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AREA INTEGRAL ASSOCIATED WITH SINGULAR MEASURES ON THE UNIT SPHERE ON C^n

YOON JAE YOO

1. Introduction. The purpose of this paper is to study some problems relating to the Lusin area integral [**8**]. In [**1**], P. Ahern and A. Nagel introduced a modified area integral, which is given by, for $0 < p < 2$,

$$G_p(f)^2(\xi) = \int_{\mathcal{A}_\alpha(\xi)} [|\nabla f(z)|^2 \rho(z)^{1-n+2(n-m)/p} + |\nabla_T f(z)|^2 \rho(z)^{-n+2(n-m)/p}]\, d\nu(z)$$

and they proved that if μ is a positive measure on the boundary of the unit ball, such that $\mu(B(\xi,\delta)) \le C\delta^m$, (hence μ may be singular) then the following singular area integral inequality, for every f in H^p, $1 < p < 2$,

$$||G_p(f)||_{L^p(d\mu)} \le C_p ||f||_{H^p}.$$

The proof proceeds in two steps. First they showed in [**1**] that the term involving the tangential part of the gradient is essentially dominated by the other term. To treat the other part they applied an analogue, for domains in C^n, of the tent space T^1_∞, which is introduced by R.R. Coifman, Y. Meyer and E. Stein [**2, 3**].

In this paper the result of Ahern-Nagel will be extended to the case $0 < p \le 2$. Here the main tool is not T^1_∞ space but T^p_2 space.

2. Preliminaries and terminologies. For two complex n vectors $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$, the inner product $\langle z, w\rangle$ is given by $\langle z, w\rangle = \sum_{i=1}^n z_i \bar{w}_i$, and the corresponding norm will be $|z| = (\sum_{i=1}^n |z_i|^2)^{1/2}$. For ξ, η in the unit sphere S of the unit ball $B = \{|z| < 1\}$ and $\delta < 0$, let $\rho(\xi,\eta) = |1 - \langle \xi, \eta\rangle|$ and $B(\xi,\delta) = \{\eta \in S : \rho(\eta,\delta) = |1 - \langle \eta, \xi\rangle| < \delta\}$.

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Then it is well known [**3, 7**] that ρ defines a pseudo-metric on S and that the triple $(S, \rho, d\sigma)$ is a space of homogeneous type. Here $d\sigma$ denotes the area measure on S. Observe that $\sigma(B(\xi,\delta))$ is roughly proportional to δ^n for small $\delta > 0$. In this setting, we introduce an approach region associated with these balls. For $\alpha > 1$ and $\xi \in S$, let $\mathcal{A}_\alpha(\xi) = \{z \in B : |1 - \langle z, \xi\rangle| < \alpha(1-|z|)\}$. Then $\mathcal{A}_\alpha(\xi)$ is called an admissible approach region. This terminology is due to Koranyi [**6**]. Throughout this paper $d\nu$ denotes the Lebesgue measure on C^n.

As usual, throughout this paper C will denote a constant not necessarily the same at each occurrence.

3. Results. For a closed subset $F \subset S$, and $\alpha > 1$, let $\mathcal{R}^\alpha(F) = \cup\{\mathcal{A}_\alpha(\xi) : \xi \in F\}$. The tent $T(O)$ over an open set $O = F^c$ is defined by the complement of $\mathcal{R}^4(F)$.

Let f be a function defined on the unit ball B. Define a functional $A(f)$, for $\xi \in S$, by

$$A(f)(\xi) = \bigg[\int_{\mathcal{A}_4(\xi)} |f(z)|^2 \frac{d\nu(z)}{(1-|z|)^{n+1}}\bigg]^{1/2}.$$

Then f is said to be in T_2^p if $A(f) \in L^p(d\sigma)$.

