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REPRESENTATIONS OF FINITE POSETS AND NEAR-ISOMORPHISM OF FINITE RANK BUTLER GROUPS

D. ARNOLD AND M. DUGAS

Introduction. This paper contains a functorial interpretation of finite rank Butler groups, up to near isomorphism, as representations of finite *posets* (partially ordered sets) over $Z/p^m Z$. The representation setting clarifies the complexity and, in special cases, the structure of these groups. In particular, for $m = 1$ the theory of representations over a field is available. As an application, indecomposable almost completely decomposable *acd* groups with arbitrarily large finite rank and fixed typeset are constructed. Examples of this sort, so far as we know, are new. In the other direction, rigid uniform *acd* groups are classified up to near isomorphism by invariants in [**17**]. These invariants classify associated representations up to isomorphism.

A *Butler group* is a pure subgroup of a **cd** group, a finite direct sum of torsion-free abelian groups of rank 1 [**16**]. Each Butler group G has a finite typeset generating a finite distributive lattice T of *types* (isomorphism classes of rank-1 groups). Thus, G is in $\mathbf{B_T}$, the quasi-homomorphism category of Butler groups with types in T. There is a category equivalence from B_T to $\text{Rep}\,(Q, JI(T)^{op})$, the category of Q-representations of the opposite of the poset $JI(T)$ of join-irreducible elements of T [**14, 15**].

Group-theoretic properties are lost by passing to the quasi-homomorphism category. For example, an indecomposable group need not be *strongly indecomposable*, indecomposable relative to quasi-isomorphism. On the other hand, determining isomorphism of **acd** groups, Butler groups quasi-isomorphic to *cd* groups, leads to number-theoretic problems [**17**] that we wish to avoid.

Near-isomorphism, a generalization of genus class for lattices over Z-orders [**20**] is an equivalence relation on torsion-free abelian groups of finite rank lying between isomorphism and quasi-isomorphism.

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An indecomposable group remains indecomposable relative to near-isomorphism, as discussed in Section 1.

The Butler groups considered herein have a restricted typeset. Let $\mathbf{T_n} = \{\tau_0, \tau_1, \tau_2, \dots, \tau_n\}$, where $\tau_1, \tau_2, \dots, \tau_n$ are pairwise incomparable types with $\tau_i \cap \tau_j = \tau_0$ for each $i \neq j$. Define $\mathbf{B(T_n)}$ to be the category of Butler groups with typeset contained in T_n.

Included in the $B(T_n)$'s are classical examples of *acd* groups as well as strongly indecomposable $G[A]$'s characterized by invariants in [**11** and references]. For $H \in B(T_n)$, let $\mathbf{C_H}$ denote $H(\tau_1) + \cdots + H(\tau_n)$. Classification in $B(T_n)$, up to near isomorphism, is reduced to classification in $\mathbf{B(T_n, p^m)}$, the category of groups H in $B(T_n)$ such that H/C_H is a finite group bounded by p^m (Section 1).

In Section 2 we further assume that there is a prime p with $(\tau_i)_p$ finite for each $\tau_i \in T_n$. In this case each $H \in B(T_n)$ is *p-locally free*, $\mathbf{H_p} = Z_p \otimes H$ is a free module over the localization $\mathbf{Z_p}$ of the integers at p. This is not a serious restriction, being equivalent to $pA_i \neq A_i$ for A_i a subgroup of Q with type τ_i.

The category of $Z/p^m Z$-representations of the poset of $n+1$ pairwise incomparable elements is denoted by $\mathbf{Rep_{n+1}(Z/p^m Z)}$. Objects of this category are $U = (U, U_1, \dots, U_{n+1})$, where U is a finite rank free $Z/p^m Z$-module with each U_i a submodule of U, and morphisms are module homomorphisms preserving the $n+1$ distinguished submodules.

The main theorem is the existence of an additive functor $F : B(T_n, p^m) \to \text{Rep}_{n+1}(Z/p^m Z)$ sending near isomorphism classes of groups to isomorphism classes of representations (Theorem 2.3). This functor is full on the subcategory of *acd* groups. Hence, for *acd* groups H and H' in $B(T_n, p^m)$, H is nearly isomorphic to H' if and only if $F(H) \approx F(H')$. Moreover, an *acd* group H is indecomposable if and only if $F(H)$ is indecomposable. Included is a characterization of the representations in the image of F.

Indecomposable groups H in $B(T_n, p^m)$ can be constructed by constructing indecomposable representations in the image of F. As an illustration, for $p \neq 2$ we construct indecomposable *acd* groups of arbitrarily large finite rank in $B(T_n, p^m)$ for $n \geq 3$ and sufficiently large m and in $B(T_n, p)$ for $n \geq 4$ (Corollaries 2.5 and 2.6). The relative simplicity of these constructions demonstrates the efficacy of the point of view of representations.

