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WEAK AND SUPPORT-OPEN TOPOLOGIES ON $C(X)$

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ABSTRACT. This paper studies the weak topology w on the set $C(X)$ of all continuous real valued functions on X, with respect to the dual of $C_k(X)$ where $C_k(X)$ has the compact-open topology. A related set-open topology s on $C(X)$ is introduced using sets that are supports of the continuous linear functionals in the dual of $C_k(X)$. It is shown that s is finer than w, and that under certain conditions if a set-open topology on $C(X)$ is finer than s, then the dual of this function space is equal to the dual of $C_k(X)$. Characterizations of metrizability and completeness of these function spaces are given in terms of topological properties of X. Also an Ascoli theorem is established for $C_s(X)$, and from this it follows that $C_k(X)$ is the k-extension of $C_s(X)$ for certain X.

1. Introduction. The set $C(X)$ of all continuous real-valued functions on a Tychonoff space X has a number of natural topologies. Three commonly used topologies on $C(X)$ are the point-open topology p, the compact-open topology k and the topology of uniform convergence u. The corresponding topological spaces are respectively denoted by $C_p(X)$, $C_k(X)$ and $C_u(X)$. It is easily seen that $p = k$ if and only if the compact subsets of X are finite, and that $k = u$ if and only if X is compact. These two conditions are quite extreme in nature. So there are considerable differences among these topologies. The gap between k and u has been especially felt in measure theory, and consequently in the last four decades there have been quite a few topologies introduced that lie between k and u, such as the strict topology, the σ-compact-open topology, the topology of uniform convergence on σ-compact subsets, and the topology of uniform convergence on bounded subsets (see, for example, [**6, 23, 8, 9, 14, 16, 3** or **22**]).

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In this paper we concentrate on topologies for $C(X)$ that lie between p and k. One of the two topologies we study here is the weak topology on $C(X)$ that is induced by the set of all continuous linear functionals on $C_k(X)$. This topology has been used, for example, in [**27**]. To help understand this weak topology on $C_k(X)$, we introduce a new related set-open topology in which the sets are the compact support sets of the continuous linear functionals on $C_k(X)$. We compare these topologies to each other and to other standard topologies on $C(X)$. Of special interest is the interaction of properties of $C(X)$ under these topologies with certain topological properties of X.

Let $M(X)$ (respectively, $M_p(X)$) be the set of all real-valued continuous linear functionals on $C_k(X)$ (respectively, on $C_p(X)$). The set $M(X)$ is called the dual of $C_k(X)$ and can be identified with the set of all regular finite Borel measures on X with compact support. Our definitions of the two topologies mentioned above are based on the fact that $C_p(X)$ can be viewed in the following two different ways. First, the point-open topology can be viewed as a 'set-open topology' where a subbasic open set looks like $[A, V] = \{f \in C(X) : f(A) \subseteq V\}$ where A is a finite subset of X and V is an open subset of the reals. But also the point-open topology can be viewed as the weak topology on $C(X)$ induced by $M_p(X)$.

The following conventions are used throughout the paper. All spaces are Tychonoff spaces. If X and Y are two spaces with the same underlying set, then $X = Y$, $X \leq Y$, $X < Y$ indicate, respectively, that X and Y have the same topology, that the topology on Y is finer than or equal to the topology on X, and that the topology on Y is strictly finer than the topology on X. The symbols $\mathbf{R}$ and $\mathbf{N}$ denote the spaces of real and natural numbers, respectively. Also $\beta\mathbf{N}$ denotes the Stone-Čech compactification of $\mathbf{N}$, and $\mathbf{N}^* = \beta\mathbf{N}\backslash\mathbf{N}$. The cardinality of a set X and the dimension of a linear space X are denoted by $\operatorname{card}(X)$ and $\dim X$, respectively. The constant zero function on $C(X)$ is denoted by f_0. Finally, for $f \in C(X)$, $A \subseteq X$ and $\varepsilon > 0$, define $\langle f, A, \varepsilon\rangle = \{g \in C(X) : |f(x) - g(x)| < \varepsilon \text{ for all } x \in A\}$; and $p_A(f) = \sup\{|f(x)| : x \in A\}$ whenever f is bounded on A. For functional analytic properties on function spaces, such as barreled or bornological spaces, one may refer to [**20, 24** or **4**].

