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THE POINT SPECTRA AND REGULARITY FIELDS OF NON-SELF-ADJOINT QUASI-DIFFERENTIAL OPERATORS

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ABSTRACT. In this paper the general ordinary quasi-differential expressions of nth order with complex coefficients are considered, and a number of results concerning the location of the point spectra and regularity fields of the operators generated by such expressions are obtained. Some of these are extensions or generalizations of those in the symmetric case in [**9, 10**] and [**11**], while others are new.

1. Introduction. The minimal operator T_0 and T_0^+ generated by a general quasi-differential expressions M and its formal adjoint M^+, respectively, form an adjoint pair of closed, densely defined operators in the underlying L_w^2-space, that is, $T_0 \subset (T_0^+)^*$. The operators which fulfill the role that the self-adjoint and maximal symmetric operators play in the case of a formally symmetric expression M are those which are regularly solvable with respect to T_0 and T_0^+. Such an operator S satisfies $T_0 \subset S \subset (T_0^+)^*$, and for some $\lambda \in \mathbf{C}$, $(S - \lambda I)$ is a Fredholm property of zero index; this means that S has the desirable Fredholm property that the equation $(S - \lambda I)u = f$ has a solution if and only if f is orthogonal to the solutions of $(S^* - \bar{\lambda} I)v = 0$, and furthermore the solution spaces of $(S - \lambda I)u = 0$ and $(S^* - \bar{\lambda} I)v = 0$ have the same finite dimension. This notion was originally due to Visik [**12**].

The main objectives of this paper are to investigate the location of the point spectra and regularity fields of general ordinary quasi-differential operators. Also, the results concerning the differential operator generalize all of those given in [**10, 11**] for the symmetric case and in [**8**] for the nonsymmetric case, by removing the condition on the regularity field.

We deal throughout with a quasi-differential expression M of arbitrary order n defined by a general Shin-Zettl matrix given in [**7**] and [**8**], and the minimal operator T_0 is generated by $(1/w)M[.]$ in $L_w^2(I)$,

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where w is a positive weight function on the underlying interval I. The left-hand end point of I is assumed to be regular but the right-hand end point may be either regular or singular.

2. Preliminaries. In this section we give some of the definitions and results which will be needed later; see [**1, 3** and **4**].

The domain and range of a linear operator T acting in a Hilbert space H will be denoted by $D(T)$ and $R(T)$, respectively, and $N(T)$ will denote its null space. The *nullity* of T, written $\text{nul}\,(T)$, is the dimension of $N(T)$ and the *deficiency* of T, written $\text{def}\,(T)$, is the codimension of $R(T)$ in H; thus, if T is densely defined and $R(T)$ is closed, then $\text{def}\,(T) = \text{nul}\,(T^*)$. The *Fredholm domain* of T is (in the notation of [**3**]) the open subset $\Delta_3(T)$ of $\mathbf{C}$ consisting of those values $\lambda \in \mathbf{C}$ which are such that $(T - \lambda I)$ is a Fredholm operator, where I is the identity operator on H. Thus $\lambda \in \Delta_3(T)$ if and only if $(T - \lambda I)$ has closed range and finite nullity and deficiency. The *index* of $(T - \lambda I)$ is the number

$$\text{ind}\,(T - \lambda I) = \text{nul}\,(T - \lambda I) - \text{def}\,(T - \lambda I),$$

this being defined for $\lambda \in \Delta_3(T)$.

Two closely densely defined operators A and B acting in H are said to form an adjoint pair if $A \subset B^*$ and consequently $B \subset A^*$; equivalently, $(Ax, y) = (x, By)$, for all $x \in D(A)$ and $y \in D(B)$, where $(.,.)$ denotes the inner product on H.

