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A NOTE ON SOME UNCOMPLEMENTED SUBSPACES

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ABSTRACT. We show that nest algebras are, in general, not complemented as subspaces in the Banach space of all bounded linear operators on a given Hilbert space.

1. All the subspaces in this note are closed subspaces. One of the most useful features of Hilbert spaces is that every subspace in a Hilbert space is complemented. For Banach spaces, the situation is quite different. Let T be the unit circle and m the normalized Lebesgue measure on T. Let $H = L^p(T,m)$ and $K = H^p(T,m)$, $1 \leq p \leq \infty$, be the usual Hardy spaces on the unit circle. It is known that K is complemented in H when $1 < p < \infty$ and not complemented when $p = 1$ or ∞. If we let X be a compact Hausdorff space, $C(X)$ be the set of all continuous functions on X and $A \subseteq C(X)$ a uniform algebra, it is not known whether or not A is always uncomplemented as a subspace of $C(X)$. Glicksberg [**2**], Pelczinsky [**4**] and Sidney [**5**] made some significant progress in this direction, but the general question still remains open. In this note we investigate the same problem for nest algebras, which many believe are a noncommutative analogue of Dirichlet algebras.

Let H be a Hilbert space and (BH) be the set of all bounded linear operators on H. A nest $\mathcal{N}$ is a totally ordered set of orthogonal projections. The corresponding nest algebra is

$$\operatorname{Alg}\mathcal{N} = \{A \in B(H) \mid P^{\perp}AP = 0,\ \forall\, P \in \mathcal{N}\}.$$

If we let $H = L^2(T,m)$, where T denotes the unit circle with normalized Lebesgue measure m, $\{e_n \mid n \in Z\}$ denote the usual orthonormal base for $L^2(T,m)$ (where $e_n(z) = z^n$, $z \in T$, $n \in Z$), P_n denote the orthogonal projection of H onto the subspace $[e_n, e_{n+1}, \dots]$, $n \in Z$, where $[\cdot]$ denotes the closed linear span and $\mathcal{N} = \{P_n\}$, $n \in Z$, then $\operatorname{Alg}\mathcal{N}$ is the set of bounded linear operators with lower triangular

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matrix representations with respect to $\{e_n \mid n \in Z\}$. Each function $\phi \in L^\infty$ corresponds to a multiplication operator on H, which we denote L_ϕ. Let H^∞ be the usual Hardy space on the unit circle, $U = L_{e_1}$ the bilateral shift and M the set of multiplication operators, that is, $M = \{L_\phi \mid \phi \in L^\infty(T, m)\}$.

Lemma 1.1 [**1**]. *M is a maximal abelian von Neumann subalgebra of $B(H)$ generated by U.*

Remark 1. For any maximal abelian subalgebra A of an algebra B, $A' = A$. Thus $M' = M$.

Lemma 1.2 [**3**]. *There is no bounded linear projection of $L^\infty(T, m)$ onto $H^\infty(T, m)$.*

Lemma 1.3. $||L_\phi|| = ||\phi||_\infty$ *for all* $\phi \in L^\infty(T, m)$.

Lemma 1.4. *Let $\mathcal{N} = \{P_n\}$, $n \in Z$ be the nest of $H = L^2(T, m)$ as above and $\phi \in L^\infty(T, m)$, then $L_\phi \in \operatorname{Alg}\mathcal{N}$ if and only if $\phi \in H^\infty(T, m)$.*

Theorem 1.5. *Let $H = L^2(T, m)$, $P_n = [e_n, e_{n+1}, \dots]$, $\mathcal{N} = \{P_n\}$, $n \in Z$. Then there is no bounded linear projection of $B(H)$ onto $\operatorname{Alg}\mathcal{N}$.*

Proof. Suppose there existed such a projection P. Let N be the additive group of all positive integers and Λ a Banach limit on N. Then Λ is a state on $l^\infty(N)$ with the following property: Given any $(a_n) \in l^\infty(N)$, $\Lambda((a_n)) = \Lambda((a_{n+1}))$. We now define a new projection $\tilde{P}$ in the following way:

For $A \in B(H)$ define the operator $\tilde{P}(A)$ as follows: for any $x, y \in H$,

$$(\tilde{P}(A)x, y) = \Lambda((U^n P(U^{-n} A U^n) U^{-n} x, y)).$$

Thus, we have a map $\tilde{P} : A \mapsto \tilde{P}(A)$. It is routine to verify that $\tilde{P}$ is a well-defined bounded linear map. We assert that

(1) $\tilde{P}(A) \in \operatorname{Alg}\mathcal{N}$ for all $A \in B(H)$.

(2) $\tilde{P}(A) = A$ for all $A \in \operatorname{Alg}\mathcal{N}$.

(3) $\tilde{P}(A) \in M$ for all $A \in M$.