This theme is continued in [**2**], wherein it is shown that the *acd* groups in $B(T_n, p^m)$ have *finite representation type* (finitely many isomorphism classes of indecomposables) if and only if $n = 2$ or else $n = 3$ and $m \leq 2$. Moreover, the category of *acd* groups in $B(T_n, p^m)$ is equivalent to the category of lattices over a certain multiple pullback ring with near-isomorphism of groups corresponding to genera of lattices.

The structure of groups in $B(T_2)$ is known. Each such group is the direct sum of indecomposable *acd* groups of rank ≤ 2 ([**24**], the statement appears in [**5**], but the proof therein is flawed). It follows readily from this classification that a complete set of near-isomorphism invariants for a group H in $B(T_2)$ is rank H, rank $H(\tau_i)$ for $1 \leq i \leq 2$, and the isomorphism class of the torsion subgroup of H/C_H, a finite group. Details of this argument are not included.

The special class $B(T_3)$ is examined in more detail in Section 3. Strongly indecomposables in $B(T_3)$ are rank-1 groups and groups of the form $G[A_1, A_2, A_3]$. If $\tau_1 + \tau_2 + \tau_3 = \text{type}\, Q$, then each *acd* $H \in B(T_3)$ is the direct sum of indecomposables of rank ≤ 3 (Theorem 3.1). Included is a complete set of near isomorphism invariants for these groups. Corollary 3.3 is a description of those H's in $B(T_3, p)$ with $F(H)$ indecomposable. This description is derived from the complete list of indecomposables in $\text{Rep}_4(Z/pZ)$ given in [**13**]. It follows that $\text{rank}\, H \leq 3$ for each indecomposable *acd* group in $B(T_3, p)$ (compare Corollary 2.6).

A complete classification of indecomposable groups in $B(T_3)$ remains elusive. The indecomposable uniform *acd* groups in $B(T_3, p^2)$ can be listed, up to near-isomorphism [**2**]. On the other hand, even the uniform *acd* groups in $B(T_3, p^m)$ for $m \geq 4$ have infinite representation type.

As for non-*acd* groups, the techniques of this paper are less useful for purposes of classification up to near isomorphism. In particular, the functor F is not full for such groups (remark following Theorem 2.3). There are indecomposables with arbitrarily large finite rank in $B(T_n, p)$ that are not *acd* groups for $n \geq 3$ (Corollary 2.4). Furthermore, there are indecomposables H of arbitrarily large finite rank in $B(T_3)$ with $H = C_H$ constructed from representations over a subring R of Q [**1**]. Specifically, if A_i is a subgroup of Q with type τ_i and each A_i is an $R = \text{End}\,(A_0)$-module, then there is a full embedding $F_A : B(T_n, p) \to \text{Rep}_n(R)$ [**1**, Corollary 1.3]. Examples of indecomposable groups $H = C_H$, for $n = 3$, arise from a difficult construction of indecomposables in the image of F_A.

We are grateful to the referee for a careful reading of the original manuscript and for numerous suggestions. Undefined notation and terminology is as in [**4**] for torsion-free abelian groups of finite rank, [**6**] for representations of finite posets, and [**10, 11**] for finite rank Butler groups.

1. Near-isomorphism, isomorphism at p and reductions. Two finite rank torsion-free abelian groups G and H are *isomorphic at p* if there is a monomorphism $f : G \to H$ such that $H/f(G)$ is finite with order relatively prime to p. Equivalently there is an integer a prime to p and $f : G \to H$ and $g : H \to G$ with $gf = a$ and $fg = a$ [**4**, Theorem 7.16]. The groups G and H are *nearly isomorphic* if and only if they are isomorphic at p for each prime p [**21**]. Equivalently, $G^m \approx H^m$ for some m [**4**, Theorem 13.9].

Given finite rank torsion-free abelian groups H and H', $\mathbf{H}$ *is a summand of* $\mathbf{H}'$ *at* $\mathbf{p}$ if there is an integer a prime to p and $f : H \to H'$ and $g : H' \to H$ with $gf = a$ (alternately, $H/gf(H)$ is finite with order prime to p). Then H is a *near-summand* of H' if H is a summand of H' at p for each prime p. If H is a near summand of H', then $H' = G \oplus K$ for some G nearly isomorphic to H [**4**, Corollary 12.9a]. In particular, if H and H' are nearly isomorphic, then H is indecomposable if and only if H' is indecomposable.