2. Weak and support-open topologies. Let L be a subset of $M(X)$. For each finite subset F of L, define a seminorm p_F on $C(X)$ by $p_F(f) = \max\{|\lambda(f)| : \lambda \in F\}$ for each $f \in C(X)$. Then the collection of seminorms $\{p_F : F \text{ is a finite subset of } L\}$ generates a locally convex topology on $C(X)$, called the weak topology on $C(X)$ generated by L. For each $x \in X$, let ϕ_x be the evaluation functional defined on $C(X)$ by $\phi_x(f) = f(x)$ for each $f \in C(X)$. Now if $L = \{\phi_x : x \in X\}$, then the linear space generated by L is $M_p(X)$, and the weak topology on $C(X)$ generated by L is the same as the point-open topology on $C(X)$. The set $\{\phi_x : x \in X\}$ is actually linearly independent, and consequently $\dim M_p(X) = \operatorname{card}(X)$. This in turn implies that $\dim M(X) \geq \operatorname{card}(X)$. On the other hand, if $L = M(X)$, then the weak topology on $C(X)$ generated by L is called the weak topology on $C_k(X)$. We denote this topology by w and the corresponding space by $C_w(X)$. It is clear that $C_p(X) \leq C_w(X) \leq C_k(X)$. In particular, $C_w(X)$ is a Hausdorff space since $C_p(X)$ is a Hausdorff space.

Since $C_w(X)$ is a locally convex space, its topology is determined by the neighborhoods at f_0. Given a finite subset F of $M(X)$ and an $\varepsilon > 0$, define $V_{F,\varepsilon} = \{f \in C(X) : |\lambda(f)| < \varepsilon \text{ for each } \lambda \in F\}$. Then the collection $\{V_{F,\varepsilon} : F \text{ is a finite subset of } M(X) \text{ and } \varepsilon > 0\}$ forms a neighborhood base at f_0 for $C_w(X)$.

Next we define the support-open topology on $C(X)$. To do this, we need to first discuss the concept of support sets. The following ideas play a key role. If $\lambda \in M(X)$ and $A \subseteq X$, then λ is said to be supported on A if whenever $f \in C(X)$ with $f|_A = 0$ then $\lambda(f) = 0$. The set K_λ, defined by $K_\lambda = \cap\{K : K \text{ is compact in } X \text{ and } \lambda \text{ is supported on } K\}$, is compact; and K_λ is nonempty if and only if $\lambda \neq 0$. Now K_λ is called the support of λ and is the smallest compact set on which λ is supported. Note that $\lambda \in M_p(X)$ if and only if K_λ is a finite set.

An element λ of $M(X)$ is called positive provided that $\lambda(f) \geq 0$ for all $f \in C(X)$ with $f \geq 0$. Let $M^+(X)$ be the set of all positive elements of $M(X)$. It can be shown that if λ is positive and if $f \geq g$ on K_λ, then $\lambda(f) \geq \lambda(g)$. For each $\lambda \in M(X)$ and $f \in C(X)$ with $f \geq 0$, define $\lambda^+(f) = \sup\{\lambda(g) : 0 \leq g \leq f\}$, $\lambda^-(f) = \sup\{-\lambda(g) : 0 \leq g \leq f\}$ and $|\lambda|(f) = \sup\{\lambda(g) : |g| \leq f\}$. These give elements of $M(X)$: extend λ^+, λ^- and $|\lambda|$ to all of $C(X)$ by linearity and by using $f = f^+ - f^-$ with their obvious meanings. A reference for these ideas is [**1**]; also see [**15**] or [**13**].

We define a space X to be a support space provided that $X = K_\lambda$ for some $\lambda \in M^+(X)$. If a subspace Y of X is a support space, then we call Y a support set in X. In particular, Y is a support set in X if and only if $Y = K_\lambda$ for some $\lambda \in M^+(X)$. We denote the family of all support sets in X by $s(X)$.

Characterizations of the concept of support space can be made by both measure-theoretic and topological ideas. The measure-theoretic characterization of support space is given by the following theorem (see [**10**, 7.6.1 and 7.6.5]).

Theorem 2.1. *A space X is a support space if and only if X is compact and there exists a regular finite Borel measure on X which is strictly positive on each nonempty open subset.*

A topological characterization of support space is given by the following property due to J.L. Kelley [**11**]. For each nonempty finite family $\mathcal{F} = \{U_1, \dots, U_n\}$ of nonempty open subsets of X, define $\mathrm{cal}\,\mathcal{F}$ to be the largest integer k such that $\cap\{U_i : i \in S\} \neq \varnothing$ for some $S \subseteq \{1, \dots, n\}$ with $\mathrm{card}\,(S) = k$. Then for each family $\mathcal{U}$ of nonempty open subsets of X, define $\kappa(\mathcal{U}) = \inf\{\mathrm{cal}\,\mathcal{F}/n : n \in \mathbf{N}$ and $\mathcal{F} = \{U_1, \dots, U_n\} \subseteq \mathcal{U}\}$. Now we say that X has property K provided that the set of all nonempty open subsets of X can be written as a countable union $\cup_{n=1}^{\infty} \mathcal{U}_n$ where each $\kappa(\mathcal{U}_n) > 0$. The next theorem relates this property to that of support spaces (see [**7**, Theorem 6.4]).