A function $a(z)$ defined on B is said to be a $(p,2)$-atom if

(i) $a(z)$ is supported on the tent $T(B(\xi,\delta))$ of a ball $B(\xi,\delta)$, and

(ii)
$$\int_{T(B(\xi,\delta))} \frac{|a(z)|^2\, d\nu(x)}{1-|z|} \le [\sigma(B(\xi,\delta))]^{1-2/p}.$$

Note that a constant function is also an atom.

Theorem 1. *Let $f \in T_2^p$, $0 < p \le 1$. Then there exist a constant C_p, a sequence $\{a_j\}$ of $(p,2)$-atoms, and a sequence $\{\lambda_j\}$ of positive numbers so that*

$$|f(z)| \le \sum_{j=1}^\infty \lambda_j |a_j(z)| \quad \textit{and} \quad \sum_{j=1}^\infty \lambda_j^p \le C_p ||A(f)||^p_{L^p(d\sigma)}.$$

Fundamental arguments of the proof of Theorem 1 are due to those in [**3**]: Let F be a closed subset of S. Let γ be fixed and $0 < \gamma < 1$.

Then we say that a point $\xi \in s$ has a global γ-density with respect to F if $\sigma[F \cap B(\xi,\delta)]/(\sigma[B(\xi,\delta)]) \geq \gamma$ for all $\delta > 0$. Let $\gamma(F)$ be the set of all the points of a global γ-density with respect to F. Note that $\gamma(F)$ is a closed set and $\gamma(F)^c = \{\xi \in S : \mathcal{M}(\chi_{F^c})(\xi) > 1-\gamma\}$, where χ_{F^c} is the characteristic function of the open set F^c and $\mathcal{M}$ denotes Hardy-Littlewood's maximal function.

Lemma 1. *Let F be a closed subset of S. Then there is a constant C_γ so that $\sigma[\gamma(F)^c] \leq C_\gamma \sigma(F^c)$.*

Proof. Since Hardy-Littlewood's maximal function $\mathcal{M}$ is of weak type (1,1), there exists a constant C so that $\sigma[\{\xi \in S : \mathcal{M}(\chi_{F^c}(\xi) > 1-\gamma\}] \leq (C/(1-\gamma))||\chi_{F^c}||_{L^1(d\sigma)} = (C/(1-\gamma))\sigma(F^c)$. But the left side of the above inequality is equal to $\sigma[\gamma(F)^c]$ and so the proof is completed. □

Lemma 2. *Suppose $\alpha > 1$ is given. Then there exist constants $C_{\alpha,\gamma}$ and γ, $0 < \gamma < 1$, sufficiently close to 1, so that whenever F is a closed subset of S and Φ is a nonnegative function defined on the unit ball B, then*

$$\int_{\mathcal{R}^\alpha(\gamma(F))} \Phi(z)(1-|z|)^n \, d\nu(z) \leq C_{\alpha,\gamma} \int_F \int_{A_4(\xi)} \Phi(z)\, d\nu(z)\, d\sigma(\xi).$$

Proof. Fubini's theorem gives

$$\int_F \int_{\mathcal{A}_4(\xi)} \Phi(z)\, d\nu(z)\, d\sigma(\xi) = \int_S \Phi(z) \left[\int_F \chi_{\mathcal{A}_4(\xi)}(z)\, d\sigma(\xi) \right] d\nu(z),$$

and so, for given $z \in \mathcal{R}^{(\alpha)}(\gamma(F))$, it will suffice to show that there exists a constant $C_{\alpha,\gamma}$ so that

$$\int_F \chi_{\mathcal{A}_4}(\xi)(z)\, d\sigma(\xi) \geq C_{\alpha,\gamma}(1-|z|)^n. \tag{1}$$