The *field of regularity* $\Pi(A)$ of A is the set of all $\lambda \in \mathbf{C}$ for which there exists a positive constant $k(\lambda)$ such that

$$||(A - \lambda I)x|| \geq k(\lambda)||x|| \quad \text{for all } x \in D(A),$$

or equivalently, on using the Closed-Graph theorem

$$\text{nul}\,(A - \lambda I) = 0 \quad \text{and} \quad R(A - \lambda I) \quad \text{is closed.}$$

The *joint field of regularity* $\Pi(A, B)$ of A and B is the set of $\lambda \in \mathbf{C}$ which are such that $\lambda \in \Pi(A)$, $\bar{\lambda} \in \Pi(B)$ and both $\text{def}\,(A - \lambda I)$ and $\text{def}\,(B - \bar{\lambda} I)$ are finite. An adjoint pair A and B is said to be *compatible* if $\Pi(A, B) \neq \varnothing$.

Definition 2.1. A closed operator S in H is said to be *regularly solvable* with respect to the compatible adjoint pair A and B if $A \subset S \subset B^*$ and $\Pi(A,B) \cap \Delta_4(S) \in \varnothing$, where

$$\Delta_4(S) = \{\lambda : \lambda \in \Delta_3(S),\ \operatorname{ind}(S - \lambda I) = 0\}.$$

The terminology "regularly solvable" comes from Visik's paper [**12**].

Definition 2.2. The resolvent set $\rho(S)$ of a closed operator S in H consists of the complex numbers λ for which $(S - \lambda I)^{-1}$ exists, is defined on H and is bounded. The complement of $\rho(S)$ in $\mathbf{C}$ is called the *spectrum of* S and is written $\sigma(S)$. The point spectrum $\sigma_p(S)$, continuous spectrum $\sigma_c(S)$ and residual spectrum $\sigma_r(S)$ are the following subsets of $\sigma(S)$:

$$\sigma_p(S) = \{\lambda \in \sigma(S) : (S - \lambda I) \text{ is not injective}\},$$

i.e., the set of eigenvalues of S

$$\begin{aligned}\sigma_c(S) = \{\lambda \in \sigma(S) : (S - \lambda I) &\text{ is injective},\\ &R(S-\lambda I) \subsetneqq \overline{R(S-\lambda I)} = H\};\\ \sigma_r(S) = \{\lambda \in \sigma(S) : (S - \lambda I) &\text{ is injective},\\ &\overline{R(S-\lambda I)} \neq H\}.\end{aligned}$$

For a closed operator S we have

$$\sigma(S) = \sigma_p(S) \cup \sigma_c(S) \cup \sigma_r(S).$$

An important subset of the spectrum of a closed densely defined T in H is the so-called *essential spectrum*. The various essential spectra of T are defined as in [**3**, Chapter IX] to be the sets

$$\sigma_{ek}(T) = \mathbf{C} \backslash \Delta_k(T), \qquad k = 1, 2, 3, 4, 5; \tag{2.1}$$

$\Delta_3(T)$ and $\Delta_4(T)$ have been defined earlier.

The sets $\sigma_{ek}(T)$ are closed and $\sigma_{ek}(T) \subset \sigma_{ej}(T)$ if $k < j$, the inclusion being strict in general. We refer the reader to [**1**] and [**3**, Chapter IX] for further information about the sets $\sigma_{ek}(T)$.

3. Quasi-differential expressions. The quasi-differential expressions are defined in terms of a Shin-Zettl matrix A on an interval I. The set $Z_n(I)$ of Shin-Zettl matrices on I consists of $(n \times n)$-matrices $A = \{a_{rs}\}$, $1 \le r,\ s \le n$ whose entries are complex-valued functions on I which satisfy the following conditions:

$$\begin{aligned} a_{rs} &\in L^1_{\text{loc}}(I) & & 1 \le r,\ s \le n, n \ge 2 \\ a_{r,r+1} &\neq 0 \quad \text{a.e. on } I & & 1 \le r \le n-1 \\ a_{rs} &= 0 \quad \text{a.e. on } I & & 2 \le r+1 < s \le n. \end{aligned} \tag{3.1}$$

