ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 1, Winter 1995

ON THE BEHAVIOR OF SOME EXPLICIT SOLUTIONS OF THE HARMONIC MAPS EQUATION

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1. Introduction and definitions. Harmonic maps of the Minkowski space are the critical points $u : \mathbf{M} \to \mathbf{N}$ of the energy functional

$$\int_{\mathbf{M}} \eta^{\alpha\beta} g_{ij} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}\, dx \tag{1.1}$$

where

$$\begin{aligned}&\mathbf{M}(n,1) \text{is the } n+1-\text{dimensional Minkowski space}\\ &\text{with Lorentzian metric } \eta^{\alpha\beta} = (1,-1,\dots,-1) \text{ and local}\\ &\text{coordinates } x^0 = t, x^1, \dots, x^n,\end{aligned} \tag{1.2}$$

$$\begin{aligned}&\mathbf{N} \text{is an } m-\text{dimensional Riemannian manifold with}\\ &\text{local coordinates } (u^1,\dots,u^m) \text{ and metric form}\\ &ds^2 = g_{ij}(u)du^i du^j.\end{aligned} \tag{1.3}$$

The Euler-Lagrange equations describing the critical points of (1.1) are

$$\frac{\partial^2 u^i}{\partial t^2} - \sum_{p=1}^{n} \frac{\partial^2 u^i}{\partial x^{p2}} + \Gamma^i_{jk}(u)\left\{\frac{\partial u^j}{\partial t}\frac{\partial u^k}{\partial t} - \sum_{p=1}^{n} \frac{\partial u^j}{\partial x^p}\frac{\partial u^k}{\partial x^p}\right\} = 0, \tag{1.4}$$

$1 \le i, j, k \le m$, Γ^i_{jk} are the Christoffel symbols corresponding to the metric in (1.3) and summation over repeated indices is understood.

There has been a lot of research done regarding different aspects of harmonic maps [**1, 2, 3** and references therein]. Here we look at the behavior of some special solutions of (1.4) defined below.

Received by the editors on September 2, 1992, and in revised form on January 29, 1993.

We call a function $u = u(t,x)$ a null-solution of (1.4) if it satisfies

$$\frac{\partial^2 u^i}{\partial t^2} - \sum_{p=1}^{n} \frac{\partial^2 u^i}{\partial x^{p2}} = \frac{\partial u^j}{\partial t}\frac{\partial u^k}{\partial t} - \sum_{p=1}^{n} \frac{\partial u^j}{\partial x_p}\frac{\partial u^k}{\partial x_p} = 0. \tag{1.5}$$

A typical example of a null-solution is a function $u = f(\langle A, x\rangle)$, $\langle A, A\rangle = 0$ where

$$\langle A, x\rangle = A_0 x^0 - A_1 x^1 - A_2 x^2 - \cdots - A_n x^n \tag{1.6}$$

and $f(\cdot)$ is a function of a scalar variable.

Here is the problem which ignited our interest that led to this paper. Let $f(s)$ and $g(s)$ be two C_0^2 functions of a single variable satisfying the following conditions:

$$\begin{aligned} g(s) &= 0 \quad \text{for } s \in (-\infty, -3) \cup (-1, \infty) \\ f(s) &= 0 \quad \text{for } s \in (-\infty, 1) \cup (3, \infty) \end{aligned} \tag{1.7}$$

with $S = \{s \in [-3,-1], g'(s) = 0\} \cup \{s \in [1,3], f'(s) = 0\}$ being finite. Then for $n = 1$ the function $u(t,x) = f(x+t) + g(x-t)$ is a null solution of (1.4) for $0 \le t \le 1$ and describes two solitary waves moving towards each other.

At time $t = 1$ the waves collide. We are interested in what happens after the collision. In Section 2 we consider this problem in the case of one space dimension and in Section 3 we generalize it to the higher dimensions. In Section 4 we look at some similar waves which are not null-solutions.

Some results and proofs described in Sections 2 and 3 can be generalized to a more general class of equations satisfying Klainerman's null-condition [**2**], though we will not do it here.

2. The case $n = 1$. We start this section with two examples that will make the subsequent computations much more clear.

Example 2.1. Consider the case when $\mathbf{N}$ is the two-dimensional hyperbolic manifold, i.e.,

$$\mathbf{N} = \left\{ u^1, u^2 \mid u^1 \in \mathbf{R}, u^2 + 1 > 0,\, ds^2 = \frac{du^{1^2} + du^{2^2}}{(u^2+1)^2} \right\}. \tag{2.1}$$

