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LIAPUNOV-RAZUMIKHIN FUNCTIONS AND AN INSTABILITY THEOREM FOR AUTONOMOUS FUNCTIONAL DIFFERENTIAL EQUATIONS WITH FINITE DELAY

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1. Introduction and notation. It is well known that Liapunov's direct method sometimes provides a useful tool in the study of instability of functional differential equations (FDEs). See, for example, [**1**] and [**3, 4**]. However, an obstacle often is encountered when one tries to apply this method; namely, it frequently is difficult—if not impossible—to construct appropriate Liapunov functions or functionals in order to make use of known instability theorems. The purpose of this paper is to provide an instability theorem that eliminates some of the obstacles imposed by this difficulty. In particular, we employ Liapunov-Razumikhin techniques and omega limit set properties in order to present an instability result (Theorem 2.1) for autonomous FDEs with finite delay. An example is given to illustrate that this theorem often is straightforward to apply—when applicable—and can be used to retrieve and extend previously known instability results.

We use the standard notation for finite delay FDEs. Let $|\cdot|$ denote any convenient norm on the real Euclidean space R^n of (column) n-vectors. Further, let $r \geq 0$ be given, and let $C = C([-r, 0], R^n)$ with

$$||\phi|| = \max_{-r \leq s \leq 0} |\phi(s)|, \qquad \phi \in C.$$

For $H > 0$, define $C_H \subset C$ by

$$C_H = \{\phi \in C : ||\phi|| < H\}.$$

If $x : [-r, A) \to R^n$ is continuous, $0 < A \leq \infty$, then, for each t in $[0, A)$, x_t in C is defined by

$$x_t(s) = x(t+s), \qquad -r \leq s \leq 0.$$

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Let G be an open subset of C and consider the autonomous system of FDEs with finite delay

$$x' = f(x_t), \tag{1.1}$$

where $f : G \to R^n$ is continuous and maps closed and bounded sets into bounded sets. Further, we assume that solutions depend continuously on initial data. It follows from these conditions on f that each initial value problem

$$x' = f(x_t), \qquad x_0 = \phi \in G \tag{1.2}$$

has a unique solution defined on some interval $[0, A)$, $0 < A \leq \infty$. This solution will be denoted by $x(\phi)(\cdot)$ so that $x_0(\phi) = \phi$. If x is a solution of (1.1) defined and bounded on $[0, A)$ with $A < \infty$, then either x_t tends to the boundary of G as $t \to A^-$ or x can be extended as a solution past A. In particular, if x is a solution of (1.1) contained in a closed and bounded subset of G on any finite interval for which it is defined, then x can be defined as a solution of (1.1) on $[0, \infty)$. For details, we refer to [**4**].

Let ϕ in C be such that $x(\phi)(t)$ is defined for all $t \geq 0$. Then the omega limit set, $\Omega[x_t(\phi)]$ (or, simply, $\Omega[\phi]$), of ϕ with respect to (1.1) is defined by

$$\Omega[\phi] = \{\psi \in C : x_{t_n}(\phi) \to \psi \text{ for some sequence } \{t_n\} \uparrow \infty\}.$$

A set S is said to be *positively invariant with respect to* (1.1) if, for each ϕ in S, $x_t(\phi) \in S$ for all $t \geq 0$ for which $x_t(\phi)$ is defined. S is *invariant with respect to* (1.1) if the mapping $x_t : S \to S$ is defined and onto for $t \geq 0$, i.e., $x_t(S) = S$ for each t in $[0, \infty)$. In particular, if S is invariant, then (i) for each ϕ in S, $x_t(\phi)$ is defined and in S on $[0, \infty)$ and (ii) for each fixed $t > 0$ and ψ in S, there exists ξ in S such that $x_t(\xi) = \psi$.

By a *Liapunov function* $V : R^n \to R$, we mean a locally Lipschitzian function V such that (a) $V(0) = 0$ and (b) if $0 \neq x(t_0)$ is such that x is differentiable at t_0, then $(d/dt)V[x(t)]$ exists at $t = t_0$. In particular,

$$\begin{aligned}\frac{d}{dt}V[x(t)]_{t=t_0} &= \operatorname{grad} V[x(t)] \cdot x'(t)_{t=t_0} \\ &= \sum_{i=1}^{n} \frac{\partial V[x(t_0)]}{\partial x_i} f_i(x_{t_0}).\end{aligned}$$

For a Liapunov function V, we define the functional $V'_{(1.1)} : G \to R$ by

$$V'_{(1.1)}[\phi] = \lim_{h\to 0+} \sup \frac{1}{h}([\phi(0) + hf(\phi)] - V[\phi(0)]).$$

It is well known (see, for example, [**4**] or [**6**]) that if $x : R \to R$ is differentiable at t_0 with $x_{t_0} = \phi$, then

$$V'_{(1.1)}[\phi] = \frac{d}{dt}V[x(t)]_{t=t_0} = \sum_{i=1}^{n} \frac{\partial V[x(t_0)]}{\partial x_i} f_i(x_{t_0}),$$

where $x_{t_0} = \phi$.