For $A \in Z_n(I)$, the quasi-derivatives associated with A are defined by:

$$\begin{aligned} y^{[0]} &:= y \\ y^{[r]} &:= a^{-1}_{r,r+1}\left\{(y^{[r-1]})' - \sum_{s=1}^{r} a_{rs} y^{[s-1]}\right\} \\ y^{[n]} &:= (y^{[n-1]})' - \sum_{s=1}^{n} a_{ns} y^{[s-1]}, \end{aligned} \tag{3.2}$$

$1 \le r \le n-1$, where the prime $'$ denotes differentiation.

The quasi-differential expression M associated with A is given by

$$M[y] := i^n y^{[n]}, \qquad n \ge 2, \tag{3.3}$$

this being defined on the set

$$V(M) := \{y : y^{[r-1]} \in AC_{\text{loc}}(I),\ r = 1, 2, \dots, n\}, \tag{3.4}$$

where $AC_{\text{loc}}(I)$ denotes the set of functions which are absolutely continuous on every compact subinterval of I.

The formal adjoint M^+ of M is defined by the matrix $A^+ \in Z_n(I)$ given by

$$A^+ := -L^{-1} A^* L, \tag{3.5}$$

where A^* is the conjugate transpose of A and L is the nonsingular $n \times n$ matrix,

$$L = \{(-1)^n \delta_{r,n+1-s}\}, \qquad 1 \le r,\ s \le n, \tag{3.6}$$

δ being the Kronecker delta. If $A^+ = \{a_{rs}^+\}$, then it follows that

$$a_{rs}^+ = (-1)^{r+s+1}\bar{a}_{n-s+1,n-r+1}, \quad \text{for each } r \text{ and } s. \tag{3.7}$$

The quasi-derivatives associated with A^+ are therefore

$$\begin{aligned} y_+^{[0]} &:= y, \\ y_+^{[r]} &:= \bar{a}_{n-r,n-r+1}^{-1}\Big\{(y_+^{[r-1]})' \\ &\qquad - \sum_{s=1}^{r}(-1)^{r+s+1}\bar{a}_{n-s+1,n-r+1}y_+^{[s-1]}\Big\} \\ y_+^{[n]} &:= (y_+^{[n-1]})' \\ &\qquad - \sum_{s=1}^{n}(-1)^{n+s+1}\bar{a}_{n-s+1,1}y_+^{[s-1]}. \end{aligned} \tag{3.8}$$

Note that $(A^+)^+ = A$ and so $(M^+)^+ = M$. We refer to [**7, 8**] and [**13**] for a full account of the above and subsequent results on quasi-differential equations.

Let the interval I have endpoints $a, b, -\infty \le a < b \le \infty$, and let w be a function which satisfies

$$w > 0 \quad \text{a.e. on } I, \qquad w \in L^1_{\text{loc}}(I). \tag{3.9}$$

The equation

$$M[u] = \lambda w u, \qquad \lambda \in \mathbf{C} \quad \text{on } I, \tag{3.10}$$

is said to be *regular* at the left endpoint $a \in \mathbf{R}$ if for all $X \in (a, b)$,

$$a \in \mathbf{R}; \qquad w, a_{rs} \in L^1[a, X], \qquad r, s = 1, 2, \dots, n. \tag{3.11}$$

Otherwise (3.10) is said to be *singular* at a. Similarly we define the terms regular and singular at b. If (3.10) is regular at both endpoints, then it is said to be regular; in this case we have

$$a, b \in \mathbf{R}; \qquad w, a_{rs} \in L^1(a, b), \qquad r, s = 1, 2, \dots, n. \tag{3.12}$$

We shall be concerned with the case when a is a regular endpoint of (3.10), the endpoint b being allowed to be either regular or singular. Note that, in view of (3.7), an endpoint of I is regular for (3.10) if and only if it is regular for the equation

$$M^+[v] = \bar{\lambda} w v \qquad \lambda \in \mathbf{C} \quad \text{on } I. \tag{3.13}$$