Let $z \in \mathcal{R}^{(\alpha)}(\gamma(F))$. Then there exists $\xi \in \gamma(F)$ so that $z \in \mathcal{A}_{(\alpha)}(\xi)$. Now it is obvious by geometric observation that

$$\sigma[B(\xi, \alpha(1-|z|)) \cap B(z/|z|, 1-|z|)^c] \leq C_\alpha \sigma[B(\xi, \alpha(\alpha - |z|))] \tag{2}$$

for some $C_\alpha < 1$. However, it can easily be verified that

$$
\begin{aligned}
(3)\quad \sigma[F \cap B(z/|z|, 1-|z|)] + \sigma[B(\xi, \alpha(1-|z|)) \cap B(z/|z|, 1-|z|)^c] \\
&\leq \sigma[F \cap B(\xi, \alpha(1-|z|)) \cap B(z/|z|, 1-|z|)] \\
&\quad + \sigma[F \cap B(\xi, \alpha(1-|z|)) \cap B(z/|z|, 1-|z|)^c] \\
&= \sigma[F \cap B(\xi, \alpha(1-|z|))].
\end{aligned}
$$

By the property of a global γ-density, (2) and (3) imply that

$$
\begin{aligned}
\sigma[F \cap B(z/|z|, 1-|z|)] &\leq [F \cap B(\xi, \alpha(1-|z|))] \\
(4)\qquad &\quad - \alpha[B(\xi, 1-|z|) \cap B(z/|z|, 1-|z|)]^c \\
&\geq (\gamma - C_\alpha)_\sigma[B(\xi, \alpha(1-|z|)]
\end{aligned}
$$

for γ sufficiently close to 1. If $\xi \in B(z/|z|, 1-|z|)$, then $|1-\langle z, \xi\rangle| \leq 2(|1-\langle z, z/|z|\rangle| + |1-\langle z/|z|, \xi\rangle|) \leq 4(1-|z|)$, and so $z \in \mathcal{A}_4(\xi)$. Thus, from (4) it follows that

$$
\begin{aligned}
(5)\qquad \int_F \chi_{\mathcal{A}_4(\xi)}(z)\, d\sigma(\xi) &\geq \sigma[F \cap B(z/|z|, 1-|z|)] \\
&\geq C_{\alpha,\gamma}\sigma[B(\xi, \alpha(1-|z|)].
\end{aligned}
$$

But in (5) we know that $\sigma[B(\xi, \alpha(1-|z|)] \approx (1-|z|)^n$ and this gives the inequality (1). ◻

Lemma 3. *Suppose O is an open set of S. If $z \in T(O)$, then $B(z/|z|, 1-|z|) \subset O$.*

Proof. Let $z \in T(O)$. Then $z \notin \mathcal{A}_4(\xi)$ for all $\xi \in F = O^c$. That is, $|1-\langle z, \xi\rangle| \geq 4(1-|z|)$ for all $\xi \in F$. On the other hand, if $\xi \in B(z/|z|, 1-|z|)$, then $|1-\langle z, \xi\rangle| \leq 2[1-\langle z/|z|, \xi\rangle| + |1-\langle z, z/|z|\rangle|] < 4(1-|z|)$. Thus, $\xi \in O$. ◻

Finally, we need a covering lemma of Whitney type [**4**]:

Lemma 4. *Let $O \subset S$ be an open set. Then there are positive constants M, $A > 1$, $B > 1$ and $C < 1$, which depend only on the dimension, and a sequence $\{B(\xi_i, \delta_i)\}$ of balls such that $\cup_{i=1}^{\infty} B(\xi_i, \delta_i) =$*

O, $B(\xi_i, B\delta_i) \subset O$, $B(\xi_i, A\delta_i) \cap O^c \neq \varnothing$, the balls $B(\xi_i, C\delta_i)$ are pairwise disjoint, and no point in O lies in more than M of the balls $B(\xi_i, B\delta_i)$.