The equations (1.4) take the form:

$$
\begin{aligned}
&\frac{\partial^2 u^1}{\partial t^2} - \frac{\partial^2 u^1}{\partial x^2} - \frac{2}{u^2+1}\langle \nabla u^1, \nabla u^2\rangle = 0 \\
&\frac{\partial^2 u^2}{\partial t^2} - \frac{\partial^2 u^2}{\partial x^2} + \frac{1}{u^2+1}\langle \nabla u^1, \nabla u^1\rangle \\
&\qquad\qquad\qquad - \frac{1}{u^2+1}\langle \nabla u^2, \nabla u^2\rangle = 0,
\end{aligned}
\tag{2.2}
$$

where $\langle \nabla u^i, \nabla u^j\rangle = (\partial u^i/\partial t)(\partial u^j/\partial t) - (\partial u^i/\partial x)(\partial u^j/\partial x)$. The initial data is chosen to be

$$
\begin{aligned}
u|_{t=0} &= \begin{pmatrix} f(x)+g(x) \\ -1+\sqrt{1+a^2-[f(x)+g(x)+a]^2} \end{pmatrix} \\
u_t|_{t=0} &= \begin{pmatrix} f'(x)-g'(x) \\ \frac{-[f(x)+g(x)+a][f'(x)-g'(x)]}{\sqrt{1+a^2-[f(x)+g(x)+a]^2}} \end{pmatrix}
\end{aligned}
\tag{2.3}
$$

with $f(x)$ and $g(x)$ as in (1.7), a a constant and the vector u satisfying

$$
(u^1+a)^2 + (u^2+1)^2 = 1+a^2. \tag{2.4}
$$

Geometrically, the conditions (2.3) simply means that the initial data lie on a geodesic. Equations (2.3) and (2.4) guarantee the existence of a solution in the form

$$
u(t,x) = \begin{pmatrix} f(x+t)+g(x-t) \\ -1+\sqrt{1+a^2-[f(x+t)+g(x-t)+a]^2} \end{pmatrix} \tag{2.5}
$$

for $0 \le t \le 1$, whereas (2.4) allows us to reduce the Cauchy problem (2.2) and (2.3) to a much simpler one

$$
\begin{gathered}
\Box w + \frac{2w}{1+a^2-w^2}\langle \nabla w, \nabla w\rangle = 0 \\
w|_{t=0} = f(x)+g(x), \qquad w_t|_{t=0} = f'(x)-g'(x)
\end{gathered}
\tag{2.6}
$$

where the $w = u^1 + a$. Using $\tilde{w} = \ln((b+w)/\sqrt{b^2-w^2})^{1/b} = (1/2b)\ln((b+w)/(b-w))$ with $b = \sqrt{1+a^2}$, (2.6) can be reduced to the following form:

$$
\begin{aligned}
&\square \tilde{w} = 0\\
&\tilde{w}|_{t=0} = \ln\left(\frac{b + f(x) + g(x)}{\sqrt{b^2 - [f(x)+g(x)]^2}}\right)^{1/b}\\
&\tilde{w}_t|_{t=0} = \frac{f'(x) - g'(x)}{b^2 - [f(x)+g(x)]^2}.
\end{aligned} \tag{2.7a}
$$

Note that for (2.7a) to make sense we must have $b \geq \max(|f(x)| + |g(x)|)$. One can easily solve (2.7a) to obtain:

$$
\begin{aligned}
\tilde{w} &= \frac{1}{2}\ln\left[\frac{f(x+t) + g(x+t) + b}{\sqrt{b^2 - [f(x+t)+g(x+t)]^2}}\right]^{1/b}\\
&+ \frac{1}{2}\ln\left[\frac{f(x-t) + g(x-t) + b}{\sqrt{b^2 - [f(x-t)+g(x-t)]^2}}\right]^{1/b}\\
&+ \frac{1}{2}\int_{x-t}^{x+t} \frac{f'(\xi) - g'(\xi)}{b^2 - [f(\xi)+g(\xi)]^2}\, d\xi.
\end{aligned} \tag{2.7b}
$$

To see what is happening, let us look at the following pictures, Figures 1 and 2:

where

$$
\begin{aligned}
&\mathcal{C} = \{(t,x) \mid -3 < x - t \leq -1, 1 \leq t + x \leq 3\};\\
&\mathcal{A} = \{(t,x) \mid t > 0, -3 \leq x - t \leq -1\}\backslash\mathcal{C};\\
&\mathcal{B} = \{(t,x) \mid t \geq 0, 1 \leq t + x \leq 3\}\backslash\mathcal{C};\\
&\mathcal{D} = \{(t,x) \mid t > 0, x \in \mathbf{R}\}\backslash\{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}\}.
\end{aligned} \tag{2.8}
$$

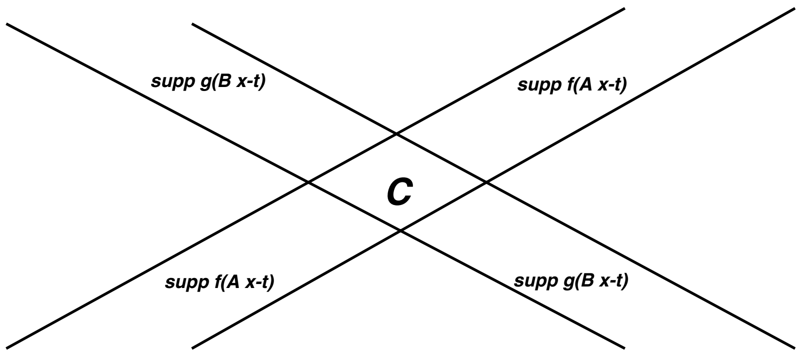

FIGURE 1.

