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ON THE STABILITY OF A WAVETRAIN CAUSED BY INTERACTING WAVE MODES

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ABSTRACT. This paper consists of an analysis of the stability of Wilton ripples to long wave perturbations. Both longitudinal and transverse perturbations are considered and regions of instability are identified.

1. Introduction. Probably the simplest interaction which is present in the theory of capillary-gravity waves is the one which occurs between the fundamental mode and its second harmonic. This leads to the creation of the so-called Wilton ripples and it is the purpose of this paper to study the stability of these waves. Both transverse and longitudinal perturbations will be considered. It will be shown that the set of wavenumbers of the unstable perturbations usually consists of an open interval but may exceptionally consist of the union of two intervals.

These waves have attracted the attention of a number of researchers. In the 1960's, Nayfeh wrote a number of papers [**9, 10**] dealing with the Wilton ripple phenomenon. He used the method of multiple scales to obtain power series expansions for the wave profiles. Later Chen and Saffman [**3**] considered the problem of perfectly general capillary-gravity wave interactions. They employed a weakly nonlinear theory to obtain formal series expansions for the wavetrains but their results were largely confirmed by the work of Jones and Toland [**8, 11**] who considered the same problem from a rigorous mathematical point of view, making use of the tools of applied functional analysis and bifurcation theory.

Recently a certain amount of work has been carried out on the stability of resonant waves. In [**7**] it was shown that the nature of the stability of deep water Wilton ripples is dependent on the nature of the roots of a certain quartic equation. In that paper an approximate form of that equation was used. However, in this paper we use the full form of the equation and hence more accurate results are obtained. The

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methods employed, both in this paper and in [**7**], are a development of those of Benjamin and Feir [**2**] who first showed that a uniform train of deep water gravity waves is unstable in the presence of sidebands and also of Zakharov [**12**] who first showed that the evolution of a wavetrain may be described up to cubic order by the nonlinear Schrödinger equation. The nonlinear Schrödinger equation, or modifications of it, has been used by a large number of researchers to describe the evolution and stability of various types of waves, see [**4, 5, 6**].

2. The mathematical set up. In this section we shall sketch the derivation of the third order evolution equations which describe the motion of Wilton ripples. Our method closely follows that of [**7**]. Consider the irrotational motion of an ideal fluid which is contained in a deep channel and acted upon by the forces of gravity and surface tension. The equations which describe the motion are

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \qquad z \le H, \tag{2.1a}$$

$$\phi \to 0, \qquad z \to -\infty, \tag{2.1b}$$

$$\phi_z - H_t - \phi_x H_x - \phi_y H_y = 0, \qquad z = H \tag{2.1c}$$

$$\begin{aligned} &\phi_t + (\phi_x^2 + \phi_y^2 + \phi_z^2)/2 + gH \\ &\quad - \frac{S(H_{xx}(1+H_y^2) + H_{yy}(1+H_x^2) - 2H_x H_y H_{xy})}{(1+H_x^2+H_y^2)^{3/2}} = 0, \qquad z = H. \end{aligned} \tag{2.1d}$$

Here $\phi(x, y, z, t)$ is the stream function; $H(x, y, t)$ is the elevation of the free surface, g is the force of gravity; S is the surface tension; (2.1c) is the kinematic condition and (2.1d) is the result of applying Bernoulli's condition along the free surface.

We shall be considering the motion of a wave train caused by the interaction of a sinusoidal wave of wavenumber k and frequency w and its second harmonic. It is a straightforward exercise to show that such waves are solutions of the linearized form of (2.1) provided $w^2 = 3gk/2$ and $S = g/2k^2$. In order to develop a "weakly nonlinear" theory of these waves, we shall introduce a parameter ε which satisfies $|\varepsilon| \ll 1$ and which acts as a measure of the wave steepness. We also introduce the following "slow variables":

$$
\begin{aligned}
X = \varepsilon x, \qquad Y = \varepsilon y, \qquad T = \varepsilon t,\\
T_1 = \varepsilon^2 t \quad \text{and set} \quad E \equiv e^{i(kx-wt)}.
\end{aligned}
\tag{2.2}
$$