Let $L^2_w(a,b)$ denote the usual weighted L^2-space with inner product

$$(f,g) := \int_a^b f(x)\overline{g(x)} w(x)\, dx, \tag{3.14}$$

and norm $||f|| := (f,f)^{1/2}$; this is a Hilbert space on identifying functions which differ only on null sets. Set

(3.15)
$$D := \{u : u \in V(M), u \text{ and } (1/w)M[u] \in L^2_w(a,b\},$$
(3.16)
$$D^+ := \{v : v \in V(M^+), v \text{ and } (1/w)M^+[v] \in L^2_w(a,b)\}.$$

The subspaces D and D^+ of $L^2_w(a,b)$ are domains of the so-called *maximal operators* T and T^+, respectively, defined by

$$Tu := \frac{1}{w}M[u], \quad u \in D \quad \text{and} \quad T^+v := (1/w)M^+[v], \quad v \in D^+.$$

For the regular problem the *minimal operators* T_0 and T_0^+ are the restrictions of $(1/w)M[.]$ and $(1/w)M^+[v]$ to the subspaces
(3.17)
$$D_0 := \{u : u \in D, u^{[r-1]}(a) = u^{[r-1]}(b) = 0,\ r = 1, 2, \dots, n\}$$
$$D_0^+ := \{v : v \in D^+, v_+^{[r-1]}(a) = v_+^{[r-1]}(b) = 0,\ r = 1, 2, \dots, n\},$$

respectively. The subspaces D_0 and D_0^+ are dense in $L^2_w(a,b)$ and T_0 and T_0^+ are closed operators (see [**13**, Section 3]).

In the singular problem we first introduce operators T_0' and $(T_0^+)'$; T_0' being the restriction of $(1/w)M[.]$ to

$$D_0' := \{u : u \in D, \operatorname{supp} u \subset (a,b)\},$$

and with $(T_0^+)'$ defined similarly. These operators are densely-defined and closable in $L_w^2(a,b)$; and we defined the minimal operators T_0 and T_0^+ to be their respective closures (see [**13**, Section 5]).

We denote the domains of T_0 and T_0^+ by D_0 and D_0^+, respectively. It can be shown that

$$u \in D_0 \Longrightarrow u^{[r-1]}(a) = 0, \qquad r = 1, 2, \dots, n, \tag{3.18}$$

$$v \in D_0^+ \Longrightarrow v_+^{[r-1]}(a) = 0, \qquad r = 1, 2, \dots, n \tag{3.19}$$

because we are assuming that a is a regular endpoint. Moreover, in both the regular and singular problems, we have

$$T_0^* = T^+, \qquad T_0^+ = T^*; \tag{3.20}$$

see [**13**, Section 5], in the case when $M = M^+$ and compare with the treatment in [**13**, Section III.10.3] in the general case.

We see from (3.20) that $T_0 \subset T = (T_0^+)^*$ and hence T_0 and T_0^+ form an adjoint pair of closed, densely-defined operators in $L_w^2(a,b)$. By [**3**, Corollary III.3.2], $\operatorname{def}(T_0 - \lambda I) + \operatorname{def}(T_0^+ - \bar{\lambda} I)$ is constant on the joint field of regularity $\Pi(T_0, T_0^+)$, and we have shown in [**5**] that

$$n \le \operatorname{def}(T - \lambda I) + \operatorname{def}(T^+ - \bar{\lambda} I) \le 2n, \quad \text{for all } \lambda \in \Pi(T_0, T_0^+). \tag{3.21}$$

In the regular problem,

$$\operatorname{def}(T_0 - \lambda I) + \operatorname{def}(T_0^+ - \bar{\lambda} I) = 2n, \quad \text{for all } \lambda \in \Pi(T_0, T_0^+). \tag{3.22}$$