For purposes of classification up to near-isomorphism, it is sufficient to assume that $H \in B(T_n)$ has no rank-1 summands of type τ_0 and that H/C_H is finite. To see this, first note that if H is in $B(T_n)$, then $H = H' \oplus H_0$ for some τ_0-homogeneous completely decomposable group H_0 with H' the purification of C_H in H, $H'/C_{H'}$ finite, and H' having no rank-1 summands of type τ_0 [**4**, Theorem 4.6]. Given another $G \in B(T_n)$, G is nearly isomorphic to H if and only if $G_0 \approx H_0$ and G' is nearly isomorphic to H'. This follows from [**4**, Corollaries 12.9b and 7.17a] and the fact that nearly isomorphic homogeneous completely decomposable groups are isomorphic.

Given $H \in B(T_n)$ with no rank-1 summands of type τ_0 and a prime p, let $\mathbf{H}(\mathbf{p})$ be the subgroup of H with $H(p)/C_H$ the p-component of H/C_H.

Theorem 1.1. *Assume that H and H' are in $B(T_n)$ with no rank-1 summands of type τ_0, and let p be a prime.*

(a) *H and H' are isomorphic at p if and only if $H(p)$ and $H'(p)$ are isomorphic at p.*

(b) *If $H(p)$ is indecomposable, then H is indecomposable.*

Proof. (a) is a consequence of the observation that H is isomorphic at p to $H(p)$.

To prove (b), assume that $H = G \oplus K$. Then $C_H = C_G \oplus C_K$ and so $H/C_H = G/C_G \oplus K/C_K$ with $H(p)/C_H = G(p)/C_G \oplus K(p)/C_K$. Furthermore, $H(p) = G(p) \oplus K(p)$ as $G(p) \cap K(p)$ is contained in $G \cap K = 0$. Since $H(p)$ is indecomposable and $\operatorname{rank} H = \operatorname{rank} H(p)$, it follows that H is also indecomposable. □

2. $B(T_n, p^m)$, near isomorphism, and $Z/p^m Z$-representations. Fix a prime p and let $\mathbf{B(T_n, p^m)/p^m}$ be the category with objects the same as $B(T_n, p^m)$, as defined in the introduction, but with morphism sets denoted by $\operatorname{Hom}(H, H')/\operatorname{Hom}(C_H, p^m C_{H'})$. Morphism sets are well defined since each element of $\operatorname{Hom}(C_H, p^m C_{H'})$ extends to a unique element of $\operatorname{Hom}(H, H')$. This category is an additive quotient category of $B(T_n, p^m)$ since C_H is a functor. Direct sums exist in $B(T_n, p^m)/p^m$; the proof is the same as that given in [**4**, Theorem 7.11]. The identity functor on objects induces an additive functor $B(T_n, p^m) \to B(T_n, p^m)/p^m$.

The proof of Lemma 2.1 is a mild variation of standard arguments given in [**4**, Sections 7 and 12]. Here $\operatorname{Hom}(C_H, p^m C_{H'})$ replaces $\operatorname{Hom}(H, p^m H')$.

Lemma 2.1. *Let p be a prime with $(\tau_i)_p$ finite for each $\tau_i \in T_n$ and H, H' groups in $B(T_n, p^m)$.*

(a) *H is a summand of H' in $B(T_n, p^m)/p^m$ if and only if H is a summand of H' at p.*

(b) *H is a near summand of H' if and only if C_H is a near summand of $C_{H'}$ and H is a summand of H' at p.*

Proof. (a) $(\rightarrow)$. Assume there is $f : H \rightarrow H'$ and $g : H' \rightarrow H$ with $gf = 1_H + p^m h$ for $h : C_H \rightarrow C_H$. Then kernel $(gf : C_H \rightarrow C_H)$ is p-divisible. Since H is p-locally free, gf must be monic. Hence, $H/gf(H)$ is finite [**4**, Proposition 6.1a]. In fact, $gf(H)$ is p-pure in H. This is because if $x, y \in H$ with $px = gf(y) = y + h(p^m y)$, then $y = px - v \in pH + C_H$ for $v = h(p^m y) \in C_H$. Hence, $y = px - h(p^m y) = px - ph(p^m x) + p^m h(v) \in pH$. Thus, $x = gf(y/p) \in gf(H)$ as desired. It now follows that $H/gf(H)$ is finite and prime to p.

$(\leftarrow)$. Suppose that $f : H \rightarrow H'$ and $g : H' \rightarrow H$ and a is an integer prime to p with $gf = a$. Then H is a summand of H' in $B(T_n, p^m)/p^m$, since a represents a unit in $\text{Hom}\,(H, H)/\text{Hom}\,(C_H, p^m C_H)$.

(b) $(\leftarrow)$ follows from the observation that C_H is isomorphic to H at q for each prime $q \neq p$ while $(\rightarrow)$ is a consequence of the fact that C_H is a functor. □

The next lemma and the remark following Theorem 2.3 were suggested by the referee.