The next step is to develop ϕ and H in ascending powers of ε. Taking (2.1a), (2.1b) into account, it turns out that, as far as the relevant order,

$$
\begin{aligned}
\phi =& \varepsilon\Bigg(A + \varepsilon(A_2 - izA_x)\\
&\qquad + \varepsilon^2\left(A_3 - izA_{2X} - \frac{z^2}{2}A_{XX} - \frac{z}{2k}A_{YY}\right)\Bigg)Ee^{kz}\\
&+ \varepsilon\Bigg(B + \varepsilon(B_2 - izB_X)\\
&\qquad + \varepsilon^2\left(B_3 - izB_{2X} - \frac{z^2}{2}B_{XX} - \frac{z}{4k}B_{YY}\right)\Bigg)E^2e^{2kz}\\
&+ \varepsilon^2 C_3E^3e^{3kz} + \varepsilon^2 C_4 E^4 e^{4kz} + \text{(c.c.)}
\end{aligned}
\tag{2.3}
$$

and

$$
\begin{aligned}
H = \varepsilon\left(\frac{ik}{w}A + \varepsilon\alpha_2 + \varepsilon^2\alpha_3\right)E + \varepsilon\left(\frac{ik}{w}B + \varepsilon\beta_2 + \varepsilon^2\beta_3\right)E^2\\
+ \varepsilon^2\gamma_3E^3 + \varepsilon^2\gamma_4E^4 + \text{(c.c.)}.
\end{aligned}
\tag{2.4}
$$

Here the coefficients A, B, etc., are functions of the slow variables only and (c.c.) denotes complex conjugate. The next step is the somewhat tedious one of substituting (2.3) and (2.4) into (2.1cd) and equating coefficient $\varepsilon^i E^j$ to zero. We merely list the results at each stage.

The terms which are linear in ε are satisfied automatically. The coefficients of $\varepsilon^2 E$ yield that

$$
\begin{aligned}
A_T + \frac{5w}{6k}A_X = -k^2A^*B \quad \text{and}\\
kA_2 + iw\alpha_2 = \frac{2ik^3}{w}A^*B - \frac{7i}{6}A_X.
\end{aligned}
\tag{2.5}
$$

(The asterisk denotes complex conjugate).

Consideration of the terms involving $\varepsilon^2 E^2$ yields

$$B_T + \frac{7w}{6k} B_X = \frac{k^2}{2} A^2 \quad \text{and} \qquad (2.6)$$
$$kB_2 + iw\beta_2 = -\frac{3ik^3}{4w} A^2 - \frac{i}{12} B_X.$$

At $\varepsilon^2 E^3$ and $\varepsilon^2 E^4$ we have

$$\begin{aligned} C_3 &= -\frac{9ik^2}{w} AB, \qquad \gamma_3 = \frac{6k^3}{w^2} AB, \\ C_4 &= -\frac{4ik^2}{w} B^2 \quad \text{and} \quad \gamma_4 = \frac{2k^3}{w^2} B^2. \end{aligned} \qquad (2.7)$$

When we consider the terms involving $\varepsilon^3 E$ we obtain two equations involving A_3 and α_3. If we eliminate A_3 and α_3 from these equations, we obtain an equation involving the other coefficients which can be simplified by means of (2.5)–(2.7). A similar calculation involving the terms of order $\varepsilon^3 E^2$ yields an analogous equation. Then, if the variables involved are rendered dimensionless by means of appropriate scalings, the consequence is that the motion is described by the following set of equations

$$A_T + \frac{5}{6} A_X = -A^* B, \qquad (2.8a)$$

$$B_T + \frac{7}{6} B_X = \frac{A^2}{2}, \qquad (2.8b)$$

$$2A_{T_1} + 2A_{2T} + \frac{5}{3} A_{2X} - \frac{11}{36} i A_{XX} - \frac{5i}{6} A_{YY} + \frac{3}{2} i |A|^2 A$$
$$-13i|B|^2 A + 2BA_2^* + \frac{i}{3} BA_X^* + 2B_2 A^* - \frac{i}{2} B_X A^* = 0, \qquad (2.8c)$$

and

$$2B_{T_1} + 2B_{2T} + \frac{7}{2} B_{2X} - \frac{23}{72} i B_{XX} - \frac{7i}{12} B_{YY}$$
$$-\frac{39}{2} i |A|^2 B - 8i|B|^2 B - 2AA_2 + \frac{19}{6} i A A_X = 0. \qquad (2.8d)$$

The preceding set of four equations in four unknowns describes up to cubic order the evolution of a deep water wavetrain consisting of waves caused by the interaction of the fundamental wave and its second harmonic. They are the counterparts to the single nonlinear Schrödinger equation which models the motion in the nonresonant case. The equations are the same as those given in [**7**] with the correction of a misprint.