Theorem 2.2. *A space is a support space if and only if it is a compact space having property K.*

A space is said to have the countable chain condition (abbreviated to ccc) if every family of pairwise disjoint nonempty open subsets is countable. It is easily seen that every space having property K has ccc, and that every separable space has property K. In particular, a compact separable space is a support space. This means that in a metrizable space every compact subset is a support set.

Like separability and ccc, property K is preserved by open subsets, by countable unions, by closures, and by continuous images. It is also preserved by taking a coarser topology. For these results see [**7**, pp. 34, 35 and 127, or **19**].

The support-open topology is an example of a 'set-open topology' in the following general sense. If α is a compact family from X, that is, a family of compact subsets of X, then let $C_\alpha(X)$ denote the space $C(X)$ with the topology having subbasic open sets of the form $[A,V] = \{f \in C(X) : f(A) \subseteq V\}$ where $A \in \alpha$ and V is open in $\mathbf{R}$. The set-open topology on $C_\alpha(X)$ is a completely regular topology that is coarser than or equal to the topology of uniform convergence on members of α (i.e., the topology generated by sets of the form $\langle f, A, \varepsilon\rangle$ for $f \in C(X)$, $A \in \alpha$ and $\varepsilon > 0$; or equivalently, the topology generated by the seminorms p_A for $A \in \alpha$). Now if $\alpha = s(X)$, we call the topology on $C_\alpha(X)$ the support-open topology, and denote this space by $C_s(X)$.

Following the ideas in [**17**], we use an 'admissible' family α to establish needed properties of $C_\alpha(X)$. If α is a compact family from X, we say that α is admissible provided that (1) for every $A, B \in \alpha$, there exists $c \in \alpha$ such that $A \cup B \subseteq C$, and (2) for each $A \in \alpha$, each closed subset B of A and each open set U in X with $B \subseteq U$, there exists a finite $\beta \subseteq \alpha$ such that $B \subseteq \cup\beta \subseteq U$.

Lemma 2.3. *If α is an admissible compact family from X, then for each $A \in \alpha$ and each finite open cover $\mathcal{U}$ of A in X, there exists a finite $\beta \subseteq \alpha$ such that $A \subseteq \cup\beta$ and β refines $\mathcal{U}$.*

Proof. Let $A \in \alpha$, and let $\mathcal{U}$ be a finite open cover of A in X. Since A is compact, there exist closed subsets $B_1, \dots, B_n$ of A such that $A = B_1 \cup \dots \cup B_n$ and, for each $n = 1, \dots, n$, $B_i \subseteq U_i$ for some $U_i \in \mathcal{U}$. Then, for each i, there exists a finite $\beta_i \subseteq \alpha$ such that $B_i \subseteq \cup\beta_i \subseteq U_i$. So if $\beta = \beta_1 \cup \dots \cup \beta_n$, then $A \subseteq \cup\beta$ and β refines $\mathcal{U}$. □

Lemma 2.4. *If α is an admissible compact family from X, then the collection $\{\langle f, A, \varepsilon\rangle : A \in \alpha$ and $\varepsilon > 0\}$ forms a neighborhood base at f for $C_\alpha(X)$.*

Proof. Let $f \in C(X)$, $A \in \alpha$ and $\varepsilon > 0$. Since A is compact, there exist $x_1, \dots, x_n \in A$ such that $f(A) \subseteq V_1 \cup \dots \cup V_n$, where each V_i is the interval $(f(x_i) - \varepsilon/2, f(x_i) + \varepsilon/2)$. Then, by Lemma 2.3, there exists a finite $\beta \subseteq \alpha$ such that $A \subseteq \cup\beta$ and β refines

$\{f^{-1}(V_i) : i = 1, \dots, n\}$. Now it is easily seen that $f \in \cap\{[B, V_i] : B \in \beta, f(B) \subseteq V_i, i = 1, \dots, n\} \subseteq \langle f, A, \varepsilon\rangle$, which shows that $\langle f, A, \varepsilon\rangle$ is a neighborhood of f in $C_\alpha(X)$. The fact that each basic neighborhood of f in $C_\alpha(X)$ contains some $\langle f, A, \varepsilon\rangle$ now follows from property (1) of the admissibility of α. $\square$

Lemma 2.5. *Let α be an admissible compact family from X, and let β be any family of closed subsets of X. Then $C_\beta(X) \leq C_\alpha(X)$ if and only if every member of β is contained in a finite union of members of α.*

Proof. For the sufficiency, let $f \in [A, V]$ where $A \in \alpha$ and V is open in $\mathbf{R}$. Then there exist $B_1, \dots, B_n \in \beta$ with $A \subseteq B_1 \cup \dots \cup B_n$. Since β is admissible, for each $i = 1, \dots, n$, there exists a finite $\beta_i \subseteq \beta$ such that $A \cap B_i \subseteq \cup\beta_i \subseteq f^{-1}(V)$. It then follows that $f \in \cap\{[B, V] : B \in \beta_1 \cup \dots \cup \beta_n\} \subseteq [A, V]$, so that $[A, V]$ is open in $C_\beta(X)$. The necessity follows from the fact that X is completely regular (see [**18**, Theorem 1.1.1]). $\square$

We now apply these lemmas to the case that $\alpha = s(X)$. Lemma 2.4 has the following consequence.