Remark. We have chosen the above definition of Liapunov function in order to use $V[x] = |x|$, while at the same time to allow for straightforward calculations. An excellent discussion regarding the computation of $V'_{(1.1)}[\phi]$ can be found in [**6**, Section 32].

Definition 1.1. The zero solution, $x = 0$, of (1.1) is *stable* if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $||\phi|| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \geq 0$. The zero solution is said to be *unstable* if it is not stable.

2. The main theorem.

Theorem 2.1. *Suppose there exists a Liapunov function $V : G \to R^+$ such that $V[0] = 0$ and $V[x] > 0$ if $x \neq 0$. If either*

$$\text{(i)} \quad V'_{(1.1)}[\phi] > 0 \quad \text{for all } \phi \text{ in } G \text{ for which } V[\phi(0)] = \max_{-r\leq s\leq 0} V[\phi(s)] > 0 \tag{2.1}$$

or

$$\text{(ii)} \quad V'_{(1.1)}[\phi] > 0 \quad \text{for all } \phi \text{ in } G \text{ for which } V[\phi(0)] = \min_{-r\leq s\leq 0} V[\phi(s)] > 0, \tag{2.2}$$

then $x = 0$ is unstable.

Proof. Let $\varepsilon > 0$ be given, where $C_\varepsilon \subseteq G$, and consider any δ with $0 < \delta < \varepsilon$.

Case (i). Let ϕ in C be chosen so that $V[\phi(0)] = \max_{r\le s\le 0} V[\phi(s)]$ and $\delta/2 \le |\phi(s)| < \delta$ for all s in $[-r, 0]$. Thus, $||\phi|| < \delta$ and $V'_{(1.1)}[\phi] > 0$. We claim that there exists $t^* > 0$ such that $|x(\phi)(t^*)| = \varepsilon$. Suppose not. Then $x(t)$ and $x'(t)$ are defined and bounded on $[0,\infty)$ and $V[x(\phi)(t)]$ is increasing at $t = 0$. It follows that $V[x(\phi)(t)] = \max_{-r\le s\le 0} V[x_t(\phi)(s)]$ for all $t \ge 0$; in particular, $V[x(\phi)(t)]$ is strictly increasing on $[0,\infty)$. Therefore, there exists p in R such that $V[x(\phi)(t)] \to p$ as $t \to \infty$. Also, the *positive orbit* $0^+[\phi] = 0^+[x_t(\phi)] = \{x_t(\phi) : t \ge 0\}$ (through ϕ) is precompact and the ω limit set, $\Omega[\phi]$, is nonempty. So, there exist ψ in $\Omega[\phi]$ and a sequence $\{t_n\} \uparrow \infty$ such that

$$x_{t_n}(\phi) \to \psi \quad \text{as } n \to \infty.$$

Note that $x_{t_n}(\phi)(s) \to \psi(s)$ and $n \to \infty$ for all s in $[-r, 0]$. It follows that $V[\psi(s)] \equiv p$ on $[-r, 0]$. Since $\Omega[\phi]$ is positively invariant, $x_t(\psi) \in \Omega[\phi]$ for all $t \ge 0$. Hence, $V[x_t(\psi)(s)] \equiv p$ on $[-r, 0]$ for all $t \ge 0$; that is, V is identically constant along $x(\psi)(t)$. But $V[\psi(0)] = \max_{-r\le s\le 0} V[\psi(s)]$ implies $V'_{(1.1)}[x_t(\psi)] > 0$ for all $t \ge 0$, which is a contradiction, and case (i) is proven.

Case (ii). We first note that, for any initial function ξ in C, there does not exist $t_0 > 0$ such that $V[x(\xi)(t_0)] = \min_{-r\le s\le 0} V[x(\xi)(t_0 + s)] \ne 0$. Otherwise, since $V[x(\xi)(\cdot)]$ is differentiable at such a t_0, we would have $(d/dt)V[x(\xi)(t)]_{t=t_0} \le 0$, which contradicts the hypothesis for this case.