Lemma 2.2. *Let M be an $r \times s$ matrix over a field F with $r \leq s$ and S an infinite subset of F. Then elements of S can be added to entries of M to form an $r \times s$ matrix M' such that each $r \times r$ submatrix of M' is nonsingular.*

Proof. We induct on r, the case $r = 1$ is clear. Now assume that the lemma is true for $1 \leq r - 1 < s$. In particular, we may assume that the submatrix of M obtained by deleting the last row has the property that each $(r-1) \times (r-1)$ submatrix is nonsingular. Use the Laplacian expansion along the last row of an $r \times r$ submatrix N of M to write $\det N$ as a sum of entries of the last row times their $(r-1) \times (r-1)$ cofactors in N. By assumption, each of these cofactors is nonzero. Now S is infinite and there are only finite many such N's. An induction on s now shows that elements of S can be added to the entries of the last row of M to guarantee that each $\det N$ is nonzero. □

Theorem 2.3. *Assume that p is a prime and that $(\tau_i)_p$ is finite for each $\tau_i \in T_n$. Then there is a functorial embedding F :*

$B(T_n, p^m)/p^m \to \mathrm{Rep}_{n+1}(Z/p^m Z)$ *defined by*

$$F(H) = (C_H/p^m C_H, (H(\tau_1) + p^m C_H)/p^m C_H, \\ \dots, (H(\tau_n) + p^m C_H)/p^m C_H, p^m H/p^m C_H)$$

that is full for acd groups.

(a) *Let* $V = (U, U_1, \dots, U_{n+1}) \in \mathrm{Rep}_{n+1}(Z/p^m Z)$. *Then* $V = F(H)$ *for some* $H \in B(T_n, p^m)$ *if and only if* $U = U_1 + \cdots + U_n$; $U, U_1, \dots, U_n$ *are free* $Z/p^m Z$*-modules;* $U_i \cap U_{n+1} = 0$ *for each* $i \leq n$; *and* $\mathrm{rank}\, U_i + \mathrm{rank}\, U_j \leq \mathrm{rank}\, U$ *for each* $i \neq j \leq n$.

(b) *If* $H, H' \in B(T_n, p^m)$ *with* H *a summand of* H^1 *at* p, *then* $F(H)$ *is a summand of* $F(H')$. *In particular, if* H *is nearly isomorphic to* H', *then* $F(H)$ *is isomorphic to* $F(H')$. *Moreover, if* $F(H)$ *is indecomposable, then* H *is indecomposable.*

(c) *Further assume that* H *and* H' *are acd groups. Then* H *and* H' *are nearly isomorphic if and only if* $F(H)$ *and* $F(H')$ *are isomorphic. Furthermore,* H *is indecomposable if and only if* $F(H)$ *is indecomposable.*

Proof. For $g : H \to H'$, let $F(g) : F(H) \to F(H')$ be induced by g. Then F is a well defined additive functor. This is because g preserves each of the coordinate spaces. Note that $F : \mathrm{Hom}\,(H, H')/\mathrm{Hom}\,(C_H, p^m C_{H'}) \to \mathrm{Hom}\,(F(H), F(H'))$ is an embedding since $F(g) = 0$ if and only if $g \in \mathrm{Hom}\,(C_H, p^m C_{H'})$.

Next assume that H and H' are *acd* groups and $\alpha : F(H) \to F(H')$ is a representation morphism. Both $F(H)$ and $F(H')$ are of the form $(U_1 \oplus \cdots \oplus U_n, U_1, \dots, U_n, U_{n+1})$. Since C_H and $C_{H'}$ are completely decomposable, α can be lifted to $f : C_H \to C_{H'}$. Write $H = C_H + (1/p^m)E$ for a subgroup E of C_H and observe that $p^m H/p^m C_H = (p^m C_H + E)/p^m C_H$. Since $f(p^m H)$ is contained in $p^m H'$, f extends to $f : H \to H'$ as desired.

(a) Write $F(H) = (U, U_1, \dots, U_{n+1})$. Then $C_H = H(\tau_1) + \cdots + H(\tau_n)$ implies that $U = U_1 + \cdots + U_n$. Also, $U = C_H/p^m C_H$ and $U_i = (H(\tau_i) + p^m C_H)/p^m C_H \approx H(\tau_i)/p^m H(\tau_i)$ are free $Z/p^m Z$-modules. Moreover, $\mathrm{rank}\, U = \mathrm{rank}\, C_H$ and $\mathrm{rank}\, U_i = \mathrm{rank}\, H(\tau_i)$ as H is p-locally free. Now $U_i \cap U_{n+1} = 0$, since $(H(\tau_i) + p^m C_H) \cap p^m H = p^m H(\tau_i) + p^m C_H = p^m C_H$ for each i. Also, $H(\tau_i) \cap H(\tau_j) = 0$

yields $\operatorname{rank} H(\tau_i) + \operatorname{rank} H(\tau_j) = \operatorname{rank}(H(\tau_i) + H(\tau_j)) \leq \operatorname{rank} C_H$. Consequently, we have $\operatorname{rank} U_i + \operatorname{rank} U_j \leq \operatorname{rank} U$.