Proof of Theorem 1. Define, for each integer k,

$$O_k = F_k^c = \{\xi \in S; A(f)(\xi) > 2^k\}.$$

Let $O_k^* = \gamma(F_k)^c$. Then by the property of a global γ-density (with γ sufficiently close to 1), it follows that $O_k^* = \{\xi \in S : \mathcal{M}(\chi_{O_k})(\xi) > 1 - \lambda\}$. From Lemma 1 it follows that $\sigma[O_k^*] \leq C_r \sigma[O_k]$. Observe that for each k, $O_{k+1} \subset O_k$, $O_k \subset O_k^*$, $T(O_k) \subset T(O_k^*)$, and $\cup_{k=-\infty}^{\infty} T(O_k^*)$ contains the support of f. Since $\gamma(F_k)$ is a closed subset of S, O_k^* is an open set. Let $O_k^* = \cup_{j=1}^{\infty} B(\xi_{k,j}, \delta_{k,j}) \equiv \cup_{j=1}^{\infty} B_{k,j}$ be a Whitney decomposition of the open set O_k^*.

Let $\widetilde{B}_{k,j} = B(\xi_{k,j}, CM\delta_{k,j})$, where M is given in Lemma 4 and C will be chosen sufficiently large in a moment. By Lemma 3, we know that $z \in T(O_k^*)$ implies that $B(z/|z|, 1 - |z|) \subset O_k^*$. Let $z/|z| \in B_{k,j_0}$ for some j_0. If $\eta \in B(\xi_{k,j_0}, M\delta_{k,j_0}) \cap \gamma(F_k)$, then

$$\begin{aligned} 1 - |z| &\leq |1 - \langle z/|z|, \eta\rangle \\ &\leq 2[1 - \langle z/|z|, \xi_{k,j_0}\rangle| + |1 - \langle \xi_{k,j_0}, \eta\rangle|] \\ &\leq 2(1 + M)\delta_{k,j_0}. \end{aligned} \tag{6}$$

Hence, if $\xi \in B(z/|z|, 1 - |z|)$, then it follows from (6) that

$$\begin{aligned} |1 - \langle \xi_{k,j_0}, \xi\rangle| &\leq 2[1 - \langle \xi_{k,j_0}, z/|z|\rangle + |1 - \langle z/|z|, \xi\rangle|] \\ &< 2[\delta_{k,j_0} + 1 - |z|] \\ &< 2[\delta_{k,j_0} + 2(1 + M)\delta_{k,j_0}] \\ &= (6 + 2M)\delta_{k,j_0} \end{aligned}$$

If we choose C so that $6 + 2M < CM$, it follows that $B(z/|z|, 1 - |z|) \subset B(\xi_{k,j_0}, CM\delta_{k,j_0}) \equiv \widetilde{B}_{k,j_0}$ and so $T(B(z/|z|, 1 - |z|)) \subset T(\widetilde{B}_{k,j_0})$. Thus we can write $T(O_k^*) \cap T(O_{k+1}^*)^c = \cup_{j=1}^{\infty} \Delta_{k,j}$, where

$$\Delta_{k,j} = T(\widetilde{B}_{k,j}) \cap [T(O_k^*) \cap T(O_{k+1}^*)^c].$$

We distinguish two cases (A) and (B):

Case (A). For every k, suppose that $O_k^* \neq S$. If we let $\chi_{k,j}$ be the characteristic function of the set $\Delta_{k,j}$, then we have

$$|f(z)| \leq \sum_{k,j} |f(z)|\chi_{k,j}(z) \equiv \sum_{k,j} |a_{k,j}|\lambda_{k,j}, \tag{7}$$

where

$$a_{k,j}(z) = f(z)\chi_{k,j}\sigma[\widetilde{B}_{k,j}]^{1/2-1/p}\left[\int_{\Delta_{k,j}} |f(z)|^2 \frac{d\nu(z)}{1-|z|}\right]^{-1/2},$$

and

$$\lambda_{k,j} = \sigma[\widetilde{B}_{k,j}]^{-1/2+1/p}\left[\int_{\Delta_{k,j}} |f(z)|^2 \frac{d\nu(z)}{1-|z|}\right]^{1/2}.$$