Theorem 2.6. *For any space X, $s(X)$ is an admissible compact family from X, and so the support-open topology on $C_s(X)$ is a locally convex topology generated by the collection of seminorms $\{p_A : A \in s(X)\}$.*

Proof. To show that $s(X)$ is admissible, let $A \in s(X)$, let B be a closed subset of A, and let U be an open subset of X with $B \subseteq U$. Since B is compact, there exists an open subset V of X with $B \subseteq V$ and $\overline{V} \subseteq U$. Now $V \cap A$ has property K, so that $\overline{V \cap A}$ also has property K. Then since $\overline{V \cap A} \subseteq A$, it is compact; and hence $\overline{V \cap A} \in s(X)$. Finally, note that $\overline{V \cap A} \subseteq \overline{V} \subseteq U$. This shows that $s(X)$ is admissible, and hence the last part of the theorem follows from Lemma 2.4. $\square$

By definition, the support-open topology s on $C(X)$ is coarser than or equal to the compact-open topology k. The next theorem shows that the support-open topology on $C(X)$ is finer than or equal to the weak topology on $C_k(X)$.

Theorem 2.7. *Every member of $M(X)$ is continuous on $C_s(X)$.*

Proof. Let $\lambda \in M(X)$. Now $\lambda = \lambda^+ - \lambda^-$; let A be the support K_{λ^+} of λ^+. Then, for any $\varepsilon > 0$, choose δ such that $0 < \delta < \varepsilon/(2\lambda^+(1)+1)$. For any $f \in \langle f_0, A, \delta\rangle$, $|\lambda^+(f)| \le \lambda^+(|f|) \le \lambda^+(\delta) < \varepsilon$. Then since $\langle f_0, A, \delta\rangle$ is a neighborhood of f_0 in $C_s(X)$, λ^+ must be continuous on $C_s(X)$. A similar argument shows that λ^- is continuous on $C_s(X)$, so that λ is indeed continuous on $C_s(X)$. $\square$

Note that Theorem 2.7 says that the dual of $C_s(X)$ is equal to $M(X)$. We now show that $s(X)$ is, in a sense, the smallest admissible compact family α from X such that the dual of $C_\alpha(X)$ is equal to $M(X)$.

Theorem 2.8. *Let α be an admissible compact family from X. Then the following are equivalent.*

(a) $C_s(X) \le C_\alpha(X)$.

(b) *$s(X)$ refines α.*

(c) *The dual of $C_\alpha(X)$ is equal to $M(X)$.*

Proof. Lemma 2.5 shows that (b) implies (a), and Theorem 2.7 shows that (a) implies (c). So to show that (c) implies (b), let $A \in s(X)$. Then $A = K_\lambda$ for some $\lambda \in M^+(X)$. Since λ is continuous on $C_\alpha(X)$, it is easy to show that λ is supported on some $A' \in \alpha$ (see [**15**, Lemma 1.1 and **13**, Lemma 2.1]). But then $K_\lambda \subseteq A'$, so that $s(X)$ refines α. $\square$

It is Theorem 2.8 that motivates our introduction of the support-open topology. This theorem shows that, for studying the dual of $C_k(X)$, it is the set of all support sets in X that plays a vital role rather than the set of all compact subsets of X.

3. Comparison of topologies and examples. In this section we compare the weak and the support-open topologies to each other and to other standard topologies. First, using basic facts and Theorem 2.7, we have for any space X,

$$C_p(X) \leq C_w(X) \leq C_s(X) \leq C_k(X) \leq C_u(X).$$

The next two theorems indicate when these inequalities are equalities. The proof of the first theorem follows from Lemma 2.5. In this theorem, 'X is compactly supported' means that every compact subset of X is contained in a support set in X.

Theorem 3.1. *For any space X, $C_s(X) = C_k(X)$ if and only if X is compactly supported.*

Since $C_k(X) = C_u(X)$ if and only if X is compact, we have the following corollary.