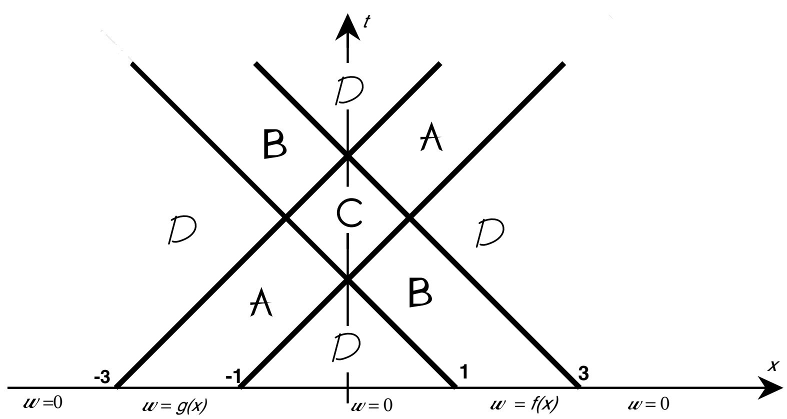

FIGURE 2.

For $(t,x) \in \mathcal{C}$ we can simplify (2.7b) in the following manner:
(2.9a)

$$\begin{aligned}\tilde{w} &= \frac{1}{2}\ln\left[\frac{f(x+t)+b}{\sqrt{b^2-[f(x+t)]^2}}\right]^{1/b} + \frac{1}{2}\ln\left[\frac{g(x-t)+b}{\sqrt{b^2-[g(x-t)]^2}}\right]^{1/b} \\ &\quad - \frac{1}{2}\int_{x-t}^{1}\frac{g'(\xi)}{b^2-[g(\xi)]^2}\,d\xi + \frac{1}{2}\int_{1}^{x+t}\frac{f'(\xi)}{b^2-[f(\xi)]^2}\,d\xi \\ &= \frac{1}{2}\ln\left(\sqrt{\frac{b+f(x+t)}{b-f(x+t)}}\right)^{1/b} + \frac{1}{2}\ln\left(\sqrt{\frac{b+g(x-t)}{b-g(x-t)}}\right)^{1/b} \\ &\quad - \frac{1}{4b}\int_{x-t}^{1}\left[\frac{1}{b-g(\xi)}+\frac{1}{b+g(\xi)}\right]g'(\xi)\,d\xi \\ &\quad + \frac{1}{4b}\int_{1}^{x+t}\left[\frac{1}{b-f(\xi)}+\frac{1}{b+f(\xi)}\right]f'(\xi)\,d\xi \\ &= \frac{1}{2b}\ln\frac{b+f(x+t)}{b-f(x+t)} + \frac{1}{2b}\ln\frac{b+g(x-t)}{b-g(x-t)}.\end{aligned}$$

Substituting the expression for $\tilde{w}$ we eventually get

$$\frac{b+w}{b-w} = \frac{(b+f(x+t))(b+g(x-t))}{(b-f(x+t))(b-g(x-t))} \tag{2.9b}$$

because the terms $g(x+t)$, $f(x-t)$ vanish in $\mathcal{C}$ and the domain of integration of the integral in (2.7b) is reduced to $(x-t,-1)\cup(1,x+t)$ with $f(\xi)$ vanishing on $(x-t,-1)$ and $g(\xi)$ vanishing on $(1,x+t)$. The value of w in the other regions is given by the following table:

TABLE 1.

region	terms in (2.7b) that vanish or cancel	terms in (2.7b) that do not disappear	w
$\mathcal{A}$	$f(x+t), f(x-t), g(x+t)$	$g(x-t)$	$g(x-t)$
$\mathcal{B}$	$g(x+t), g(x-t), f(x-t)$	$f(x+t)$	$f(x+t)$
$\mathcal{D}$	$g(x+t), g(x-t), f(x+t)$ $f(x-t)$	0	0

We can interpret this in the following manner: the solution of (2.2), (2.3) and (2.4) consists of two solitary waves

$$\begin{pmatrix} f(x+t) \\ -1+\sqrt{b^2-[f(x+t)+a]^2} \end{pmatrix}$$

and

$$\begin{pmatrix} g(x-t) \\ -1+\sqrt{b^2-[g(x-t)+a]^2} \end{pmatrix}$$

running towards each other colliding, interacting and departing as if nothing happened. We eventually obtain

$$u(t,x) = \begin{cases} \begin{pmatrix} f(x+t) \\ \sqrt{b^2-[f(x+t)+a]^2}-1 \end{pmatrix} \\ \quad + \begin{pmatrix} g(x-t) \\ \sqrt{b^2-[g(x-t)+a]^2}-1 \end{pmatrix} & (t,x) \notin \mathcal{C} \\ \begin{pmatrix} w-a \\ -1+\sqrt{1+a^2-w^2} \end{pmatrix} \end{cases}$$

with w satisfying (2.9b).