Theorem 3.1. *Suppose $f \in L^1_{\text{loc}}(I)$, and suppose that the conditions (3.1) are satisfied. Then, given any complex numbers $c_j \in \mathbf{C}$, $j = 0, 1, \dots, n-1$ and $x_0 \in (a,b)$, there exists a unique solution of $M[\phi] = wf$ in (a,b) which satisfies*

$$\phi^{[j]}(x_0) = c_j, \qquad j = 0, 1, \dots, n-1.$$

Proof. See [**3**] and [**9**, Part II, Theorem 16.2.2]. □

Theorem 3.2 (cf. [**8**, Theorem II.2.5]). *Let M be a regular quasi-differential expression of order n on $[a,b]$. For $f \in L^2_w(a,b)$, the equation $(1/w)M[\phi] = f$ has a solution $\phi \in V(M)$ satisfying*

$$\phi^{[r]}(a) = \phi^{[r]}(b) = 0, \qquad r = 0, 1, \dots, n-1$$

if and only if f is orthogonal in $L^2_w(a,b)$ to the solution space of $M^+[\psi] = 0$, i.e.,

$$R[T_0(M) - \lambda I)] = N[T(M^+) - \bar{\lambda} I]^{\perp}. \tag{3.23}$$

Corollary 3.3 (cf. [**8**, Corollary II.2.6]). *As a result from Theorem* 3.2*, we have that*

$$R[T_0(M) - \lambda I]^{\perp} = N[T(M^+) - \bar{\lambda} I]. \tag{3.24}$$

4. The spectra of $T_0(M)$ and $T_0(M^+)$. In this subsection we deal with the various components of the spectra of $T_0(M)$ and $T_0(M^+)$.

Theorem 4.1. *The point spectra $\sigma_p(T_0(M))$ and $\sigma_p(T_0(M^+))$ of $T_0(M)$ and $T_0(M^+)$ are empty.*

Proof. Let $\lambda \in \sigma_p(T_0(M))$. Then there exists a nonzero element $\phi \in D_0(M)$, such that

$$(T_0(M) - \lambda I)\phi = 0.$$

In particular, this gives that

$$\begin{aligned} &M[\phi] = \lambda w \phi, \\ &\phi^{[r]}(a) = \phi^{[r]}(b) = 0, \qquad r = 0, 1, \dots, n-1. \end{aligned}$$

From Theorem 3.1, it follows that $\phi \equiv 0$ and hence $\sigma_p[T_0(M)] = \varnothing$. Similarly $\sigma_p[T_0(M^+)] = \varnothing$. $\square$

Theorem 4.2. (i) $\rho[T_0(M)] = \varnothing$,

(ii) $\sigma_p[T_0(M)] = \sigma_c[T_0(M)] = \varnothing$,

(iii) $\sigma[T_0(M)] = \sigma_r[T_0(M)] = \mathbf{C}$.

Proof. (i) Since $R[T_0(M)-\lambda I]$ is a proper closed subspace of $L_w^2(a,b)$, then the resolvent set $\rho[T_0(M)]$ is empty.

(ii) Since $R[T_0(M)-\lambda I]$ is closed, then the continuous spectrum of $T_0(M)$ is the empty set, i.e., $\sigma_c[T_0(M)] = \varnothing$.

(iii) From (i), (ii) and Lemma 4.1, it follows that

$$\sigma[T_0(M)] = \sigma_r[T_0(M)] = \mathbf{C}. \qquad \square$$

Corollary 4.3. (i) $\sigma_c[T(M)] = \sigma_r[T(M)] = \varnothing$,

(ii) $\sigma[T(M)] = \sigma_p[T(M)] = \mathbf{C}$,

(iii) $\rho[T(M)] = \varnothing$.