Corollary 3.2. *For any space X, $C_s(X) = C_u(X)$ if and only if X is a support space.*

For the next result, we need to introduce some terms. First, a space is called pseudofinite if all of its compact subsets are finite. Next, in a locally convex space, a set is called bounded if it is absorbed by every neighborhood of the origin. Finally, a subset A of a locally convex space is called precompact if for every neighborhood U of the origin, there exist $a_1, \dots, a_n \in A$ such that $A \subseteq \cup\{a_i + U : 1 \leq i \leq n\}$. Every bounded set is precompact (see [**21**, p. 50]).

The equivalence of conditions (b) and (c) in the next theorem is an extension of Theorem 9 in [**27**].

Theorem 3.3. *For any space X, the following are equivalent.*

(a) $C_p(X) = C_w(X)$.

(b) $C_w(X) = C_s(X)$.

(c) *X is pseudofinite.*

Proof. If X is pseudofinite, then $C_p(X) = C_k(X)$, so that (c) implies both (a) and (b). Now suppose that (a) is true. Then the members of

$M(X)$ are continuous on $C_p(X)$, so that the support sets in X are finite. Since the closure of a countable set has property K, every compact subset of X must be finite. Thus (a) implies (c).

Finally, suppose that (b) is true. We argue as in the proof of Theorem 9 in [**27**]. Suppose that there exists an infinite support set K of X. Let $B = \{f \in C(X) : \sup\{|f(x)| : x \in X\} \leq 1\}$, which is a bounded set in $C_s(X) = C_w(X)$. So B is precompact in $C_w(X)$. Let $V = \{f \in C(X) : p_K(f) < 1\}$, which is a neighborhood of f_0 in $C_s(X) = C_w(X)$. Then since B is precompact, there exist $f_1, \ldots, f_n \in B$ such that $B \subseteq \cup\{(f_i + V) : 1 \leq i \leq n\}$. Now K is infinite, so we may choose n distinct points $x_1, \ldots, x_n$ from K. Since X is completely regular, there exists a continuous $f : X \to [0,1]$ such that, for $1 \leq j \leq n$, $f(x_j) = -1$ if $f_j(x_j) \geq 0$ and $f(x_j) = 1$ if $f_j(x_j) < 0$. Then $f \in B$, but $p_K(f - f_j) \geq 1$ for all j. Then $f \notin \cup\{(f_i + V) : 1 \leq i \leq n\}$, which is a contradiction. □

Using the previous theorems, we illustrate with examples all of the possible combinations of equalities and inequalities among the five topologies that we have been considering.

Example 3.4. $C_p(X) = C_w(X) = C_s(X) = C_k(X) = C_u(X)$ if and only if X is a finite space.

Example 3.5. $C_p(X) = C_w(X) = C_s(X) = C_k(X) < C_u(X)$ if and only if X is an infinite pseudofinite space. Such spaces include infinite discrete spaces and countably infinite subspaces of $\beta\mathbf{N}$.

Example 3.6. $C_p(X) < C_w(X) < C_s(X) = C_k(X) = C_u(X)$ if and only if X is an infinite support space. Such spaces include infinite separable compact spaces, for example $\beta\mathbf{N}$ and cubes (powers of $[0,1]$) with no more than $2^{\aleph_0}$ factors.

Example 3.7. $C_p(X) < C_w(X) < C_s(X) < C_k(X) = C_u(X)$ if and only if X is compact, but not a support space. Such spaces include compact spaces which do not have ccc, for example $\mathbf{N}^*$ and the space ω_1^+ of ordinals less than or equal to the first uncountable ordinal ω_1.

Example 3.8. $C_p(X) < C_w(X) < C_s(X) = C_k(X) < C_u(X)$ if and only if X is compactly supported, but not compact or pseudofinite. Such spaces include noncompact nondiscrete metric spaces and the space ω_1 of countable ordinals.

Example 3.9. $C_p(X) < C_w(X) < C_s(X) < C_k(X) < C_u(X)$ if and only if X is neither compact nor compactly supported. Such spaces include the disjoint topological sum $X_1 \oplus X_2$ and the topological product $X_1 \times X_2$, where X_1 satisfies Example 3.7 and X_2 satisfies Example 3.8; for example $\mathbf{N}^* \oplus \mathbf{R}$, $\mathbf{N}^* \times \mathbf{R}$, $\omega_1^+ \oplus \omega_1 = 2\omega_1$ and $\omega_1^+ \times \omega_1$.

4. Metrizability. In this section we characterize when $C_w(X)$ and $C_s(X)$ are metrizable in terms of topological properties on X. Since the weak topology on $C_k(X)$ and the point-open topology are weak topologies on $C(X)$ generated by collections of continuous linear functionals on $C_k(X)$ that are total, it follows (see [**25**, p. 157]) that $C_w(X)$ ($C_p(X)$, respectively) is metrizable if and only if $M(X)$ ($M_p(X)$, respectively) has a countable Hamel basis. In particular, the characterization of $C_w(X)$ being metrizable resembles the characterization of $C_p(X)$ being metrizable. Now $C_p(X)$ is metrizable (first countable) if and only if X is countable (see [**18**]). The next theorem is an analogous theorem for $C_w(X)$.