3. Stability of a wavetrain. A set of solutions to the system (2.8) is

$$\begin{aligned} &(3.1) \qquad & A &= i \exp\left[i\left(\pm\frac{T}{2} + w_1T_1\right)\right], \\ & & B &= \pm\frac{i}{2}\exp\left[2i\left(\pm\frac{T}{2} + w_1T_1\right)\right] \\ &(3.2) \qquad & A_2 &= i(\pm 1 - b)\exp\left[i\left(\pm\frac{T}{2} + w_1T_1\right)\right], \\ & & B_2 &= i\left(\frac{11}{4} \mp \frac{b}{2}\right)\exp\left[2i\left(\pm\frac{T}{2} + w_1T_1\right)\right], \end{aligned}$$

where $w_1 = 29/8 \mp b/2$ and b is a real arbitrary constant.

It is possible to use the relationships (2.5)–(2.7) to recover the corresponding wave profile H. If this is done, there results that

$$\begin{aligned} H = &-2\varepsilon\cos\chi + 2\varepsilon^2 b\cos\chi \mp \varepsilon\cos 2\chi - \varepsilon^2(4 \mp b)\cos 2\chi \\ &\mp 6\varepsilon^2\cos 3\chi - \varepsilon^2\cos 4\chi + 0(\varepsilon^3) \end{aligned} \tag{3.3}$$

where

$$\chi = x - wt \pm \frac{wT}{2} + \left(\frac{29}{8} \mp \frac{b}{2}\right)wT_1.$$

Thus, it may be seen that physically b represents the magnitude of the coefficient of the second-order occurrence of the fundamental mode. In order to study the stability of these waves, we shall make perturbations to the leading order terms as follows:

$$\begin{aligned} A &= i(1+\alpha')\exp\left[i\left(\pm\frac{T}{2} + w_1T_1\right) + i\theta'\right], \\ B &= \pm\frac{i}{2}(1+\beta')\exp\left[2i\left(\pm\frac{T}{2} + w_1T_1\right) + i\psi'\right]. \end{aligned} \tag{3.4}$$

We shall be considering plane wave perturbations so that

$$\begin{pmatrix} \alpha' \\ \beta' \\ \theta' \\ \psi' \end{pmatrix} = \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \\ \bar{\theta} \\ \bar{\psi} \end{pmatrix} \exp[i(\delta X + \sigma Y - \tau T_1)] \tag{3.5}$$

where the physical perturbation is represented by the real part of (3.5). The next step is to substitute (3.5) into the equations (2.8) and linearize. The result is that the amplitudes of the perturbations satisfy a set of four simultaneous equations. These equations can only be consistent if the following determinant is equal to zero:

(3.6)

$$\begin{vmatrix} \tau \mp 6\delta & -P_1 - 36(11 \mp 2b) & \pm 9\delta & 36(1 \mp b) \\ -P_1 - 108 & \tau \mp 6\delta & 18(15 \mp 2b) & \pm 9\delta \\ 456\delta & 288(\pm 1 - b) & \pm 2\tau & \mp P_2 + 72(\pm 7 + 2b) \\ 72(\pm 43 - 4b) & 456\delta & \mp P_2 + 72(\pm 11 + 2b) & \pm 2\tau \end{vmatrix}.$$

(In order to facilitate notation, τ has been scaled to $\tau/72$.)

In (3.6), $P_1 = 11\delta^2 + 30\sigma^2$ and $P_2 = 23\delta^2 + 42\sigma^2$.

We shall confine ourselves to discussing the instabilities which can occur when the perturbations are either in the same direction as the wave or normal to it.