Proof. From Theorem 3.2 and since $T(M) = [T_0(M^+)]^*$ it follows that $R[T(M)-\lambda I]$ is closed, for every $\lambda \in \mathbf{C}$; see [**3**, Theorem I.3.7]. Also, we have

$$\text{nul}\,[T(M)-\lambda I] = \text{def}\,[T_0(M^+)-\bar{\lambda} I] = n,$$

and

$$\text{def}\,[T(M)-\lambda I] = \text{nul}\,[T_0(M^+)-\bar{\lambda} I] = 0.$$

(i) Since $R[T(M)-\lambda I]$ is closed and $\text{def}\,[T(M)-\lambda I] = 0$, then $R[T(M)-\lambda I] = H$ and this yields that

$$\sigma_c[T(M)] = \sigma_r[T(M)] = \varnothing.$$

(ii) Since $\text{nul}\,[T(M)-\lambda I] = n$, for every $\lambda \in \mathbf{C}$, then we have that $\sigma_p[T(M)] = \mathbf{C}$. It also follows that $\sigma[T(M)] = \mathbf{C}$ and hence $\rho[T(M)] = \varnothing$. $\square$

Lemma 4.4 (cf. [**3**, Lemma IX.9.1]). *If $I = [a,b]$, with $-\infty < a < b < \infty$, then for any $\lambda \in \mathbf{C}$, the operator $[T_0(M)-\lambda I]$ has closed range, zero nullity and deficiency n. Hence*

$$\sigma_{ek}[T_0(M)] = \begin{cases} \varnothing & k = 1,2,3, \\ \mathbf{C} & k = 4,5. \end{cases}$$

5. The field of regularity of $T_0(M)$ and $T_0(M^+)$. We now obtain some results which in fact are a natural consequence of those in Section 4.

Theorem 5.1. (i) $\Pi(T_0(M)] = \Pi[T_0(M^+)] = \mathbf{C}$ for every $\lambda \in \mathbf{C}$,

$$\operatorname{def}\,[T_0(M) - \lambda I] = \operatorname{def}\,[T_0(M^+) - \bar{\lambda} I] = n.$$

(ii) $\Pi[T(M)] = \Pi[T(M^+)] = \varnothing$, and for every $\lambda \in \mathbf{C}$,

$$\operatorname{nul}\,[T(M) - \lambda I] = \operatorname{nul}\,[T(M^+) - \bar{\lambda} I] = n.$$

Proof. (i) We have from Theorem 3.2 and Lemma 4.1 that, for every $\lambda \in \mathbf{C}$, $(T_0(M) - \lambda I)^{-1}$ exists and its domain $R[T_0(M) - \lambda I]$ is a closed subspace of $L^2_w(a, b)$. Hence, since $T_0(M)$ is a closed operator, then $(T_0(M) - \lambda I)^{-1}$ is also closed, and so it follows from the Closed Graph theorem that $(T_0(M) - \lambda I)^{-1}$ is bounded and hence $\Pi[T_0(M)] = \mathbf{C}$. From Theorem 3.2, $R[T_0(M) - \lambda I]^{\perp}$ is the n-dimensional subspace of $L^2_w(a, b)$. Thus

$$\operatorname{def}\,[T_0(M) - \lambda I] = \dim R[T_0(M) - \lambda I]^{\perp} = n,$$

for every $\lambda \in \mathbf{C}$. Similarly for M^+.

(ii) As $\Pi[T_0(M^+)] = \mathbf{C}$, we have for every $\lambda \in \mathbf{C}$ that $(T_0(M^+) - \bar{\lambda} I)$ has closed range and so, since $T(M) = [T_0(M^+)]^*$, $[T(M) - \lambda I]$ has closed range, see [**3**, Theorem I.3.7]. Furthermore, from (i),

$$\operatorname{nul}\,[T(M) - \lambda I] = \operatorname{def}\,[T_0(M^+) - \bar{\lambda} I] = n.$$

Hence $\lambda \notin \Pi[T(M)]$ and so part (ii) of the theorem follows. □

Corollary 5.2. *The operators $T_0(M)$, $T_0(M^+)$ form a compatible adjoint pair with* $\Pi[T_0(M), T_0(M^+)] = \mathbf{C}$.

