (3), *where* $p(z) = p_n(z - z_1)^{m_1}(z - z_2)^{m_2}(z - z_3)^{m_3}$, *is given by*

$$\begin{aligned} K(t,s) =& a \sum_{j=0}^{m_1-1} \left(\sum_{i=j}^{m_1-1} b_i \binom{m_3+i-j-1}{m_3-1} \right) \\ & \times (z_3 - z_1)^j \binom{t-s-1}{j} z_1^{t-s-j-1} \\ & + c \sum_{j=0}^{m_2-1} \left(\sum_{i=j}^{m_2-1} d_i \binom{m_3+i-j-1}{m_3-1} \right) \\ & \times (z_3 - z_2)^j \binom{t-s-1}{j} z_2^{t-s-j-1} \\ & - \sum_{j=0}^{m_3-1} \Bigg[a \Bigg(\sum_{i=0}^{m_1-1} b_i \binom{m_3+i-j-1}{m_3-1} \Bigg) (z_1 - z_3)^j \\ & \qquad + c \Bigg(\sum_{i=0}^{m_2-1} d_i \binom{m_3+i-j-1}{m_3-1} \Bigg) (z_2 - z_3)^j \Bigg] \\ & \times \binom{t-s-1}{j} z_3^{t-s-j-1} \end{aligned}$$

where

$$\begin{aligned} a &= \frac{(-1)^{m_2+m_3}}{p_n(z_2 - z_1)^{m_1+m_2-1}(z_3 - z_1)^{m_3}} \\ b_i &= \binom{m_1+m_2-i-2}{m_2-1} \left(\frac{z_2 - z_1}{z_3 - z_1} \right)^i, \qquad 0 \le i \le m_1 - 1 \\ c &= \frac{(-1)^{m_1+m_3}}{p_n(z_1 - z_2)^{m_1+m_2-1}(z_3 - z_2)^{m_3}} \\ d_i &= \binom{m_1+m_2-i-2}{m_1-1} \left(\frac{z_1 - z_2}{z_3 - z_2} \right)^i, \qquad 0 \le i \le m_2 - 1. \end{aligned}$$

Proof. By (2) with $M = (E - z_3)^{m_3}$ and $N = p_n(E - z_1)^{m_1}(E - z_2)^{m_2}$, we get, using (5) and Theorem 2,

$$K(t,s)=\sum_{\tau=s+1}^{t-1}\left\{\frac{(-1)^{m_2}}{p_n(z_2-z_1)^{m_1+m_2-1}}\sum_{i=0}^{m_1-1}\binom{m_1+m_2-i-2}{m_2-1}\right.$$

$$\times(z_2-z_1)^i\binom{t-\tau-1}{i}z_1^{t-\tau-i-1}$$

$$+\frac{(-1)^{m_1}}{p_n(z_1-z_2)^{m_1+m_2-1}}\sum_{i=0}^{m_2-1}\binom{m_1+m_2-i-2}{m_1-1}$$

$$\left.\times(z_1-z_2)^i\binom{t-\tau-1}{i}z_2^{t-\tau-i-1}\right\}\binom{\tau-s-1}{m_3-1}z_3^{\tau-s-m_3}$$

$$=\frac{(-1)^{m_2}}{p_n(z_2-z_1)^{m_1+m_2-1}}\sum_{i=0}^{m_1-1}\left\{\binom{m_1+m_2-i-2}{m_2-1}\right.$$

$$\left.\times(z_2-z_1)^i\sum_{\tau=s+1}^{t-1}\binom{t-\tau-1}{i}\binom{\tau-s-1}{m_3-1}z_1^{t-\tau-i-1}z_3^{\tau-s-m_3}\right\}$$

$$+\frac{(-1)^{m_1}}{p_n(z_1-z_2)^{m_1+m_2-1}}\sum_{i=0}^{m_2-1}\left\{\binom{m_1+m_2-i-2}{m_1-1}\right.$$

$$\left.\times(z_1-z_2)^i\sum_{\tau=s+1}^{t-1}\binom{t-\tau-1}{i}\binom{\tau-s-1}{m_3-1}z_2^{t-\tau-i-1}z_3^{\tau-s-m_3}\right\}.$$