Conversely, let $(U, U_1, \dots, U_{n+1})$ be as given and

$$(*) \qquad 0 \to V \to W = \oplus\{U_i : 1 \leq i \leq n\} \to U \to 0$$

a split exact sequence of free $Z/p^m Z$-modules induced by the inclusion of each U_i in U. In particular, $V \cap U_i = 0$.

There is an exact sequence of Butler groups

$$0 \to K \to L = \oplus\{H_i : 1 \leq i \leq n\} \to C \to 0,$$

such that:

(i) Each H_i is τ_i-homogeneous completely decomposable with $\operatorname{rank} H_i = \operatorname{rank} U_i$;

(ii) K is τ_0-homogeneous completely decomposable with $\operatorname{rank} K = \operatorname{rank} V$;

(iii) $K \cap (H_i \oplus H_j) = 0$ for each $i \neq j$;

(iv) The induced exact sequence $0 \to K/p^m K \to L/p^m L \to C/p^m C \to 0$ is isomorphic to $(*)$, in particular, $\operatorname{rank} C = \operatorname{rank} U$.

To construct this sequence, choose H_i τ_i-homogeneous completely decomposable with $H_i/p^m H_i = U_i$. Next pull back a basis of V and purify to find a K pure in L with $(K + p^m L)/p^m L = V$. Then $\operatorname{rank} K = \operatorname{rank} V$. Each $K \cap H_i = 0$, since $V \cap U_i = 0$ and K is pure in L. Therefore, K is a τ_0-homogeneous Butler group, hence completely decomposable. This is because each nonzero element of K has at least two nonzero coordinates in L and $\tau_0 = \tau_i \cap \tau_j$ for each $i \neq j$.

We next show that K can be chosen with $K \cap (H_i \oplus H_j) = 0 \in L$ for each $i \neq j$, noting that K is only unique mod $p^m L$. Now $Q \otimes K$ is a subspace of $Q \otimes L$, and as such can be viewed as the row space of an $r \times s$ Q-matrix M with $r = \operatorname{rank} K \leq s = \operatorname{rank} L$. Let $S = \{p^m, p^{m+1}, \dots\}$, an infinite subset of Q. Elements of S can be added to the entries of M to create an $r \times s$ Q-matrix M' such that each $r \times r$ submatrix of M' is nonsingular (Lemma 2.2). Let K' denote the row space of M'. Then $K'' = K' \cap L$ is a pure subgroup of L with $(K'' + p^m L)/p^m L = (K + p^m L)/p^m L$ by the choice of S. To see that $K'' \cap (H_i \oplus H_j) = 0$, first notice that $\operatorname{rank} U = s - r \geq \operatorname{rank} U_i + \operatorname{rank} U_j = \operatorname{rank} H_i + \operatorname{rank} H_j$.

Hence, $s - \operatorname{rank} H_i - \operatorname{rank} H_j \geq r$. Since each $r \times r$ submatrix of M' is nonsingular, no nonzero element of K' has $\geq r$ zero coordinates. It now follows that $K'' \cap (H_i \oplus H_j) = 0$.

To complete the construction of a sequence satisfying (i)–(iv), define $C = L/K$. Then C is a Butler group with $\operatorname{rank} C = \operatorname{rank} U$. Since K is pure in L, tensoring by $Z/p^m Z$ preserves exactness so that $C/p^m C = U$.

The proof is concluded by constructing $H \in B(T_n, p^m)$ with $F(H) = (U, U_1, \dots, U_{n+1})$. Identify H_i with its image in C, via (iii). Then $U_i = (H_i + p^m C)/p^m C$ and $C = H_1 + \cdots + H_n$ is a Butler group with H_i contained in $C(\tau_i)$. Let E be a subgroup of C with $E/p^m C = U_{n+1}$. Define $H = C + (1/p^m)E$, a Butler group with H/C finite and bounded by p^m. Then $p^m H/p^m C = U_{n+1}$.