Proof. From part (i) of Theorem 5.1, it follows that $\Pi[T_0(M), T_0(M^+)] = \mathbf{C}$. Using (3.20), the corollary follows. □

Theorem 5.3 (cf. [**8**, Proposition III.3.24]). *If for some $\lambda_0 \in \mathbf{C}$, there are n linearly independent solutions of*

$$M[\phi] = \lambda_0 w_\phi \quad \textit{and} \quad M^+[\psi] = \bar{\lambda}_0 w\psi,$$

in $L^2_w(a,b)$. Then all solutions of

$$M[\phi] = \lambda w\phi \quad \textit{and} \quad M^+[\psi] = \bar{\lambda} w\psi$$

are in $L^2_w(a,b)$ for all $\lambda \in \mathbf{C}$.

From Corollary 5.2 and Theorem 5.3 we have the following lemma.

Lemma 5.4. *If, for some $\lambda_0 \in \mathbf{C}$, there are n linearly independent solutions of*

$$M[\phi] = \lambda_0 w\phi \quad \textit{and} \quad M^+[\psi] = \bar{\lambda}_0 w\psi$$

in $L^2_w(a,b)$, then $\lambda_0 \in \Pi[T_0(M), T_0(M^+)]$; see also [**10**, Theorem 2.1] *and* [**11**, Lemma 5.1].

Theorem 5.5. *Let $T_0(M)$ and $T_0(M^+)$ be the minimal operators associated with M and M^+ defined on the interval $[a,b)$. If $\Pi[T_0(M), T_0(M^+)]$ is empty, then*

$$\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \bar{\lambda} I] \neq 2n.$$

In particular, if $\Pi[T_0(M), T_0(M^+)]$ is empty and $n = 1$, then

$$\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \bar{\lambda} I] = 1.$$

Proof. If $\operatorname{def}[T_0(M) - \lambda I] = \operatorname{def}[T_0(M^+) - \bar{\lambda} I] = n$ for some $\lambda_0 \in \mathbf{C}$, then

$$M[\phi] = \lambda_0 w\phi \quad \text{and} \quad M^+[\psi] = \bar{\lambda}_0 w\psi$$

each have $n - L^2_w(a,b)$ solutions. Hence, by Theorem 5.3, we have that all solutions of

$$M[\phi] = \lambda w\phi \quad \text{and} \quad M^+[\psi] = \bar{\lambda} w\psi$$

are in $L^2_w(a,b)$ for all $\lambda \in \mathbf{C}$, and hence by Corollary 5.2, we have that $\lambda \in \Pi[T_0(M), T_0(M^+)]$. Thus, if $\Pi[T_0(M), T_0(M^+)]$ is empty, we cannot have

$$\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \bar{\lambda} I] = 2n.$$

In particular, if $n = 1$, then the relation (3.21) gives that

$$1 \leq \operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \bar{\lambda} I] \leq 2,$$

so if $\Pi[T_0(M), T_0(M^+)]$ is empty we have

$$\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \bar{\lambda} I] = 1. \qquad \square$$

For a regularly solvable operator, we have the following general theorem:

Theorem 5.6. *Suppose for a regularly solvable extension of the minimal operator $T_0(M)$ that*

$$\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \bar{\lambda} I] = N, \qquad n \leq N \leq 2n$$

for all $\lambda \in \Pi[T_0(M), T_0(M^+)]$. Then

$$\operatorname{nul}[T(M) - \lambda I] + \operatorname{nul}[T(M^+) - \bar{\lambda} I] \leq N, \quad \textit{for all } \lambda \in \mathbf{C}.$$

If $\Pi[T_0(M), T_0(M^+)]$ is empty, then

$$\operatorname{nul}[T(M) - \lambda I] + \operatorname{nul}[T(M^+) - \bar{\lambda} I] < N.$$

Proof. Let $\operatorname{def}[T_0(M) - \lambda I] = r$, $\operatorname{def}[T_0(M^+) - \bar{\lambda} I] = s$ such that

$$\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \bar{\lambda} I] = r + s, \qquad n \leq r + s \leq 2n,$$

for all $\lambda \in \Pi[T_0(M), T_0(M^+)]$.