Let ϕ in C be chosen so that $\delta/2 \le |\phi(s)| < \delta$ for all s in $[-r, 0]$. So there exists $\alpha > 0$ such that $V[\phi(s)] > \alpha$, $-r \le s \le 0$. It follows that $V[x(\phi)(t)] > \alpha$ for all $t > 0$. If not, then there exists $t_0 > 0$ such that $V[x(\phi)(t_0)] = \min_{-r\le s\le 0} V[x(\phi)(t_0 + s)] = \alpha > 0$, which cannot occur. Again, we claim that there must exist $t^* > 0$ such that $|x(\phi)(t^*)| = \varepsilon$. Suppose not. Then, as in Case (i) above, $0^+[\phi]$ is precompact and $\Omega[\phi]$ is nonempty. Also,

$$\lim_{t\to\infty} \inf V[x(\phi)(t)] = \beta \ge \alpha \text{ exists.}$$

It follows that there exist ψ in $\Omega[\phi]$ and a sequence $\{t_n\} \uparrow \infty$ such that

$$V[x(\phi)(t_n)] \to \beta \quad \text{and} \quad x_{t_n}(\phi) \to \psi \quad \text{as } n \to \infty.$$

Note that $V[x_{t_n}(\phi)(s)] \to V[\psi(s)] \geq \beta$ as $n \to \infty$ for all s in $[-r, 0]$. In particular,

$$\beta = V[\psi(0)] = \min_{-r \leq s \leq 0} V[\psi(s)] \neq 0.$$

Now $\Omega[\phi]$ is invariant (and not just positively invariant), so by a well-known property of invariance (cf., e.g., [**5**, pp. 166–167]), $x_t(\Omega[\phi]) = \Omega[\phi]$ for each $t > 0$. In particular, the mapping $x_t : \Omega[\phi] \to \Omega[\phi]$ is onto for each $t > 0$. Fix $t_0 > 0$. Since $\psi \in \Omega[\phi]$, there exists $\xi = \xi(t_0)$ such that $x_{t_0}(\xi) = \psi$. Thus, from (2.1),

$$\beta = V[x(\xi)(t_0)] = \min_{-r \leq s \leq 0} V[x(\xi)(t_0 + s)] > 0,$$

which is impossible since $t_0 > 0$. This contradiction completes the proof of the theorem. $\square$

3. An example. It is well known (see, e.g., [**3**, Section 5.2]) that the zero solution of the linear equation

$$x'(t) = ax(t) + bx(t - r), \qquad r \geq 0, \tag{3.1}$$

is unstable if $a + b > 0$. In fact, several techniques—including the use of the characteristic equation and the use of the Liapunov functions (cf. [**3**])—have been employed to prove this. However, it becomes a more complicated matter when related nonlinear equations are considered. The situation is simplified significantly though if we apply Theorem 2.1.

Example 3.1. Consider the (possibly) nonlinear scalar equation

$$x'(t) = ah(x(t)) + bh(x(t - r)), \tag{3.2}$$

where $r \geq 0$ and $h : R \to R$ is continuous and strictly increasing with $h(0) = 0$. We will employ Theorem 2.1 to show that the zero solution $x = 0$ of (3.1) is unstable whenever $a + b > 0$. Define $V : R \to R$ by

$V(x) = |x|$, and let $x(\cdot)$ denote a solution of (3.1). We will consider several cases, but many of the details are straightforward and, therefore, omitted.

(a) Suppose $b \leq 0$. Then, for this case, $a > |b| = -b$. Suppose $t \geq 0$ is such that $V(x(t)) = \max_{-r \leq s \leq 0} V(x(t+s))$; that is, $|x(t)| = \max_{-r \leq s \leq 0} |x(t+s)| > 0$.

(i) If $x(t) > 0$, then $x(t) \geq x(t-r)$ and

$$\begin{aligned} V'(x(t)) = |x(t)|' = x'(t) &= ah(x(t)) + bh(x(t-r)) \\ &\geq ah(x(t)) + bh(x(t)) \\ &= (a+b)h(x(t)) > 0. \end{aligned}$$

(ii) If $x(t) < 0$, then $x(t) \leq x(t-r)$ and

$$\begin{aligned} V'(x(t)) = |x(t)|' = -x'(t) &= -ah(x(t)) - bh(x(t-r)) \\ &\geq -ah(x(t)) - bh(x(t)) \\ &= -(a+b)h(x(t)) > 0. \end{aligned}$$

Thus, (2.1) of Theorem (2.1) holds.

(b) Suppose $b > 0$. Let $t \geq 0$ be such that $|x(t)| = \min_{-r \leq s \leq 0} |x(t+s)| > 0$.

(i) If $x(t) > 0$, then $x(t) \leq x(t-r)$ and

$$\begin{aligned} V'(x(t)) &= ah(x(t)) + bh(x(t-r)) \\ &\geq (a+b)h(x(t)) > 0. \end{aligned}$$

(ii) If $x(t) < 0$, then $x(t) \geq x(t-r)$ and

$$\begin{aligned} V(x(t)) &= -ah(x(t)) - bh(x(t-r)) \\ &\geq -(a+b)h(x(t)) > 0. \end{aligned}$$

It follows that (2.2) of Theorem 2.1 holds.

In conclusion, we have shown that, for $a + b > 0$, condition (2.1) of Theorem 2.1 holds if $b \leq 0$ and (2.2) holds if $b > 0$. In any case, it follows from Theorem 2.1 that $x = 0$ of (3.2) is unstable.

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