3.1. *Perturbations in the x-direction only.* This corresponds to putting $\sigma = 0$ in (3.6). If we expand (3.6) and take the top sign, the condition that it should equal zero means that τ must satisfy the following quartic equation:

(3.7)

$$\begin{aligned} &4\tau^4 - (1013\delta^4 - 9792b\delta^2 + 8928\delta^2 + 62208b^2 - 264384b + 948672)\tau^2 \\ &\quad + (1710720b\delta^3 - 76496640\delta^3 - 8211456b^2\delta + 411505 92b\delta - 329391136\delta)\tau \\ &\quad + 640009\delta^8 - 1220472b\delta^6 - 27315 72\delta^6 \\ &\quad + 2509056b^2\delta^4 + 78029568b\delta^4 - 270502416\delta^4 \\ &\quad + 33592320b^3\delta^2 - 917630208b^2\delta^2 \\ &\quad + 6282883584b\delta^2 - 27673073 28\delta^2 + 2176782336b^2 \\ &\quad - 364611041 28b + 193461530112 = 0. \end{aligned}$$

Expanding (3.6) with the choice of the bottom sign yields the same equation as (3.7) with δ and b replaced by $-\delta$ and $-b$, respectively. Then, since δ and b can each take either sign, there is no loss of generality in working with (3.7) alone. In the stability discussion given in [**7**] an approximate form of equation (3.7) was used.

Equation (3.7) will be written in abbreviated form as

$$4\tau^4 - 6c\tau^2 + 4d\tau + e = 0. \tag{3.8}$$

Thus, the waves are stable if and only if all four roots of (3.8) are real and it follows from standard theory [**1**, p. 192] that this happens if and only if

$$c > 0, \qquad \Delta > 0 \quad \text{and} \quad \eta > 0 \tag{3.9}$$

where the functions $\eta(b, \delta)$ and $\eta(b, \delta)$ are defined by

$$\Delta(b, \delta) = (4e + 3c^2)^3 - 27(4ec - 4d^2 - c^3)^2 \tag{3.10}$$

and

$$\eta(b, \delta) = 9c^2 - 4e.$$

(Of course, the coefficients c, d and e are functions of b and δ but we have suppressed this for notational convenience.) Thus, to determine the nature of the stability of the waves it is necessary to determine the signs of the quantities in (3.9). The first thing to note is that a straightforward "completing the square" calculation shows that the coefficient c is positive for all choices of b and δ and hence the stability only depends on the signs of Δ and η. A further calculation shows that:

$$\begin{aligned}\eta(b, \delta) = \delta^8 &+ \frac{1}{5}(152576 - 768b)\delta^6 \\ &+ \frac{1}{25}(78145152 - 24184704\delta + 2244096b^2)\delta^4 \\ &+ \frac{1}{25}(755758080 - 1528713216b + 258895872b^2)\delta^2 \\ &+ \frac{1}{25}(-27103776768 + 10092625920b + 1890127872b^2 \\ &\qquad\qquad - 406093824b^3 + 47775744b^4).\end{aligned} \tag{3.11}$$

(We have scaled η by a positive constant in order to make the coefficient of the leading term equal to unity. Some of these calculations were carried out with the aid of MATHEMATICA.) It is clear that whatever value is taken by b, $\eta(b,\delta)$ is positive for sufficiently large δ. In fact, numerical calculations indicate that if $b < -2.934$, then η is nonnegative for all δ while if $b > -2.934$ then η is negative for certain values of δ.

The final quantity whose value influences the stablity of the waves is Δ. The complexity of Δ is such that we shall not present it here but merely remark that the leading order term is a positive multiple of δ^{24} and Δ is, like η, an even function in δ. Further, numerical calculations show that, whatever the value of b, Δ is positive for $\delta = 0$ (as well as for large $|\delta|$ of course) but always negative for some range of values of b.

Having made these observations, we are now in a position to derive the stability criteria. We shall consider a selection of values of b to show the various types of instability regions which can occur. Since Δ and η are even functions of δ, we shall confine ourselves to describing the results for positive δ.