Theorem 4.1. *For any space X, the following are equivalent.*

(a) *$C_w(X)$ is metrizable.*

(b) *$C_w(X)$ is first countable.*

(c) *$M(X)$ has a countable Hamel basis.*

(d) *X is a countable pseudofinite space.*

Proof. See [**25**, p. 159] for the equivalence of (a), (b) and (c). If X is pseudofinite, then $C_w(X) = C_p(X)$ by Theorem 3.3. So if X is also countable, then $C_w(X)$ is metrizable, which shows that (d) implies (a). Finally, to show that (a) implies (d), suppose $C_w(X)$ is metrizable. Since $\dim M(X) \geq \operatorname{card}(X)$, X must be countable. Let K be any compact subset of X. Then $M(X)$ contains a copy of $M(K)$ (see [**15**]). Therefore $\dim M(X) \geq \dim M(K)$. But $M(K)$ is the norm dual of $C(K)$ equipped with the supremum norm. Under this norm, $M(K)$ is a Banach space. Therefore, $\dim M(K)$ is either uncountable or finite. Since $\dim M(X)$ is countable, $\dim M(K)$ must be finite. But $\dim M(K) \geq \operatorname{card}(K)$, so that K must be finite. □

We now turn to the metrizability of $C_s(X)$. Since the topology of $C_s(X)$ is a set-open topology like that of $C_k(X)$, the characterization of $C_s(X)$ being metrizable resembles more the characterization of $C_k(X)$ being metrizable. Now $C_k(X)$ is metrizable (first countable) if and only if X is hemicompact (see [**2**]), where a hemicompact space is a space containing a countable family of compact subsets such that every compact subset of the space is contained in some member of this countable family. The analogous property for $C_s(X)$ is that of hemisupport space, by which we mean a space containing a countable family of support sets such that every support set in the space is contained in some member of this countable family.

Theorem 4.2. *For any space X, the following are equivalent.*

(a) *$C_s(X)$ is metrizable.*

(b) *$C_s(X)$ is first countable.*

(c) *X is a hemisupport space.*

(d) *X is a compactly supported hemicompact space.*

Proof. Since $s(X)$ is an admissible compact family from X, the equivalences of (a), (b) and (c) follow from Theorem 3.2 of [**17**] (or can be shown in a functional analytic way by modifying the proof for the corresponding result on $C_k(X)$ stated on page 63 of [**4**]). Clearly (d) implies (c). Finally, a metrizable locally convex space is bornological and thus has its Mackey topology. Since $C_s(X) \leq C_k(X)$, it follows that $C_s(X) = C_k(X)$. Then from Theorem 3.1 we have that (a) implies (d). □

Corollary 4.3. *For any space X, if $C_s(X)$ is metrizable, then*:

(a) $C_s(X) = C_k(X)$;

(b) *X is a countable union of support sets*;

(c) *X is a σ-compact space having* ccc.

Example 4.4. The space $\mathbf{N}^*$ is a compact space, so that $C_k(\mathbf{N}^*)$ is metrizable. But $\mathbf{N}^*$ does not have ccc (see Theorem 3.22 in [**26**]), so that $C_s(\mathbf{N}^*)$ is not metrizable, and therefore has topology strictly coarser than that of $C_k(\mathbf{N}^*)$.

5. Completeness and separability. A topological group E is complete provided that every Cauchy net in E converges to some element in E, where a net (x_α) in E is Cauchy if for every neighborhood U of 0 in E there is an α_0 such that $x_{\alpha_1} - x_{\alpha_2} \in U$ for all $\alpha_1, \alpha_2 \geq \alpha_0$ (for E additive).

The topology on $C_s(X)$ is generated by the uniformity of uniform convergence on support sets. When this uniformity is complete, then $C_s(X)$ is said to be uniformly complete. One can check that $C_s(X)$ is uniformly complete if and only if it is complete as an additive topological group. Also $C_s(X)$ is completely metrizable if and only if it is complete and metrizable (see [**5**, p. 34, 46]).

We begin by examining when $C_w(X)$ is complete. The completeness of $C_w(X)$ is much like the completeness of $C_p(X)$. In particular, $C_p(X)$ is complete if and only if X is a discrete space, which is precisely when $C_p(X) = \mathbf{R}^X$. This is because $C_p(X)$ is a dense subspace of $\mathbf{R}^X$, and a complete subspace of a space is a closed subset. It follows that $C_p(X)$ is completely metrizable if and only if X is a countable discrete space.