It is easy to check that $a_{k,j}$ is a $(p,2)$-atom associated with the ball $\widetilde{B}_{k,j}$. Now put $F = O_{k+1}^c$, $R^{(\alpha)}(\gamma(F)) = T(O_{k+1}^*)^c$, $\gamma(F) = (O_{k+1}^*)^c$, and $\Phi(z) = |f(z)|^2(1/(1-|z|)^{n+1})\chi_{T(\widetilde{B}_{k,j})}(z)$, and apply Lemma 2 to obtain the following inequalities

$$\begin{aligned} \int_{\Delta_{k,j}} |f(z)|^2 \frac{d\nu(z)}{1-|z|} &\leq \int_{T(\widetilde{B}_{k,j})\cap T(O_{k+1}^*)^c} |f(z)|^2 \frac{d\nu(z)}{1-|z|} \\ &\leq \int_{T(O_{k+1}^*)^c} \chi_{T(\widetilde{B}_{k,j})}(z)|f(z)|^2 \frac{d\nu(z)}{1-|z|} \\ &\leq C_{\alpha,\gamma}\int_{O_{k+1}^c}\int_{\mathcal{A}_4(\xi)} |f(z)|^2\chi_{T(\widetilde{B}_{k,j})}(z)\frac{d\nu(z)\,d\sigma(\xi)}{(1-|z|)^{1+n}} \\ &\leq C_{\alpha,\gamma}\int_{O_{k+1}^c\cap\widetilde{B}_{k,j}} A(f)^2(\xi)\,d\sigma(\xi) \\ &\leq C_{\alpha,\gamma}2^{2(k+1)}\sigma(\widetilde{B}_{k,j}). \end{aligned} \tag{8}$$

Since $\sigma(\widetilde{B}_{k,j}) \leq C\sigma(B_{k,j})$ by the doubling property of $B_{k,j}$, we have

$$\begin{aligned} \sum_{k,j} \lambda_{k,j}^p &= \sum_{k,j} \sigma(\widetilde{B}_{k,j})^{1-p/2}\left[\int_{\Delta_{k,j}} |f(z)|^2 \frac{d\nu(z)}{1-|z|}\right]^{p/2} \\ &\leq C_p \sum_{k,j} 2^{pk}\sigma(B_{k,j})^{1-p/2}\sigma(B_{k,j})^{p/2} \end{aligned}$$

$$\begin{aligned}&\le C_p \sum_{k,j} 2^{pk}\sigma(B_{k,j})\\&\le C_p \sum_{k} 2^{pk}\sigma(O_k^*)\\&\le C_p \sum_{k} 2^{pk}\sigma(O_k)\\&\le C_p\|A(f)\|^p_{L^p(d\sigma)}.\end{aligned}$$

Case (B). If the case (A) does not occur, then there is an integer n such that $O_n^* = S$. Without loss of generality, we may assume $n = 1$. Then $O_1^* = S$, and $O_k^* \neq S$ if $k > 1$. Let $\Delta_1 = B \cap T(O_2^*)^c$, $\lambda_1 = \sigma(s)^{-1/2+1/p}[\int_{\Delta_1} |f(z)|^2 d\nu(z)/(1-|z|)]^{-1/2}$, where χ_{Δ_1} is the characteristic function of Δ_1. Then it can be shown that a_1 is a $(p,2)$-atom supported on B. For $k > 1$, define $a_{k,j}$ as before. Then we have

$$\begin{aligned}|f(z)| &\le |f(z)|\chi_{\Delta_1}(z) + \sum_{k\ge 2,j} |f(z)|\chi_{k,j}(z)\\&= \lambda_1 a_1 + \sum_{k\ge 2,j}\lambda_{k,j}a_{j,k}.\end{aligned}$$