Case 1. $b = 5$. This case is depicted in Figure 1(a) which, together with the other graphs in Figure 1, shows the qualitative configuration of the curves Δ and η. We do not give a vertical scale in Figure 1 because it is only intended to depict the relative positions of the curves. Figure 1(a) shows that Δ is negative for $0.854 < \delta < 1.595$ and $7.304 < \delta < 7.649$ and positive otherwise while η is negative for $1.176 < \delta < 7.459$ and positive otherwise. Hence, the waves are *unstable* if $0.854 < \delta < 7.649$, otherwise they are *stable*. (Here and elsewhere numerical quantities are given to three decimal places.)

Case 2. $b = 1$. This is depicted in Figure 1(b). The results are that the waves are *unstable* if $0 < \delta < 10.629$; otherwise they are stable.

Case 3. $b = -1$. This is depicted in Figure 1(c). The results are that the waves are *unstable* if $0 < \delta < 20.848$; otherwise they are stable.

Case 4. $b = -5$. This is depicted in Figure 1(d). The results are that

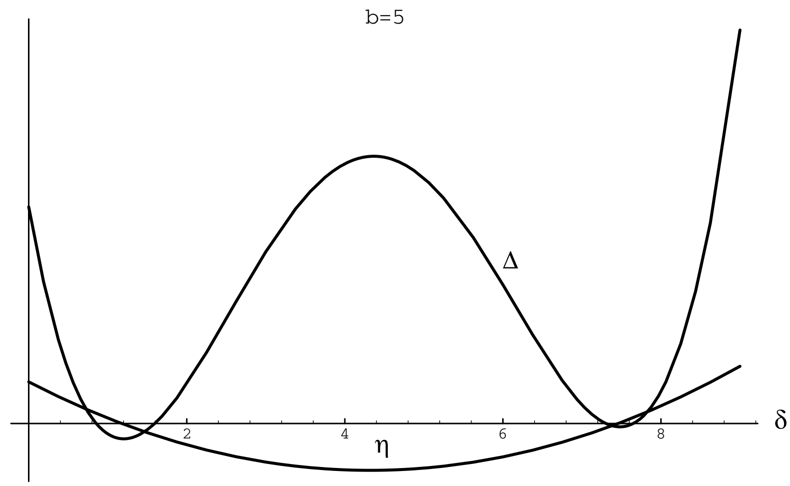

FIGURE 1(a).

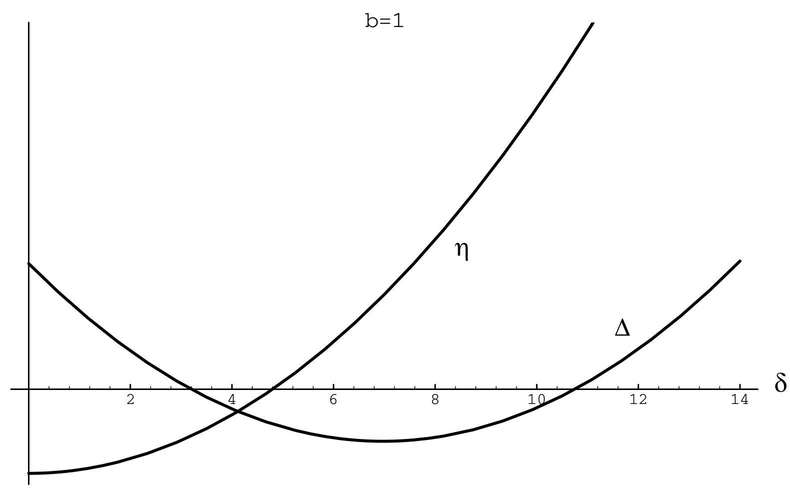

FIGURE 1(b).