We next verify that H is in $B(T_n, p^m)$. Given $0 \neq x \in C$, type x is the supremum of the inf$\{\text{type}\, H_i : i \in S\}$'s ranging over all subsets S of $\{1, 2, \dots, n\}$ with $Qx \cap (\Sigma\{H_i : i \in S\}) \neq 0$ [**7**, Theorem 1.7]. Since H_i is τ_i-homogeneous decomposable and $\tau_i \cap \tau_j = \tau_0$ for $i \neq j$, it follows that type $x \in T_n$ unless $Qx \cap H_i \neq 0$ and $Qx \cap H_j \neq 0$ for some $i \neq j$. The latter case is impossible as $H_i \cap H_j = 0 \in C$ by (iii). This shows that typeset C is contained in T_n with each $C(\tau_i)/H_i$ finite. Consequently, $C \in B(T_n)$. Since H/C is finite and bounded by p^m and $C = C(\tau_1) + \cdots + C(\tau_n)$, it follows that $H \in B(T_n, p^m)$.

Finally, each H_i is p-pure in C, as $H_i/p^m H_i \approx U_i = (H_i + p^m C)/p^m C$. Thus, $U_i = (H_i + p^m C)/p^m C = (C(\tau_i) + p^m C)/p^m C$. It follows from $U_{n+1} \cap U_i = 0$ that $p^m H(\tau_i) = p^m H \cap C(\tau_i)$ is contained in $p^m C$, whence $H(\tau_i)$ is contained in C. This shows that $H(\tau_i) = C(\tau_i)$, $U_i = (H(\tau_i) + p^m C)/p^m C$ and $C = C_H$. We now have $H \in B(T_n, p^m)$ with $F(H) = (U, U_1, \dots, U_n, U_{n+1})$, as desired.

(b) is a consequence of Lemma 2.1a and the fact that F is an additive embedding.

(c) In view of (b) and the fact that near isomorphism preserves indecomposability, it suffices to prove that if H is an *acd* group and W is a nonzero representation summand of $F(H)$, then there is an *acd* group G such that $W = F(G)$ and G is a near summand of H. Note that X is an *acd* group if and only if $F(X)$ is of the form $(U_1 \oplus \cdots \oplus U_n, U_1, \dots, U_{n+1})$. Since representations of this form are closed under summands, it follows from (a) that $W \approx F(G)$ for some *acd* group G. Moreover, G is a summand of H in $B(T_n, p^j)/p^j$ as F is a full embedding for *acd* groups. By Lemma 2.1a, G is a summand of H at p, whence C_G is a summand of C_H at p. In fact, C_G is a summand of C_H since both groups are *cd* groups. Applying Lemma 2.1b shows that G is a near summand of H, as desired. □

Remark. The embedding F is not, in general, a full embedding. For example, choose subgroups A_i of Q containing 1 with type $A_i = \tau_i \in T_3$ and a prime p with $1/p \notin A_i$ for each i. Define $H = A_1 x + A_2 y + A_3(x + py)$, a subgroup of $Qx \oplus Qy$, and let $k = Z/pZ$. Then $H = C_H$ is indecomposable but $F(H) = (kx \oplus ky, kx, ky, kx, 0)$ is decomposable.

Let $\mathbf{R} = Z/p^m Z$, $m \geq 1$, and write $\mathbf{R}^{\mathbf{r(n-1)}} = R^r e_1 \oplus \cdots \oplus R^r e_{n-1}$, a free R-module of rank $r(n-1)$, with $R^r e_i \approx R^r$. Let $\mathbf{R^r(e_1 + \cdots + e_{n-1})}$ be the image of the diagonal embedding of R^r in $R^r e_1 \oplus \cdots R^r e_{n-1}$. For an R-module homomorphism $M : R^r e_1 \to R^r e_2 \oplus \cdots \oplus R^r e_{n-1}$, define $\mathbf{(1 + M)R^r e_1} = \{(x, M(x)) : x \in R^r e_1\}$, a submodule of $R^r e_1 \oplus \cdots \oplus R^r e_{n-1}$.

Corollary 2.4. *Assume that $p \neq 2$ is a prime with $(\tau_i)_p$ finite for each $\tau_i \in T_n$ and that $n \geq 3$. For each j, $m \geq 1$ there is an indecomposable H in $B(T_n, p^m)$ such that H is not an acd group and* rank $H = j(n-1)$.

Proof. Let $R = Z/p^m Z$ and define $U \in \mathrm{Rep}_{n+1}(R)$ by

$$U = (R^{j(n-1)}, R^j e_1, \ldots, R^j e_{n-1}, R^j(e_1 + \cdots + e_{n-1}), (1 + M)R^j e_1)$$

where $M : R^j e_1 \to R^j e_2 \oplus \cdots \oplus R^j e_{n-1}$ is given by $M^{tr} = (M_2, \ldots, M_{n-1})$ and each M_i is a $j \times j$ R-matrix in Jordan canonical form with 2's on the diagonal and 1's on the superdiagonal. By Theorem 2.3a, $U = F(H)$ for some H in $B(T_n, p^m)$ that is not an *acd* group. Moreover, $(1 + M)R^j e_1$ has zero intersection with the other coordinate spaces. This is because the determinant of each M_i and $M_i - 1$ is a unit of R since $p \neq 2$.