Example 2.2. Consider the case when $\mathbf{N} = \mathbf{S}^2$, i.e.,

$$\mathbf{N} = \{(r,\theta) \mid -\pi/2 \le r \le \pi/2, 0 \le \theta \le 2\pi, ds^2 = dr^2 + \cos^2 r\, d\theta^2\}.$$

The equations (1.4) take the form

$$\begin{aligned} \Box r + \sin r \cos r \langle \nabla\theta, \nabla\theta\rangle &= 0, \\ \Box\theta - 2\tan r \langle \nabla r, \nabla\theta\rangle &= 0. \end{aligned} \tag{2.10}$$

We choose the initial data to be:

(2.11)

$$\begin{pmatrix} r \\ \theta \end{pmatrix}_{t=0} = \begin{pmatrix} f(x)+g(x) \\ \arcsin(\alpha(\tan[f(x)+g(x)])) \end{pmatrix}$$

$$\begin{pmatrix} r_t \\ \theta_t \end{pmatrix}_{t=0} = \begin{pmatrix} f'(x)-g'(x) \\ a[f'(x)-g'(x)]/(\cos^2(x)\sqrt{1-\alpha^2\tan^2(f(x)-g(x))}\,) \end{pmatrix}$$

with $f(x)$ and $g(x)$ as in (1.7a) and α a number. Again, (2.11) gives us a solution

$$\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} f(x+t)+g(x-t) \\ \arcsin(\alpha(\tan[f(x+t)+g(x-t)]), \end{pmatrix}$$

for $0 \le t \le 1$, with this formula suggesting that we look for a solution of (2.10) and (2.11) in the form

$$u(t,x) = \begin{pmatrix} w \\ \arcsin[\alpha \tan w] \end{pmatrix}$$

with $w(t,x)$ being a scalar function. Substituting the above into (2.10) and (2.11), we obtain

$$\text{(2.12)} \qquad \begin{cases} \Box w + (\alpha^2 \sin w/\cos^3 w(1-\alpha^2\tan^2 w))\langle \nabla w, \nabla w\rangle = 0, \\ w|_{t=0} = f(x)+g(x) \\ w_t|_{t=0} = f'(x) - g'(x). \end{cases}$$

Again, using $\tilde{w} = \ln\left[\sqrt{\alpha^2+1}\sin w + \sqrt{(\alpha^2+1)\sin^2 w - 1}\right]^{1/\sqrt{\alpha^2+1}}$ we obtain for $\tilde{w}$,

$$\text{(2.13)} \qquad \begin{aligned} &\Box \tilde{w} = 0 \\ &\tilde{w}|_{t=0} = \frac{1}{\sqrt{\alpha^2+1}} \ln\Big[\sqrt{\alpha^2+1}\sin(f(x)+g(x)) \\ &\qquad\qquad + \sqrt{(\alpha^2+1)\sin^2(f(x)+g(x))-1}\Big] \\ &\tilde{w}_t|_{t=0} = \frac{1}{\sqrt{\alpha^2\tan^2(f(x)+g(x))-1}}(f'(x)-g'(x)). \end{aligned}$$

Again, we can solve (2.13) and thus (2.12) to obtain

$$\text{(2.15)} \qquad w = \begin{cases} 0 & \text{in } \mathcal{D}, \\ f(x+t) & \text{in } \mathcal{B}, \\ g(x-t) & \text{in } \mathcal{A}, \\ I^{-1}(I(f(x+t)) + I(g(x-t))) & \text{in } \mathcal{C}, \end{cases}$$

with

$$\begin{aligned} I(f(\xi)) &= \int_0^{f(\xi)} \frac{\alpha^2 \sin\rho}{\cos^3\rho\sqrt{1-\alpha^2\tan^2\rho}}\,d\rho \\ &= \frac{1}{\sqrt{\alpha^2+1}} \ln\left[\sqrt{\alpha^2+1}\sin(f(\xi)) + \sqrt{(\alpha^2+1)\sin^2(f(\xi))-1}\right] \end{aligned}$$

and $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ as on the picture of Example 2.1. Correspondingly, $u = \begin{pmatrix} w \\ \arcsin[\alpha\tan w] \end{pmatrix}$ is obtained.