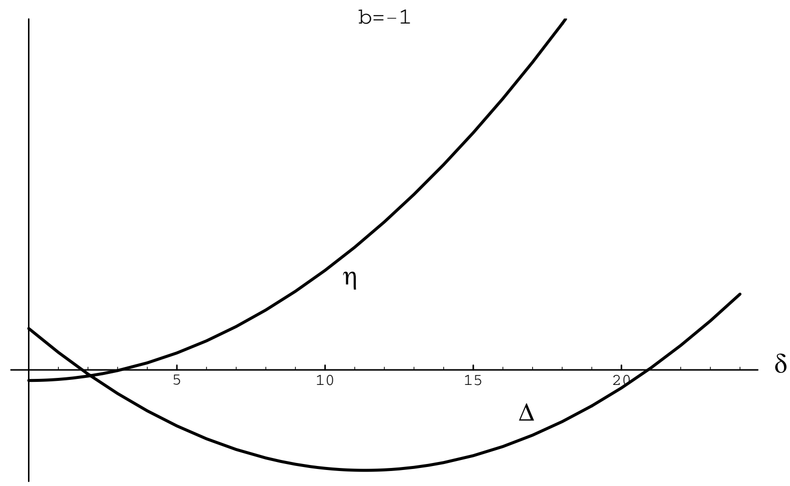

FIGURE 1(c).

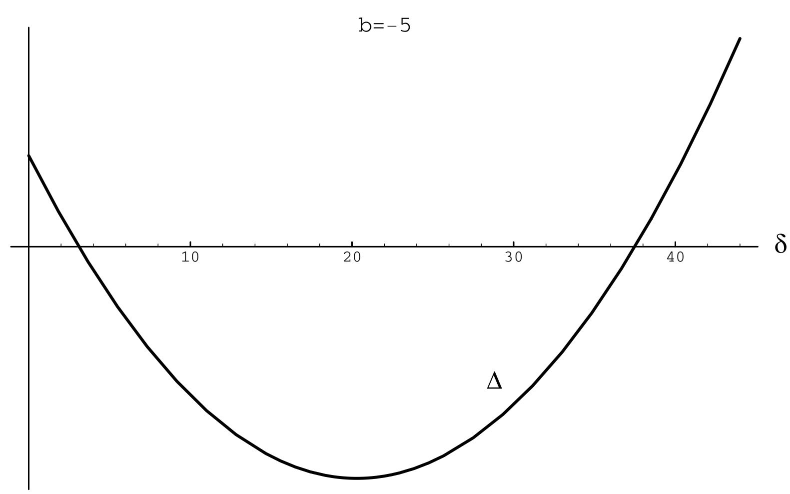

FIGURE 1(d).

the waves are *unstable* if $3.066 < \delta < 36.942$; otherwise they are stable.

3.2. *Perturbations in the y-direction only.* This case corresponds to putting $\delta = 0$ in (3.6). Then, on expanding this determinant and taking the top sign (as before, this leads to no loss of generality), we obtain:
(3.12)

$$
\begin{aligned}
4\tau^4 &- (5364\sigma^4 - 20736b\sigma^2 + 6048\sigma^2 + 62208b^2 - 264384b + 948672)\tau^2 \\
&+ 1587600\sigma^8 - 14696640b\sigma^6 - 22317120\sigma^6 \\
&+ 18662400b^2\sigma^4 + 353465856b\sigma^4 - 1454640768\sigma^4 \\
&+ 761425920b^3\sigma^2 - 1291064832b^2\sigma^2 \\
&+ 7936745472b\sigma^2 + 2642875776\sigma^2 + 2176782336b^2 \\
&- 364611041280b + 193461530112 = 0.
\end{aligned}
$$

One notes that, in contrast to (3.7), equation (3.12) contains no term in τ and this makes this case somewhat easier to analyze. We shall write (3.12) in abbreviated form as

$$4\tau^4 + p\tau^2 + q = 0. \tag{3.13}$$

It should first be noted that a straightforward calculation shows that the coefficient p is negative for all b and σ. Next, it is straightforward to show that the nature of the roots of (3.13) is governed by the following criteria:

(a) if $q < 0$, there are two real and two imaginary roots,

(b) if $q = 0$, the roots are all real,

(c) if $q > 0$ and $p^2 - 16q < 0$, the roots are all imaginary,

(d) if $q > 0$ and $p^2 - 16q \geq 0$, the roots are all real.

Before deriving the stability criteria, we remark that numerical calculations indicate that if $b > -1.440$, then q is positive for all values of σ; while if $b < -2.934$, then $p^2 - 16q$ is positive for all values of σ. (We shall not present the expression for $p^2 - 16q$ since it is rather complicated, but we remark that the leading order term is a positive multiple of σ^8.)