Then for any closed extension of $T_0(M)$ which is regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$, we have from [**3**, Theorem III.3.5] that

$$\dim\{D(S)/D_0(M)\} = \operatorname{def}[T_0(M) - \lambda I] = r,$$
$$\dim\{D(S^*)/D_0(M^+)\} = \operatorname{def}[T_0(M^+) - \bar{\lambda} I] = s.$$

Hence S and S^* are finite dimensional extensions of $T_0(M)$ and $T_0(M^+)$, respectively. Thus, from [**3**, Corollary IX.4.2], we get

$$\sigma_{ek}[T_0(M)] = \sigma_{ek}(S), \qquad k = 1, 2, 3. \tag{5.1}$$

Since $[T_0(M) - \lambda I]$ has closed range, zero nullity and deficiency r (see Lemma 4.4). Then for any $\lambda \in \mathbf{C}$, we have that

$$\Pi[T_0(M)] \cap \sigma_{ek}[T_0(M)] = \varnothing, \qquad k = 1, 2, 3.$$

By (5.1) we have that

$$\Pi[T_0(M)] \cap \sigma_{ek}(S) = \varnothing, \qquad k = 1, 2, 3.$$

Therefore,

$$\Delta_k[T_0(M)] = \Delta_k(S) = \mathbf{C}, \qquad k = 1, 2, 3.$$

Similarly,

$$\Delta_k[T_0(M^+)] = \Delta_k(S^*) = \mathbf{C}, \qquad k = 1, 2, 3.$$

Furthermore, the equations

$$M[\phi] = \lambda w \phi \quad \text{and} \quad M^+[\psi] = \bar{\lambda} w \psi$$

has at most r and s linearly independent solutions for $\lambda \in \mathbf{C}$, respectively. Hence

$$\operatorname{nul}[T(M) - \lambda I] + \operatorname{nul}[T(M^+) - \bar{\lambda} I] \le N \quad \text{for all } \lambda \in \mathbf{C}.$$

But if, for any $\lambda_0 \notin \Pi[T_0(M), T_0(M^+)]$, then either $\lambda_0 \notin \Pi[T_0(M)]$ or $\bar{\lambda}_0 \notin \Pi[T_0(M^+)]$. If $\lambda_0 \notin \Pi[T_0(M)]$, then either λ_0 is an eigenvalue of $T_0(M)$ or $R[T_0(M) - \lambda I]$ is not closed. Similarly for $\bar{\lambda}_0 \notin \Pi[T_0(M^+)]$. But $T_0(M)$ and $T_0(M^+)$ have no eigenvalues, then if

$\lambda_0 \notin \Pi[T_0(M), T_0(M^+)]$, we have $R[T_0(M) - \lambda_0 I]$ and $R[T_0(M^+) - \bar{\lambda}_0 I]$ are both not closed and so we cannot have

$$\text{nul}\,[T(M) - \lambda_0 I] + \text{nul}\,[T(M^+) - \bar{\lambda}_0 I] = N.$$

Hence,

$$\text{nul}\,[T(M) - \lambda I] + \text{nul}\,[T(M^+) - \bar{\lambda} I] < N,$$

for any $\lambda \notin \Pi[T_0(M), T_0(M^+)]$. □

Remark. It remains an open question as to how many of the solutions of $M[\phi] = \lambda w\phi$ and $M^+[\psi] = \bar{\lambda} w\psi$ may be in $L^2_w(a,b)$ for any $\lambda \in \mathbf{C}$ when $\Pi[T_0(M), T_0(M^+)]$ is empty, except that we know from above that not all of them are in $L^2_w(a,b)$.

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