It is now sufficient to show that U is an indecomposable representation, in which case H is indecomposable by Theorem 2.3b. Observe that $[\mathbf{p}] : \mathrm{Rep}_{n+1}(R) \to \mathrm{Rep}_{n+1}(Z/pZ)$, defined by $\mathbf{U}[\mathbf{p}] =$

$(U[p], U_1[p], \dots, U_{n+1}[p])$, is an additive functor with $U[p] = 0$ if and only if $U = 0$, where $\mathbf{U_i[p]}$ denotes the p-socle of U_i. Thus, $U[p]$ indecomposable implies that U is indecomposable.

Now assume that $R = Z/pZ$, $U[p] = U$, and α is an idempotent endomorphism of U. Then $\alpha = \oplus_{n-2} N$ for some idempotent matrix $N \in \mathrm{Mat}_j(R)$ with $NM_i = M_i N$ for each $2 \le i \le n-2$. This is because α preserves each of the coordinate spaces of U. Thus, N is in the centralizer of each M_i which is known to be isomorphic to $R[A] = R[x]/\langle (x-2)^j \rangle$. Hence, $N^2 = N$ is either the zero or the identity matrix, and so $\alpha = 0$ or 1. Consequently, U is indecomposable. □

Corollary 2.5. *Assume that $n \ge 3$, $p \ne 2$ is a prime, and $(\tau_i)_p$ is finite for each $\tau_i \in T_n$. Given $m \ge 1$, there is an indecomposable acd group H in $B(T_n, p^{m+1})$ with* $\operatorname{rank} H = mn$.

Proof. Let $R = Z/p^{m+1}Z$ and write the free R-modules $R^{mn} = R^m e_1 \oplus \cdots \oplus R^m e_n$ and $R^m e_i = Re_{i1} \oplus \cdots \oplus Re_{im}$. Let $M : R^m e_1 \to R^m e_2 \oplus \cdots \oplus R^m e_{n-1}$ be as specified in Corollary 2.4. Define U in $\mathrm{Rep}_{n+1}(R)$ by $U = (R^{mn}, R^m e_1, \dots, R^m e_n, V)$. Here $V = V_1 \oplus \cdots \oplus V_m \oplus V_{m+1}$, $V_t = p^{t-1} R\Delta_t$ for $R\Delta_t$ the image of a diagonal embedding of R in $RE_{1t} \oplus \cdots \oplus Re_{nt}$ for $t \le m$, and $V_{m+1} = p^m(1+M)R^m e_1$. Note that the $R^m e_n$-coordinate of V_{m+1} is 0. By Theorem 2.3a,there is an almost completely decomposable H in $B(T_n, p^{m+1})$ with $F(H) = U$.

To show that $U = F(H)$ is indecomposable, let α be an idempotent endomorphism of U. Then $\alpha = N_1 \oplus \cdots \oplus N_n$, where each $N_i : R^m e_i \to R^m e_i$ is idempotent. It is sufficient to verify that $N_1 \equiv \cdots \equiv N_n \pmod p$ and each $N_i \pmod p$ is in the centralizer of $M_i \pmod p$. If so, then $\alpha \equiv 0$ or $1 \pmod p$ and H is indecomposable, just as in the proof of Corollary 2.4.

We next prove that the N_i's are congruent mod p. Note that α preserves $p^k U = (p^k R^{mn}, p^k R^m e_1, \dots, p^k R^m e_n, p^k V)$ and $p^k V = p^k V_1 \oplus \cdots \oplus p^k V_m = p^k R\Delta_1 \oplus p^{k+1} R\Delta_2 \oplus \cdots \oplus p^m \Delta_{m+1-k}$ for each $1 \le k \le m$. Hence, $\alpha(V_t[p]) = \alpha(p^{m+1-t} V_t) = \alpha(p^m R\Delta_t)$ is contained in $V[p] \cap p^{m+1-t} V$ with $p^{m+1-t} V = p^{m+1-t} R\Delta_1 \oplus \cdots \oplus p^m R\Delta_t$ for $m \ge t \ge 1$. Equate coordinates of $\alpha(V_t[p]) = (\oplus N_i)(V_t[p])$ in R^{mn} for each $1 \leq t \leq m$, recalling the definition of $R\Delta_t$. It follows that the N_i's are congruent modulo p. Briefly, this is because for each t,

$$\begin{aligned}\alpha(p^m(e_{1t} + \cdots + e_{nt})) &= p^m N_1 e_{1t} + \cdots + p^m N_n e_{nt} \\ &= p^{m+1-t} r_1(e_{11} + e_{n1}) + \cdots \\ &\quad + p^m r_t(e_{1t} + \cdots + e_{nt}) \\ &\in V[p] \cap p^{m+1-t} V\end{aligned}$$

implies that $N_i e_{it} \equiv r_t e_{it} \pmod p$ for each $1 \leq i \leq n$.