The above examples lead us to the following

Theorem 2.1. *Let u be a harmonic map from $M(1,1)$ into a Riemannian manifold $\mathbf{N}$ which, for $0 \le t \le 1$, has the form $f(x+t)+g(x-t)$, i.e., for $0<t<1$, u satisfies* (1.5) *with initial data*

$$\begin{aligned} u|_{t=0} &= f(x)+g(x) \\ u_t|_{t=0} &= f'(x)-g'(x) \end{aligned}$$

with $f(x)$ and $g(x)$ satisfying (1.7) *and being sufficiently small. Let us also assume that*

$$f = \begin{pmatrix} f^1(x) \\ f^2(f^1(x)) \\ \vdots \\ f^m(f^1(x)) \end{pmatrix}, \qquad g = \begin{pmatrix} g^1(x) \\ g^2(g^1(x)) \\ \vdots \\ g^m(g^1(x)) \end{pmatrix} \tag{2.16}$$

i.e., all coordinates are some functions of the first one. Then there exist functions F^i of two variables such that

$$u^i = F^i(f^1, g^1), \quad 1 \le i \le m \tag{2.17}$$

$$\begin{cases} F^1(0,g^1) = g^1, & F^1(f^1,0) = f^1 \\ F^i(0,g^1) = g^i(g^1), & F^i(f^1,0) = f^i(f^1), \\ & f^i(f^1=0)=g^i(g^1=0)=0. \end{cases} \tag{2.18}$$

Remark 1. The condition (2.16) is not too restrictive. It can be easily generalized and is used for simplicity of notation. We could have assumed $f^i = f^i(\tilde{f}(x))$, $g^i = g^i(\tilde{g}(x))$ and repeated the computations with only slight modifications. One-dimensionality of $f(x)$ and $g(x)$ follow from one-dimensionality of x.

Remark 2. The last condition on $f^i(f^1 = 0) = g^i(g^1 = 0) = 0$ is here to insure that all the f^i and g^i vanish in D. It puts, however, an unnecessary constraint requiring $f^i = 0$ and $g^i = 0$ at the points of $A \cup B \cup C$ where $f^1 = 0$ and $g^1 = 0$, correspondingly. It can be avoided if we assume that at $t = 0$, $f^1 \neq 0$ in (1,3) and $g^1 \neq 0$ in $(-1, -3)$. If it is not the case, in view of the previous remark, we can always replace f^1 and g^1 with some other parameters $\tilde{f}$ and $\tilde{g}$ which do not vanish inside of $(1, 3)$ and $(-3, -1)$, correspondingly.

Proof. We show existence of the F^i by constructing them.

Differentiating (2.17) we obtain

$$\begin{aligned} \square u^i &= -4F^i_{12} f^{1'}(x+t) g^{1'}(x-t), \\ \langle \nabla u^j, \nabla u^k \rangle &= -2(F^j_1 F^k_2 + F^k_1 F^j_2) f^{1'} g^{1'} \end{aligned} \tag{2.19}$$

where

$$F^i_{12} = \frac{\partial F^i(\xi, \eta)}{\partial \xi \partial \eta}, \qquad f^{1'}(x+t) = \frac{\partial f^1}{\partial (x+t)}, \text{ etc.}$$

Combining (2.19) with (1.4), we get

$$(F^i_{12} + \Gamma^i_{jk} F^j_1 F^k_2) f^{1'} g^{1'} = 0, \tag{2.20}$$

together with the boundary conditions (2.18).

Thus, our problem is reduced to showing existence of solutions of the following system:

(2.21)

$$\begin{cases} F^i_{12} + \Gamma^i_{jk}(F^1, \dots, F^m) F^j_1 F^k_2 = 0, & F^i = F^i(\xi, \eta),\ 1 \le i \le m \\ F^1(0, \eta) = \eta, & F^1(\xi, 0) = \xi \\ F^i(0, \eta) = g^i(\eta), & F^i(\xi, 0) = f^i(\xi) \end{cases}$$

which can be viewed as a nonlinear Goursat problem.

Note that the data is given on the characteristics. There are two ways to show existence of F^is.

In the first one we can write the equations (2.21) on $\eta = 0$ in the form

$$\frac{\partial}{\partial\xi}\begin{pmatrix}\partial F^1/\partial\eta\\ \vdots\\ \partial F^m/\partial\eta\end{pmatrix}+\begin{pmatrix}\Gamma^1_{j1}\partial F^j/\partial\xi & \cdots & \Gamma^1_{jm}\partial F^j/\partial\xi\\ \Gamma^m_{j1}\partial F^j/\partial\xi & \cdots & \Gamma^m_{jm}\partial F^j/\partial\xi\end{pmatrix}\begin{pmatrix}\partial F^1/\partial\eta\\ \vdots\\ \partial F^m/\partial\eta\end{pmatrix}=0$$

with $\partial F^1/\partial\xi = 1$, $\partial F^i/\partial\xi = \partial f^i(\xi)/\partial\xi$ and

$$\begin{pmatrix}\partial F^1/\partial\eta\\ \vdots\\ \partial F^m/\partial\eta\end{pmatrix}_{\xi=0} = \begin{pmatrix}1\\ \vdots\\ \partial g^m(0)/\partial\eta\end{pmatrix}.$$

By solving the latter system on $\eta = 0$ we obtain $\partial F^i/\partial\eta$ on $\eta = 0$. Differentiating the system and solving it again we, in principle, can compute all derivatives of F^i, and if the F^i and the Γ^i_{jk}s are analytic, we can use the argument of the Cauchy-Kowalewski theorem to obtain an analytic solution near $\eta = 0$.