Theorem 5.1. *For any space X, the following are equivalent.*

(a) *$C_w(X)$ is complete.*

(b) *$C_w(X) = C_p(X) = \mathbf{R}^X$.*

(c) *X is a discrete space.*

Proof. If $C_w(X)$ is complete, then $C_w(X)$ is a product of lines (see [**12**, 20.9(2)]). Therefore, $C_w(X)$ is barreled, and hence has its Mackey topology. This means that $C_w(X) = C_s(X) = C_k(X)$. Then by Theorem 3.3, $C_w(X) = C_p(X)$.

Corollary 5.2. *For any space X, the following are equivalent.*

(a) $C_w(X)$ *is completely metrizable.*

(b) $C_w(X)$ *is metrizable and* $C_p(X)$ *is complete.*

(c) $C_w(X)$ *is complete and* $C_p(X)$ *is metrizable.*

(d) $C_w(X) = C_p(X) = \mathbf{R}^X$ *and* X *is countable.*

(e) X *is a countable discrete space.*

The completeness of $C_s(X)$ is somewhat more interesting since it resembles more the situation for $C_k(X)$. In particular, $C_k(X)$ is complete if and only if X is a k_R-space (i.e., whenever f is a real-valued function on X such that $f|_A$ is continuous for each compact subset A of X, then f is continuous). We must now define the analogous property to k_R-space using support sets. We define X to be an s_R-space provided that whenever f is a real-valued function on X such that $f|_A$ is continuous for each support set A in X, then f is continuous. Clearly, every s_R-space is a k_R-space.

Every hemicompact k_R-space is a k-space (i.e., whenever S is a subset of X such that $S \cap A$ is closed for every compact subset A in X, then S is closed). We again have an analogous concept using support sets. We define X to be an s-space provided that whenever S is a subset of X such that $S \cap A$ is closed for every support set A in X, then S is closed. It is evident that every s-space is a k-space. As every k-space is a k_R-space, so is every s-space an s_R-space. Since a countable compact space is a support space, it is easily shown that every first countable space is an s-space.

Example 5.3. The space $\mathbf{N}^*$ is a compact space (and hence a k-space and a k_R-space) that is not an s_R-space (and hence not an s-space). To show this, let p be any P-point in $\mathbf{N}^*$ and define function f on $\mathbf{N}^*$ by $f(x) = 0$ if $x \neq p$ and $f(p) = 1$. Now p is not an accumulation point of any support set in $\mathbf{N}^*$ (see Proposition 3 in [**19**]). This means that f is continuous on each support set in $\mathbf{N}^*$. Since f is not continuous, $\mathbf{N}^*$ is not an s_R-space.

The space in Example 5.3 is not compactly supported. However, if a k-space is compactly supported, then it is easy to show that it must be an s-space. Now, using this fact, along with Theorem 4.2 and the fact that a hemicompact k_R-space is a k-space, we obtain the following lemma.

Lemma 5.4. *Every hemisupport k_R-space is an s-space.*

The next theorem can be proved in a manner similar to Theorem 5.1.1 in [**18**].

Theorem 5.5. *For any space X, $C_s(X)$ is complete if and only if X is an s_R-space.*

Now Theorem 5.5, Theorem 4.2 and Lemma 5.4 together give characterizations of $C_s(X)$ being completely metrizable.

Corollary 5.6. *For any space X, the following are equivalent.*

(a) *$C_s(X)$ is completely metrizable.*

(b) *$C_s(X)$ is metrizable and $C_k(X)$ is complete.*

(c) *$C_s(X) = C_k(X)$ and is completely metrizable.*

(d) *X is a hemisupport s_R-space (s-space).*

Example 5.7. Let I_L^2 be the space $[0,1] \times [0,1]$ with the order topology from lexicographic ordering. Now I_L^2 is a compact Hausdorff space that is first countable, and hence is an s-space. Since I_L^2 does not have ccc, it is not a support space and hence is not compactly supported.

Example 5.7 shows that the completeness of $C_s(X)$ does not in general imply that $C_s(X) = C_k(X)$. It also shows that if $C_s(X)$ is complete and $C_k(X)$ is metrizable, $C_s(X)$ is in general not metrizable.

We end this section by turning to the separability of $C_w(X)$ and $C_s(X)$. This is easily characterized by using the characterizations of the separability of $C_p(X)$ and $C_k(X)$ (see Theorem 5 in [**27**] or Corollary 5.2.3 in [**18**]) and the fact that $C_p(X) \leq C_w(X) \leq C_s(X) \leq C_k(X)$.

Theorem 5.8. *For any space X, the following are equivalent.*

(a) $C_p(X)$ *is separable.*

(b) $C_w(X)$ *is separable.*

(c) $C_s(X)$ *is separable.*

(d) $C_k(X)$ *is separable.*

(e) X *has a coarser separable metrizable topology.*

6. Compact subsets and the k-extension. As we have seen in the previous section, the properties of s-spaces are analogous to the properties of k-spaces. Another example of this is that a space is an s-space if and only if it is a quotient space of some locally support space, where a locally support space is a space having a neighborhood base consisting of support sets. We need this idea in the proof of Lemma 6.1.