Note that $A \in \operatorname{Alg}\mathcal{N}$ if and only if $(Ae_i, e_j) = 0$ for all $j < i$, $i, j \in Z$. Hence, it is easy to see that if $A \in \operatorname{Alg}\mathcal{N}$, then $U^{-n}AU^n \in \operatorname{Alg}\mathcal{N}$ for all $n \in Z$. Since $P(U^{-n}AU^n) \in \operatorname{Alg}\mathcal{N}$, we have $U^nP(U^{-n}AU^n)U^{-n} \in \operatorname{Alg}\mathcal{N}$ for all $A \in B(H)$. Hence, $(\tilde{P}(A)e_i, e_j) = \Lambda((U^nP(U^{-n}AU^n)U^{-n}e_i, e_j)) = 0$ for all $j < i$, $i, j \in Z$, which implies that $\tilde{P}(A) \in \operatorname{Alg}\mathcal{N}$ for all $A \in B(H)$. Therefore, (1) is proved.

To prove (2), note that for all $A \in \operatorname{Alg}\mathcal{N}$, $U^{-n}AU^n \in \operatorname{Alg}\mathcal{N}$; thus, $P(U^{-n}AU^n) = U^{-n}AU^n$. From this, we can obtain

$$\begin{aligned}(\tilde{P}(A)x, y) &= \Lambda((U^nP(U^{-n}AU^n)U^{-n}x, y)) = \Lambda((U^nU^{-n}AU^nU^{-n}x, y)) \\ &= \Lambda((Ax, y)) = (Ax, y), \qquad \forall\, x, y \in H.\end{aligned}$$

Therefore, we have $\tilde{P}(A) = A$ for all $A \in \operatorname{Alg}\mathcal{N}$, as we desired.

We now prove (3). For all $A \in M$, by Lemma 1.1 and Remark 1, $A \in M = M'$. Thus A commutes with U (and therefore with U^{-1}), which implies $U^{-n}AU^n = A$. It follows that

$$\begin{aligned}(\tilde{P}(A)x, y) &= \Lambda((U^nP(U^{-n}AU^n)U^{-n}x, y)) = \Lambda((U^nP(A)U^{-n}x, y)) \\ &= \Lambda((U^{n+1}P(A)U^{-(n+1)}x, y)), \qquad \forall\, x, y \in H.\end{aligned}$$

(The last equality follows from the property that $\Lambda(a_n) = \Lambda(a_{n+1})$, for all $((a_n)) \in l^\infty(N)$.) Note also that

$$\begin{aligned}\Lambda((U^{n+1}P(A)U^{-(n+1)}x, y)) &= \Lambda((U^nP(A)U^{-n}U^{-1}x, U^{-1}y)) \\ &= (\tilde{P}(A)U^{-1}x, U^{-1}y) \\ &= (U\tilde{P}(A)U^{-1}x, y), \qquad \forall\, x, y \in H.\end{aligned}$$

We have $(\tilde{P}(A)x, y) = (U\tilde{P}(A)U^{-1}x, y)$ for all $x, y \in H$. Thus, $\tilde{P}(A) = U\tilde{P}(A)U^{-1}$ and so $\tilde{P}(A)$ commutes with U and U^{-1}. Since U and U^{-1} generate M in the weak operator topology, $\tilde{P}(A) \in M' = M$, which completes the proof of (3).

Now we could define a bounded linear projection $\hat{P}$ of $L^\infty(T, m)$ onto $H^\infty(T, m)$, for all $f \in L^\infty(T, m)$:

$$L_f \in M, \quad \text{hence } \tilde{P}(L_f) \in M \cap \operatorname{Alg}\mathcal{N}.$$

By Lemma 1.4, $\tilde{P}(L_f) = L_h$, for some $h \in H^\infty(T, m)$. Let $\hat{P}(f) = h$. Then it is easy to see that $\hat{P}$ is a well-defined linear operator. To see the boundedness of $\hat{P}$, observe that

$$||\hat{P}(f)||_\infty = ||h||_\infty = ||L_h|| = ||\tilde{P}L_f|| \leq ||\tilde{P}||\,||L_f|| = ||\tilde{P}||\,||f||_\infty.$$

(Here $||\cdot||$ denotes the uniform operator norm, and the second and the last equalities follow from Lemma 1.3.)

Hence, $\hat{P}$ would be a bounded linear operator from $L^\infty(T, m)$ into $H^\infty(T, m)$. Furthermore, if $h \in H^\infty(T, m)$, then $L_h \in \operatorname{Alg}\mathcal{N}$ (by Lemma 1.4). Therefore, $\tilde{P}(L_h) = L_h$, which gives us $\hat{P}(h) = h$. Thus $\hat{P}$ would be a bounded linear projection of $L^\infty(T, m)$ onto $H^\infty(T, m)$, which contradicts Lemma 1.2. □

Theorem 1.6. *Let H be any Hilbert space, $\mathcal{N}$ an arbitrary nest with the corresponding nest algebra* $\operatorname{Alg}\mathcal{N}$*, and suppose that $\mathcal{N}$ contains infinitely many orthogonal projections. Then there is no bounded linear projection of $B(H)$ onto* $\operatorname{Alg}\mathcal{N}$*.*

Proof. This general case can be easily reduced to the special case above; we omit the proof. □

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