The second method consists of rewriting (2.21) in the form:

(2.22)
$$\begin{aligned}
F^1(\xi,\eta) &= \xi+\eta\\
&\quad+\int_0^\xi\int_0^\eta \Gamma^1_{jk}(F^1,\dots,F^m)F^j_1(\xi_1,\eta_1)F^k_2(\xi_1,\eta_1)\,d\xi_1\,d\eta_1\\
F^i(\xi,\eta) &= g^i(\eta)+f^i(\xi)\\
&\quad+\int_0^\xi\int_0^\eta \Gamma^i_{jk}(F^1,\dots,F^m)F^j_1(\xi_1,\eta_1)F^k_2(\xi_1,\eta_1)\,d\xi_1\,d\eta_1.
\end{aligned}$$

where $F^i_1(\xi,\eta) = \partial F^i/\partial\xi$, $F^i_2(\xi,\eta) = \partial F^i/\partial\eta$.

We consider a ball of radius R in the Banach space C^1 endowed with the norm

$$||F|| = \max_{i,-\varepsilon\le\xi,\eta\le\varepsilon}\left\{|F^i| + \left|\frac{\partial F^i}{\partial\xi}\right| + \left|\frac{\partial F^i}{\partial\eta}\right|\right\}. \tag{2.23}$$

For ε, $g^i(\eta)$ and $f^i(\xi)$ sufficiently small, the righthand sides of (2.22) generate a contractive mapping of the ball into itself. The fixed point of this contraction gives us the required functions F^i. □

Corollary. *In terms of the picture* (2.8) *used for the previous two examples we can write* $u = f(x+t) + g(x-t)$, *provided* $(x,t) \notin \mathcal{C}$. *Two solitary waves* $f(x+t)$ *and* $g(x-t)$ *move towards each other, collide, pass through each other and depart as if nothing happened.*

What happens during the collision is not clear. The following theorem sheds some light on it.

Theorem 2.2. *Let* u *be a harmonic map from* $\mathbf{M}(1,1)$ *into* $\mathbf{N}$ *which for* $0 \leq t \leq 1$ *has the form* $f(x+t) + g(x-t)$, *i.e., for* $0 \leq t \leq 1$, u *satisfies* (1.5) *with the initial data*

$$u|_{t=0} = f(x) + g(x), \qquad u_t|_{t=0} = f'(x) - g'(x)$$

with $f(x)$ *and* $g(x)$ *satisfying* (1.7). *Let us also assume that there are sufficiently smooth invertible functions* G^i *such that*

$$f^i(x) = G^i(f^1(x)), \qquad g^i(x) = G^i(g^1(x)), \tag{2.24}$$

i.e., the image lies on a curve in $\mathbf{N}$ *given by* (2.24). *Then the image will stay on the curve* $u^i = G^i(u^1)$ *for all* $t \geq 0$ *if and only if this curve is a geodesic. Moreover, if* $u^i = G^i(u^1)$ *is a geodesic, there exists an invertible function* F *such that*

$$F(u^1) = F(f^1(x+t)) + F(g^1(x-t)). \tag{2.25}$$

Proof. Corollary to Theorem 2.1 implies that the image stays on the curve $u^i = G^i(u^1)$ for all (x,t) away from the collision region $\mathcal{C}$ (as on the picture of Example 2.1). For $(x,t) \in \mathcal{C}$ the image will stay on the curve $u^i = G^i(u^1)$ if and only if the following equations hold:

$$\square u^1 + \frac{G^{i''} + \Gamma^i_{jk} G^{j'} G^{k'}}{G^{i'}} \langle \nabla u^1, \nabla u^1 \rangle = 0, \quad 1 \leq i \leq m. \tag{2.26}$$