But the inner sums in these two terms are, from the proof of Theorem 2, the Cauchy functions for (3) with characteristic polynomials $(z-z_1)^{i+1}(z-z_3)^{m_3}$ and $(z-z_2)^{i+1}(z-z_3)^{m_3}$, respectively. Hence, by Theorem 2,

$$K(t,s)=\frac{(-1)^{m_2}}{p_n(z_2-z_1)^{m_1+m_2-1}}\sum_{i=0}^{m_1-1}\binom{m_1+m_2-i-2}{m_2-1}$$

$$\times(z_2-z_1)^i\left\{\frac{(-1)^{m_3}}{(z_3-z_1)^{i+m_3}}\sum_{j=0}^{i}\binom{i+m_3-j-1}{m_3-1}\right.$$

$$\times(z_3-z_1)^j\binom{t-s-1}{j}z_1^{t-s-j-1}$$

$$+\frac{(-1)^{i+1}}{(z_1-z_3)^{i+m_3}}\sum_{j=0}^{m_3-1}\binom{i+m_3-j-1}{i}$$

$$
\begin{aligned}
&\times (z_1 - z_3)^i \binom{t-s-1}{j} z_3^{t-s-j-1}\Bigg\}\\
&+ \frac{(-1)^{m_1}}{p_n (z_1 - z_2)^{m_1+m_2-1}} \sum_{i=0}^{m_2-1} \binom{m_1+m_2-i-2}{m_1-1}\\
&\times (z_1 - z_2)^i \Bigg\{ \frac{(-1)^{m_3}}{(z_3 - z_2)^{i+m_3}} \sum_{j=0}^{i} \binom{i+m_3-j-1}{m_3-1}\\
&\times (z_3 - z_2)^j \binom{t-s-1}{i} z_2^{t-s-j-1}\\
&+ \frac{(-1)^{i+1}}{(z_2 - z_3)^{i+m_3}} \sum_{j=0}^{m_3-1} \binom{i+m_3-j-1}{i}\\
&\times (z_2 - z_3)^i \binom{t-s-1}{j} z_3^{t-s-j-1}\Bigg\}\\
&= a \sum_{i=0}^{m_1-1} b_i \sum_{j=0}^{i} \binom{i+m_3-j-1}{m_3-1}\\
&\times (z_3 - z_1)^j \binom{t-s-1}{j} z_1^{t-s-j-1}\\
&+ c \sum_{i=0}^{m_2-1} d_i \sum_{j=0}^{i} \binom{i+m_3-j-1}{m_3-1}\\
&\times (z_3 - z_2)^j \binom{t-s-1}{j} z_2^{t-s-j-1}\\
&- a \sum_{i=0}^{m_1-1} b_i \sum_{j=0}^{m_3-1} \binom{m_3+i-j-1}{i}\\
&\times (z_1 - z_3)^j \binom{t-s-1}{j} z_3^{t-s-j-1}\\
&- c \sum_{i=0}^{m_2-1} d_i \sum_{j=0}^{m_3-1} \binom{m_3+i-j-1}{i}\\
&\times (z_2 - z_3)^j \binom{t-s-1}{j} z_3^{t-s-j-1}.
\end{aligned}
$$

Interchanging the order of summation in each of these terms, we easily get the formula for $K(t, s)$ given in the statement of the theorem. □

From Theorem 3, we get the following example.

Example 4. The Cauchy function for

$$(E-1)^2(E-2)^3(E-3)^2y(t)=0$$

is given by

$$\begin{aligned}K(t,s)&=-1-\frac{1}{4}(t-s-1)+2^{t-s}+(t-s-1)^{(2)}2^{t-s-4}\\&\quad-3^{t-s-1}+\frac{1}{4}(t-s-1)3^{t-s-2}.\end{aligned}$$

Theorem 4. *The Cauchy function for* (3) *with*

$$p(z)=p_n(z-z_1)^{m_1}(z-z_2)^{m_2}(z-z_3)\cdots(z-z_k),$$

is given by

$$\begin{aligned}K(t,s)=\sum_{j=3}^{k}\frac{z_j^{t-s-1}}{p'(z_j)}&-\sum_{j=0}^{m_1-1}\Bigg[\sum_{l=3}^{k}\frac{(z_l-z_1)^j}{p'(z_l)}\\&-\sum_{i=0}^{m_2-1}\sum_{l=3}^{k}\frac{(z_l-z_2)^i}{q'(z_l)}\binom{m_1+i-j-1}{i}\Bigg]\\&\qquad\times\frac{(-1)^i}{(z_2-z_1)^{m_1+i-j}}\binom{t-s-1}{j}z_1^{t-s-j-1}\\&-\sum_{j=0}^{m_2-1}\Bigg[\sum_{i=j}^{m_2-1}\Bigg(\sum_{l=3}^{k}\frac{(z_l-z_2)^i}{q'(z_l)}\Bigg)\binom{m_1+i-j-1}{m_1-1}\\&\qquad\times\frac{(-1)^{m_i}}{(z_1-z_2)^{m_1+i-j}}\Bigg]\binom{t-s-1}{j}z_2^{t-s-j-1}\end{aligned}$$

where $q(z)=p_n(z-z_2)^{m_2}(z-z_3)\cdots(z-z_k)$.