Again apply Lemma 2 to obtain

$$\begin{aligned}\lambda_1^p &= \sigma(S)^{-p/2+1}\left[\int_{\Delta_1}|f(z)|^2\frac{d\nu(z)}{1-|z|}\right]^{p/2}\\&\le C\sigma(S)^{-p/2+1}\left[\int_{O_2^c}\int_{\mathcal{A}_4(\xi)}|f(z)|^2\frac{d\nu(z)}{(1-|z|)^{n+1}}\right]^{p/2}\\&\le C\sigma(S)^{-p/2+1}\left[\int_{O_2^c}A(f)^2(\xi)\,d\sigma(\xi)\right]^{p/2}\\&\le C\sigma(S)\\&\le C\sigma(O_1)\\&\le C\|A(f)\|^p_{L^p(d\sigma)}.\end{aligned}$$

For $k \ge 2$ we have as before $\sum_{k,j}\lambda_{k,j}^p \le c\|A(f)\|^p_{L^p(d\sigma)}$. This completes the proof of Theorem 1. □

Define a functional W_p by, for $\xi \in S$,

$$W_p(f)(\xi) = \left[\int_{\mathcal{A}_4(\xi)} |f(z)|^2 \frac{(1-|z|)^{2(n-m)/p}}{(1-|z|)^{1+n}} \, d\nu(z) \right]^{1/2}.$$

Note that $W_p = A$ if $m = n$.

Lemma 5. *Suppose μ is a positive measure on S satisfying $\mu(B(\xi,\delta)) \le C\delta^m$. Let a be a $(p,2)$-atom, $0 < p \le 1$, supported on the tent $T(B(\xi,\delta))$. Then there exists a constant C_p so that $\int_S W_p(a)^2 \, d\mu(\xi) \le C_p$.*

Proof. Put

$$\chi(z,\xi) = \begin{cases} 1, & \text{if } z \in \mathcal{A}_4(\xi), \\ 0, & \text{otherwise.} \end{cases}$$

Then

(9)
$$\begin{aligned}
&\int_S W_p(a)^2(\xi) \, d\nu(\xi) \\
&\qquad = \int_S \int_{\mathcal{A}_4(\xi)} |a(z)|^2 \frac{(1-|z|)^{2(n-m)/p}}{(1-|z|)^{1+n}} \, d\nu(z) \, d\mu(\xi) \\
&\qquad = \int_B |a(z)|^2 \frac{(1-|z|)^{2(n-m)/p}}{(1-|z|)^{1+n}} \left[\int_S \chi(z,\xi) \, d\mu(\xi) \right] d\nu(z).
\end{aligned}$$

It is easy to check that, for fixed $z \ne 0$, $\chi(z,\xi) = 1$ only on $B(z/|z|, 6(1-|z|))$ and so

$$\int_S \chi(z,\xi) \, d\nu(\xi) \le C(1-|z|)^m,$$

for some constant C. Now it is true from Lemma 3 that $1-|z| \le 4\delta$ for $z \in T(B(\xi,\delta))$ and for small $\delta > 0$. Thus the last integral in (9) is less than

$$C \int_{T(B(\xi,\delta))} |a(z)|^2 \frac{(1-|z|)^{2(n-m)/p}}{(1-|z|)^{1+n}} \, d\nu(z),$$

which is again less than $C\mu[B(\xi,\delta)]^{1-2/p}$. Thus,

$$\begin{aligned}\int_S W_p(a)^p(\xi)\,d\mu(\xi)&\\ &\le \left[\int_S W_p(a)^2(\xi)\,d\mu(\xi)\right]^{p/2}\left[\int_S \chi_{B(\xi,\delta)}(\xi)\,d\mu(\xi)\right]^{(2-p)/2}\\ &\le C_p.\end{aligned}$$

This completes the proof. □

Theorem 2. *Let μ be a positive measure $\circ S$ which satisfies $\mu(B(\xi,\delta)) \le C\delta^m$. If $0 < p \le 1$, then wthere exists a constant C_p so that*

$$||W_p(f)||_{L^p(d\mu)} \le C_p||A(f)||_{L^p(d\sigma)} \quad \textit{for all } f \in T_2^p.$$