We can now present the stability results. As before, we take a selection of values of b, in order to show the various stability regions

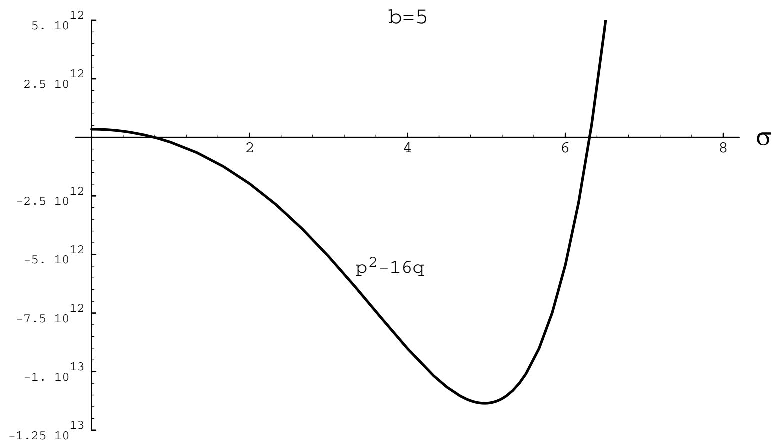

FIGURE 2(a).

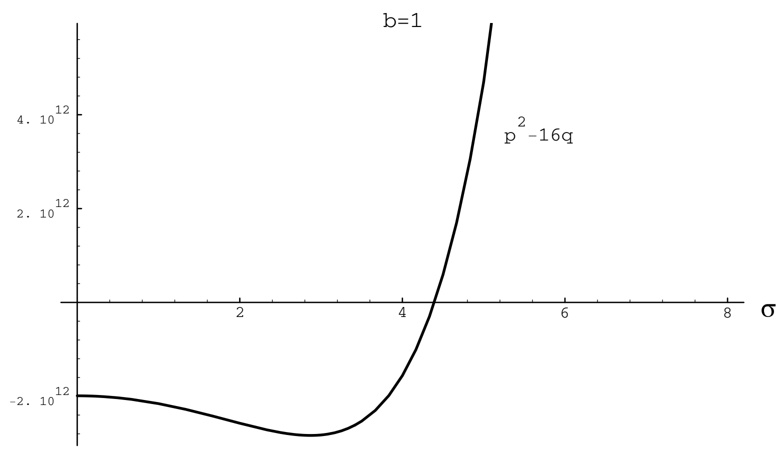

FIGURE 2(b).

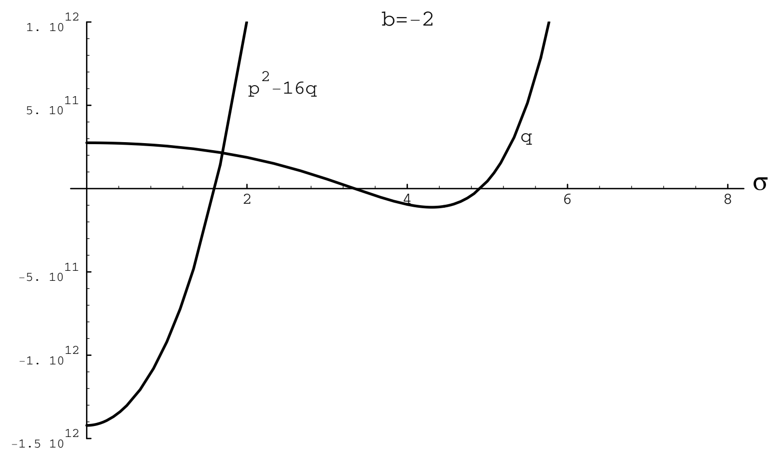

FIGURE 2(c).

which can arise. Since all the functions involved are even in σ, we confine ourselves to positive σ.

Case 1. $b = 5$. This is depicted in Figure 2(a). In this case q is always positive, while $p^2 - 16q$ changes sign at $\sigma = 0.795$ and 6.308. The conclusion is that the waves are *unstable* if $0.795 < \sigma < 6.308$ and stable otherwise.

Case 2. $b = 1$. This case is depicted in Figure 2(b). It follows that the waves are *unstable* if $0 < \sigma < 4.393$ and stable otherwise.