Proof. Take $M = p_n(E - z_2)^{m_2}(E - z_3)\cdots(E - z_k)$ and $N = (E - z_1)^{m_1}$. Then, by Theorem 1, (5) and (2), we get that

$$\begin{aligned} K(t,s) = \sum_{l=3}^{k} \frac{1}{q'(z_l)} \sum_{\tau=s+1}^{t-1} \binom{t-\tau-1}{m_1-1} z_1^{t-\tau-m_1} z_l^{\tau-s-1} \\ - \sum_{i=0}^{m_2-1} \sum_{l=3}^{k} \frac{(z_l - z_2)^i}{q'(z_l)} \sum_{\tau=s+1}^{t-1} \binom{t-\tau-1}{m_1-1} \\ \times z_1^{t-\tau-m_1} \binom{\tau-s-1}{i} z_2^{\tau-s-i-1}. \end{aligned}$$

The inside sums in the last two terms are the Cauchy functions for (3) with characteristic polynomials $(z-z_1)^{m_1}(z-z_l)$ and $(z-z_1)^{m_1}(z-z_2)^{i+1}$, respectively. Hence, using Theorems 1 and 2, we get that

$$\begin{aligned} K(t,s) = {} & \sum_{l=3}^{k} \frac{z_l^{t-s-1}}{q'(z_l)(z_l - z_1)^{m_1}} \\ & - \sum_{l=3}^{k} \frac{1}{q'(z_l)} \sum_{j=0}^{m_1-1} \frac{(z_l - z_1)^j}{(z_l - z_1)^{m_1}} \binom{t-s-1}{j} z_1^{t-s-j-1} \\ & - \sum_{i=0}^{m_2-1} \sum_{l=3}^{k} \frac{(z_l - z_2)^i (-1)^{i+1}}{q'(z_l)(z_2 - z_1)^{m_1+i}} \\ & \times \sum_{j=0}^{m_1-1} \binom{m_1+i-j-1}{i} (z_2 - z_1)^j \binom{t-s-1}{j} z_1^{t-s-j-1} \\ & - \sum_{i=0}^{m_2-1} \sum_{l=3}^{k} \frac{(-1)^{m_1}(z_l - z_2)^i}{q'(z_l)(z_1 - z_2)^{m_1+i}} \\ & \times \sum_{j=0}^{i} \binom{m_1+i-j-1}{m_1-1} (z_1 - z_2)^j \binom{t-s-1}{j} z_2^{t-s-j-1}. \end{aligned}$$

Changing the order of summation in each term and using $p'(z_l) =$

$(z_l - z_1)^{m_1} q'(z_l)$, we get that

$$\begin{aligned} K(t,s) &= \sum_{l=3}^{k} \frac{z_l^{t-s-1}}{p'(z_l)} \\ &- \sum_{j=0}^{m_1-1} \sum_{l=3}^{k} \frac{(z_l - z_1)^j}{p'(z_l)} \binom{t-s-1}{j} z_1^{t-s-j-1} \\ &+ \sum_{j=0}^{m_1-1} \sum_{i=0}^{m_2-1} \left(\sum_{l=3}^{k} \frac{(z_l - z_2)^j}{q'(z_l)} \right) \binom{m_1+i-j-1}{i} \\ &\times \frac{(-1)^i}{(z_2 - z_1)^{m_1+i-j}} \binom{t-s-1}{j} z_1^{t-s-j-1} \\ &- \sum_{j=0}^{m_2-1} \sum_{i=j}^{m_2-1} \left(\sum_{l=3}^{k} \frac{(z_l - z_2)^i}{q'(z_l)} \right) \binom{m_1+i-j-1}{m_1-1} \\ &\times \frac{(-1)^{m_1}}{(z_1 - z_2)^{m_1+i-j}} \binom{t-s-1}{j} z_2^{t-s-j-1}. \end{aligned}$$

This leads to the final result. □

Example 5. The Cauchy function for

$$(E-1)(E-2)(E+1)^3(E-1/2)^2 y(t) = 0$$

is given by

$$\begin{aligned} K(t,s) = &-\frac{1}{2} + \frac{4}{243} 2^{t-s-1} + \frac{64}{243} \left(\frac{1}{2}\right)^{t-s-1} \\ &+ \frac{32}{81}(t-s-1)\left(\frac{1}{2}\right)^{t-s-2} \\ &+ \frac{107}{486}(-1)^{t-s-1} + \frac{13}{81}(t-s-1)(-1)^{t-s-2} \\ &+ \frac{1}{27}(t-s-1)^{(2)}(-1)^{t-s-3}. \end{aligned}$$

Example 6. The Cauchy function for

$$(E-1)^2(E-2)^2(E-3)(E-4)y(t) = 0$$

is given by

$$K(t,s) = -\frac{1}{4}3^{t-s-1} + \frac{1}{36}4^{t-s-1} + \frac{17}{36} + \frac{1}{6}(t-s-1) - \frac{1}{4}2^{t-s-1} + \frac{1}{2}(t-s-1)2^{t-s-2}.$$

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UNIVERSITY OF NEBRASKA-LINCOLN, LINCOLN, NEBRASKA 68588-0323