Proof. Let $f \in T_2^p$. It follows from Theorem 1 that there exist a constant C, a sequence $\{a_j\}$ of $(p,2)$-atoms and a sequence $\{\lambda_j\}$ of positive numbers so that

$$|f(z)| \le \sum \lambda_j|a_j(z)| \quad \text{and} \quad \sum \lambda_j^p \le C||A(f)||^p_{L^p(d\sigma)}.$$

Replace $|f(z)|$ by its majorant $\sum \lambda_j|a_j(z)|$ to obtain that

$$\begin{aligned}W_p(f)^2(\xi) &= \int_{\mathcal{A}_4(\xi)}\left[\sum_j \lambda_j|a_j(z)|\right]^2\frac{(1-|z|)^{2(n-m)/p}}{(1-|z|)^{1+n}}\,d\nu(z)\\ &\le \left[\sum_j \lambda_j W_p(a_j)(\xi)\right]^2\end{aligned} \tag{10}$$

by Theorem 1 and the Schwarz inequality. Integrate both sides of (10) with respect to $d\mu(\xi)$ and apply Lemma 5 to get

$$\begin{aligned}\int_S W_p(f)^p(\xi)\,d\mu(\xi) &\le \int_S\sum_j \lambda_j^p W_p(a_j)^p(\xi)\,d\mu(\xi)\\ &\le C_p||A(f)||^p_{L^p(d\sigma)}.\end{aligned}$$

This completes the proof. $\square$

4. Generalized area integral. In this section we study a modified area integral of Lusin type. Let μ denote a positive measure on S which satisfies $\mu(B(\xi,\delta)) \leq C\delta^m$. Define, for $0 < p \leq 2$,

$$G_p(f)^2(\xi) = \int_{\mathcal{A}_4(\xi)} \bigg[|\nabla f(z)|^2 (1-|z|)^{1-n+2(n-m)/p} + |\nabla_T f(z)|^2 (1-|z|)^{-n+2(n-m)/p} \bigg] \, d\nu(z).$$

Here $\nabla_T f$ denotes the gradient of f in the tangential direction.

Let $H^p(B)$ be the family of all holomorphic functions defined on the unit ball in C^n satisfying the following growth condition $\int_S |f(r\xi)|^p d\sigma(\xi) < \infty$ for all r, $0 < r < 1$. Dentoe $||f||_{H^p}$ by

$$||f||_{H^p} = \sup_{0<r<1} \left[\int_S |f(r\xi)|^p \, d\sigma(\xi) \right]^{1/p}.$$

For the general reference about the H^p space, see [**7**].

Theorem 4. *Let μ be a positive measure on S satisfying $\mu(B(\xi,\delta)) \leq C\delta^m$. For $0 < p \leq 2$, there exists a constant C_p so that for every $f \in H^p(B)$*

$$||G_p||_{L^p(d\mu)} \leq C_p ||f||_{H^p}.$$

Proof. Due to Lemma 5.1 [**1**, p. 382], it suffices to deal with part of $G_p(f)$ which does not involve the tangential component of the gradient. Now, by Theorem 3, there exists a constant C_p so that

$$(11) \quad ||W_p[(1-|z|)|\nabla f(z)|]||_{L^p(d\mu)} \leq C_p ||A[(1-|z|)|\nabla f(z)|]||_{L^p(d\sigma)}.$$

Again, by Theorem [**8**] for the standard area integral inequality of holomorphic functions, we have

$$(12) \qquad ||A(1-|z|)\nabla f(z)||_{L^p(d\sigma)} \leq C_p ||H(f)||_{H^p}.$$

Combining inequalities (11) and (12), we obtain the conclusion. $\square$

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Department of Mathematics, Teachers College, Kyungpook National University, Taegu, Korea 702-701