We also note that the concept of a k-extension of a space has its analog for s-spaces using support sets instead of compact sets; one might call this the s-extension. All the properties of k-extensions carry over to s-extensions. We point out some of these properties for k-extensions. First, the k-extension of a space X is the space with the same underlying set X and with the topology generated by the compact sets; that is, the closed sets in the k-extension are precisely the subsets of X whose intersection with each compact subset of X is closed in X. Denote the k-extension of X by kX. Now kX is a k-space with topology finer than or equal to the topology on X; these topologies being equal if and only if X is itself a k-space. In any case, when restricted to a compact subset of X, these topologies are always equal on this subset. So kX has the same compact subsets as X.

Our main goal in this section is to establish an Ascoli theorem which characterizes the compact subsets of $C_s(X)$. The key to establishing the necessity condition is the following lemma.

Lemma 6.1. *If Z is locally compact and X is an s-space, then a function $f : Z \times X \to Y$ is continuous if and only if $f|_{Z\times A}$ is continuous for each support set A in X.*

Proof. Let X' be the disjoint topological sum of all the support sets in X, and let $\phi : X' \to X$ be the natural map. Then ϕ is a quotient map since X is an s-space. If $i : Z \to Z$ is the identity map, then $i \times \phi : Z \times X' \to Z \times X$ is a quotient map since Z is locally compact (see Theorem 2.5.10 in [**18**]). Therefore, to show that f is continuous, it suffices to show that $g = f \circ (i \times \phi)$ is continuous. So let $(z, x) \in Z \times X'$, and let V be a neighborhood of $g(z, x)$ in Y. Now x is in some summand A of X', where $\phi(A)$ is a support set in X. By hypothesis, there exist a neighborhood W of z in Z and a neighborhood U of $\phi(x)$ in X such that $f(W \times (U \cap \phi(A))) \subseteq V$. Let $U' = \phi^{-1}(U) \cap A$, which is a neighborhood of x in X'. Then $W \times U'$ is a neighborhood of (z, x) in $Z \times X'$ such that $g(W \times U') \subseteq V$. □

Theorem 6.2. *If X is an s-space, then a subset of $C_s(X)$ is compact if and only if it is closed, pointwise bounded and equicontinuous.*

Proof. If F is a closed subset of $C_s(X)$ that is pointwise bounded and equicontinuous, then F is closed in $C_k(X)$ and hence compact by the Ascoli theorem. Conversely, let F be a compact subset of $C_s(X)$. It suffices to show that the topology on $C_s(X)$ is weakly conjoining (see [**18**, Theorem 3.2.4 and Exercise 4]). That is, we need to show that if Z is a compact space, then the exponential function $E : C(Z \times X) \to C(Z, C_s(X))$ is a surjection. To this end, let $g \in C(Z, C_s(X))$. By Lemma 6.1, to show that $E^{-1}(g)$ is continuous, it suffices to show that $E^{-1}(g)|_{Z \times A}$ is continuous for each support set A in X. So let $A \in s(X)$, let $i : A \to A$ be the identity map, and let $j : A \to X$ be the inclusion map. Now the induced function $j^* : C_s(X) \to C_s(A)$ is continuous. Also $C_s(A) = C_k(A)$, so that the evaluation map $e : C_s(A) \times A \to \mathbf{R}$ is continuous. Therefore $E^{-1}(g)|_{Z \times A} = e \circ (j^* \times i) \circ (g \times i)$ is continuous. □

Corollary 6.3. *if X is an s-space, $C_s(X) \leq C_k(X) \leq kC_s(X)$.*

Corollary 6.4. *If X is a hemicompact s-space, then $C_k(X)$ is the k-extension of $C_s(X)$.*

Now using Theorem 2.8, we obtain the following result.

Corollary 6.5. *Let X be a hemicompact s-space, and let α be an admissible compact family from X. Then $C_\alpha(X) = C_k(X)$ if and only if $C_\alpha(X)$ is a k-space and the dual of $C_\alpha(X)$ is equal to the dual of $C_k(X)$.*

Example 6.6. Let I_L^2 be the space in Example 5.7. Then $C_s(I_L^2) < C_k(I_L^2)$, and $C_k(I_L^2)$ is the k-extension of $C_s(I_L^2)$. In particular, $C_s(I_L^2)$ is not a k-space. In fact, since $C_k(I_L^2)$ is completely regular, $C_s(I_L^2)$ is not a k_R-space. In this example, since I_L^2 is compact, the k-extension of the support-open topology is in fact the topology of uniform convergence.

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