The above equations can be compatible if and only if the nonlinear term is the same for all i between 1 and m which, in turn, can happen only if either

$$\langle \nabla u^1, \nabla u^1 \rangle = 0 \tag{2.27}$$

or

$$h(u^1) = \frac{G^{i''} + \Gamma^i_{jk} G^{j'} G^{k'}}{G^{i'}} \tag{2.28}$$

is independent of i. Equation (2.27) implies $\Box u^1 = \langle \nabla u^1, \nabla u^1 \rangle = 0$ which, in turn, gives us $u^1 = \varphi(x+t) + \psi(x-t)$, $\varphi'\psi' = 0$, i.e., the situation which cannot happen during collision of the solitary waves $f(x+t)$ and $g(x-t)$, i.e., for $(x,t) \in \mathcal{C}$ since it would contradict the condition on S after (1.7). So we discard this possibility and turn to (2.28) which is an equation of a geodesic parametrized by u^1, with $d^2u^1/ds^2 = -h$, where s is the arc length. So, if $u^i = G^i(u^1)$ is a geodesic, (2.26) has a solution for all $t \geq 0$. If, however, $u^i = G^i(u^1)$ is not a geodesic, (2.26) becomes contradictory for $1 \leq t \leq 3$ in the domain of collision (region $\mathcal{C}$ on the picture of Example 2.1).

One can easily see now that if $u^i = G^i(u^1)$ is a geodesic, the function

$$F(u^1) = \int_0^{u_1} d\rho e^{\int^\rho h(s)\, ds}$$

satisfies (2.25). $\Box$

3. The case $n > 1$. Technically this section is similar to the previous one though the results and interpretations are different enough to justify putting them into a separate section. We want to study here the interaction of two null-solutions $f(\langle A, X \rangle)$ and $g(\langle B, X \rangle)$, with $f(s)$ and $g(s)$ being as in (1.7), $\langle A, A \rangle = \langle B, B \rangle = 0$. Without loss of generality, we can assume that $A_0 = B_0 = 1$ and $\sum_{p=1}^n A_p^2 = \sum_{p=1}^n B_p^2 = 1$.

Theorem 3.1. *Let the functions $F^1(\xi,\eta), \ldots, F^m(\xi,\eta)$ be solutions of*

$$\begin{aligned} &F^i_{12} + \Gamma^i_{jk}(F^1, F^2) F^j_1 F^k_2 = 0, \qquad F^i = F^i(\xi,\eta), \quad 1 \leq i \leq m \\ &F^1(0,\eta) = \eta, \qquad F^1(\xi,0) = \xi \\ &F^i(0,\eta) = g^i(\eta), \qquad F^i(\xi,0) = f^i(\xi), \qquad g^i(0) = f^i(0) = 0, \end{aligned} \tag{3.1}$$

with $g^2(\eta)$ and $f^2(\xi)$ being C^1 invertible functions and the $\Gamma^i_{jk}(F^1,F^2)$ Christoffel symbols on a Riemannian manifold **N** *with local coordinates*

$(F^1, \dots, F^m)$. *Then the functions* $u^i = F^i(f^1(A \cdot x - t), g^1(B \cdot x - t))$ *determine a harmonic map from* $\mathbf{M}(n, 1)$ *into* $\mathbf{N}$ *satisfying* (1.4) *and having the form*

$$u^i(t,x) = \begin{cases} F^i(f^1(A \cdot x - t), g^1(B \cdot x - t)), & (x,t) \in \mathcal{C} \\ f^i(A \cdot x - t) + g^i(B \cdot x - t), & (x,t) \notin \mathcal{C} \end{cases}$$

with $f^1(\cdot)$, $g^1(\cdot)$ *satisfying* (1.7) *and* $\mathcal{C} = \operatorname{supp} f^1(A \cdot x - t) \cap \operatorname{supp} g^1(B \cdot x - t)$.

The situation is very similar to that described in the previous section. Both waves $f(A \cdot x - t)$ and $g(B \cdot x - t)$ move in the directions of A and B, respectively, with the only interaction occurring at $\mathcal{C}$.

Again, the remark after (2.16) applies. If the f^i and g^i are sufficiently small, then as we saw in the previous section, (3.1) has a solution.

We also have:

Theorem 3.2. *Let* $f(\cdot)$ *and* $g(\cdot)$ *be two* C_0^1 *functions of one variable satisfying* (1.7) *and also assume that there are sufficiently smooth invertible functions* G^i *such that*

$$f^i = G^i(f^1), \qquad g^i = G^i(g^1). \tag{3.2}$$

Then the functions u^1 *and* $u^i = G^i(u^1)$ *will be solutions of* (1.4) *if and only if the curve* $u^i = G^i(u^1)$ *is a geodesic. Moreover, if* $u^i = G^i(u^1)$ *is a geodesic, there exists an invertible function* F *such that*

$$F(u^1) = F(f(A \cdot x - t)) + F(g(B \cdot x - t)).$$

Proof and expression for the F are the same as in the proof of Theorem 2.2.

We see from the theorems that both waves move in the directions A and B, respecitvely, with the only interaction happening at $\mathcal{C} = \operatorname{supp} f(A \cdot x - t) \cap \operatorname{supp} g(B \cdot x - t)$. At the points away from $\mathcal{C}$ each wave behaves as if the other one does not exist.