Case 3. $b = -2$. This case is probably the most interesting and is depicted in Figure 2(c). The stability results are that the waves are *unstable* if $0 < \sigma < 1.599$ or $3.344 < \sigma < 4.909$; otherwise they are stable. Hence, there are two distinct intervals of instability.

Case 4. $b = -5$. This is depicted in Figure 2(d). (Recall that $p^2 - 16q$ is always positive in this case.) The result is that the waves are *unstable* if $2.254 < \sigma < 5.754$; otherwise they are stable.

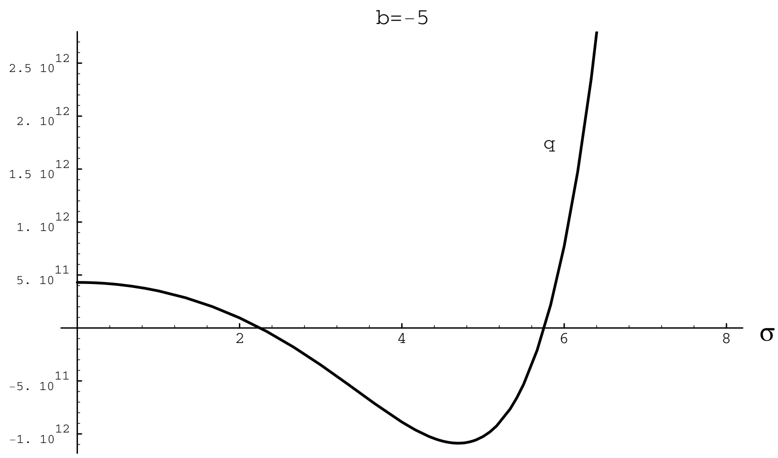

FIGURE 2(d).

4. Conclusions. We have presented expressions for the wave profiles of small amplitude Wilton ripples in the form of series expansions in power of ε, where ε represents the wave steepness. It has been shown that a large number of different wave profiles belong to the class of Wilton ripples, each corresponding to a different value of b, where b is the magnitude of the coefficient of the second order occurrence of the fundamental mode. The stability of these waves to both longitudinal and transverse perturbations is considered. In the case of perturbations in the x-direction and with positive wave number, a region of instability is always present, and it consists of an interval which (depending on b) may or may not extend up to the origin. In the case of perturbations in the y-direction, the region of stability is again usually of this form but for certain values of b, close to $b = -2$, the region of instability consists of two distinct intervals.

REFERENCES

1. S. Barnard and J.M. Child, *Higher algebra*, MacMillan, London, 1936.

2. T.B. Benjamin and J.E. Feir, *The disintegration of wavetrains on deep water* Part 1. *Theory*, J. Fluid Mech. **27** (1967), 417–430.

3. B. Chen and P.G. Saffman, *Steady gravity-capillary waves on deep water – 1. Weakly nonlinear waves*, Studies Appl. Math. **60** (1979), 183–210.

4. A. Davey and K. Stewartson, *On three dimensional packets of surface waves*, Proc. Roy. Soc. London **338** (1974), 101–110.

5. V.D. Djordjevic and L.G. Redekopp, *On two dimensional packets of capillary-gravity waves*, J. Fluid Mech. **79** (1977), 703–714.

6. H. Hasimoto and H. Ono, *Nonlinear modulation of gravity waves*, J. Phys. Soc. Japan **33** (1972), 805–811.

7. M.C.W. Jones, *Nonlinear stability of resonant capillary-gravity waves*, Wave Motion **15** (1992), 267–283.

8. M. Jones and J. Toland, *Symmetry and the bifurcation of capillary-gravity waves*, Arch. Rational Mech. Anal. **96** (1986), 29–53.

9. A.H. Nayfeh, *Finite amplitude surface waves in a liquid layer*, J. Fluid Mech. **40** (1970), 671–684.

10. ———, *Second harmonic resonance in the interaction of an air stream with capillary-gravity waves*, J. Fluid Mech. **59** (1973), 803–816.

11. J.F. Toland and M.C.W. Jones, *The bifurcation and secondary bifurcation of capillary-gravity waves*, Proc. Roy. Soc. London **399** (1985), 391–417.

12. V.E. Zakharov, *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, J. Appl. Mech. Tech. Phys. **2** (1968), 190–194.

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