If we have three null-solutions $f_i(A_i \cdot x - t)$ with the three vectors A_i linearly independent and the $f_i(\cdot)$ satisfying one of the conditions of (1.7), then $\cap_{i=1}^{3}\operatorname{supp} f_i(A_i \cdot x - t)$ has compact support and moves as a "particle." It would be very interesting to see whether there is any physical significance attached to it. Though we believe that the interaction of three null-solutions $f_i(A_i \cdot x - t)$ should be similar to the interaction of only two of them, we have not been able to prove it. The reason for that being that the differential equation in (3.4) is replaced with a more complicated one for which we could not prove existence.

4. Waves in the background. Let $\mathbf{M}(n,1)$ and $\mathbf{N}$ be as in (1.2) and (1.3), and let h be a solution of the following system:

$$\sum_{p=2}^{n}\left(\frac{\partial^2 h^i}{\partial x^{p^2}} + \Gamma^i_{jk}\frac{\partial h^j}{\partial x^p}\frac{\partial h^k}{\partial x^p}\right) = 0. \tag{4.1}$$

We call such a solution "background" and assume that h depends on a parameter w as well as the $x^2, \dots, x^n$ and satisfies (4.1) for all values of w in some interval $[w_1, w_2]$. One can easily observe then that $h(x^2, \dots, x^n, w)$ is a solution of (1.4) for $w = f(x_1 + t)$ and $w = g(x_1 - t)$. If $f(\cdot)$ and $g(\cdot)$ satisfy (1.7), then $h(x^2, \dots, x^n, f(x_1+t)+g(x_1-t))$ is also a solution of (1.4) for $0 \leq t \leq 1$. Again, we ask ourselves what happens after $t = 1$.

Since h satisfies (4.1), it will be a solution of (1.4) if and only if w satisfies

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^{1^2}} + \left\{\frac{1}{\partial h^i/\partial w}\left(\frac{\partial^2 h^i}{\partial w^2} + \Gamma^i_{jk}\frac{\partial h^j}{\partial w}\frac{\partial h^k}{\partial w}\right)\right\} \times\left[\left(\frac{\partial w}{\partial t}\right)^2 - \left(\frac{\partial w}{\partial x^1}\right)^2\right] = 0. \tag{4.2}$$

If the expression in $\{\dots\}$ depends on w only (for example, if w is a parameter along a geodesic) we can repeat the procedure desccribed before reducing (4.2) to a linear wave equation and solve it for w with the initial conditions $w|_{t=0} = f(x) + g(x)$; $w_t|_{t=0} = f'(x) - g'(x)$. Having done this, we obtain w in terms of $f(x^1 + t)$ and $g(x^1 - t)$.

One can also consider another problem. Let γ be a geodesic described by the following equation:

$$\frac{\partial^2 h^i}{\partial x^{n2}} + \Gamma^i_{jk}(h)\frac{\partial h^j}{\partial x^n}\frac{\partial h^k}{\partial x^n} = 0, \qquad h = h(x^n). \tag{4.3}$$

Also let $f(\cdot)$ and $g(\cdot)$ be two scalar functions satisfying (1.7). Then

$$
\begin{aligned}
& u = h(x^n + f(A \cdot x - t)), \\
& \qquad\qquad A = (A_1, \dots , A_{n-1}, 0), \qquad \sum_{p=1}^{n-1} A_p^2 = 1 \\
& v = h(x^n + g(B \cdot x - t)), \\
& \qquad\qquad B = (B_1, \dots , B_{n-1}, 0), \qquad \sum_{p=1}^{n-1} B_p^2 = 1
\end{aligned}
\tag{4.4}
$$

are solutions of (1.4). Away from $\operatorname{supp} f(A \cdot x - t) \cup \operatorname{supp} g(B \cdot x - t)$ each of the (4.4) functions coincides with $h(x^n)$. We can view u and v as describing distortions propagating on $h(x^n)$.

Function

$$
\begin{gathered}
\varphi = h(x^n + w(t, x)) \\
w(t, x) = f(A \cdot x - t) + g(B \cdot x - t)
\end{gathered}
\tag{4.5}
$$

for $(x, t) \notin \operatorname{supp} f(A \cdot x - t) \cap \operatorname{supp} g(B \cdot x - t)$ describes two distortions propagating on $h(x^n)$, and we are interested in their interaction. Using the ordinary chain rule we can easily see that (4.5) is a solution of (1.4) if and only if

$$
\frac{\partial^2 w}{\partial t^2} - \sum_{p=1}^{n-1} \frac{\partial^2 w}{\partial x^{p^2}} = 0.
\tag{4.6}
$$

Linearity of (4.6) implies that $w(t, x) = f(A \cdot x - t) + g(B \cdot x - t)$ for all t, x and thus the distortions do not interact.

Acknowledgments. The work was supported by the grants from National Science and Engineering Research Council of Canada. The authors also would like to thank E. Leonard for typing the manuscript.

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