It remains to show that $N_i M_i \equiv M_i N_i \pmod p$ for each $2 \leq i \leq n-1$. Note that $\alpha(p^m(1+M)R^m e_1) = \alpha(V_{m+1}) = \alpha(V_{m+1}[p])$ is contained in $V[p] \cap p^m R^{mn}$. Equating coordinates, as above, shows that $\alpha(V_{m+1})$ is contained in V_{m+1}. This uses the fact that the $R^m e_n$-coordinate of V_{m+1} is 0. Since $N_1 \equiv N_i \pmod p$ and $M_i = M_t$, the proof is complete. □

Corollary 2.6. *Let $p \neq 2$ be a prime with $(\tau_i)_p$ finite for each $\tau_i \in T_n$ with $n \geq 4$. Then there are indecomposable acd groups of arbitrarily large finite rank in $B(T_n, p)$.*

Proof. Let $k = Z/pZ$, $r \geq 1$, $n > 4$ an odd integer, and $U = (k^{rn}, k^r e_1, \ldots, k^r e_n, V) \in \mathrm{Rep}_{n+1}(k)$, where

$$\begin{aligned}V = k^r(e_1 + e_3 + \cdots + e_{n-2} + (1+M)e_{n-1}) \\ \oplus k^r(e_2 + e_4 + \cdots e_{n-3} + e_{n-1} + e_n),\end{aligned}$$

M is an $r \times r$ k-matrix with all 2's on the diagonal, all 1's on the superdiagonal, and 0's elsewhere and $(1+M)e_{n-1}$ is contained in $k^r e_{n-1} \oplus k^r e_n$. Elements of V are of the form $(x, y, \ldots, x, y, x+y, Mx+y) \in K^r e_1 \oplus \cdots \oplus k^r e_n$ for $x, y \in k^r$. To see that U is indecomposable, let $f = N_1 \oplus \cdots \oplus N_n$ be an endomorphism of U, where each N_i is an $r \times r$ k-matrix. Applying f to elements of V shows that $N = N_i$ for each i and $NM = MN$. It follows that U is indecomposable. By Theorem 2.3, $U = F(H)$ for some indecomposable *acd* group $H \in B(T_n, p)$ of rank rn.

A similar construction can be made for $n \geq 4$ even. □

Remark. $U[p]$ may be decomposable for an indecomposable U. For example, there is an indecomposable almost completely decomposable group $H \in B(T_3, p^3)$ of rank 6, given in Corollary 2.5, with $H/C_H \approx Z/p^3Z \oplus Z/p^2Z \oplus (Z/pZ)^2$ and $U = F(H) \in \operatorname{Rep}_4(Z/p^3Z)$ indecomposable. However, we show in Corollary 3.3 that each indecomposable almost completely decomposable in $\operatorname{Rep}_4(Z/pZ)$ has rank ≤ 3, in particular $U[p]$ must be decomposable.

3. $B(T_3)$**.** The next theorem is a characterization of the *acd* groups in $B(T_3)$ for the case that the types in T_3 sum to the type of Q. This proof is realized by reducing to the analogous results for $B(T_2)$ given in the introduction. Part (b) of the following theorem is stated in [**5**], but the proof is flawed. Let $\mathbf{T}(\mathbf{H})$ denote the torsion subgroup of H/C_H, a finite group for $H \in B(T_n)$.

Theorem 3.1. *Assume that* $\tau_1 + \tau_2 + \tau_3 =$ *type* Q*.*

(a) *A complete set of near isomorphism invariants for an acd group* $H \in B(T_3)$ *is* $\operatorname{rank} H$, $\operatorname{rank} H(\tau_i)$ *for* $1 \leq i \leq 3$, *and the isomorphism class of* $T(H)$.

(b) *Each indecomposable acd group* H *in* $B(T_3)$ *has* rank ≤ 3*. If* $\operatorname{rank} H = 3$, *then* H/C_H *is torsion cyclic.*

Proof. (a) These invariants are clearly preserved by near-isomorphism. Conversely, let G be another *acd* group in $B(T_3)$ with the same invariants as H. It is sufficient to assume that each $G(\tau_i) \neq 0$, i.e., $G \notin B(T_2)$. We may also assume that G and H have no rank-1 summands of type τ_0 and $T(G) = G/C_G$ is finite. Then $G'' = G \oplus C_G \approx G/G(\tau_1) \oplus G/G(\tau_2) \oplus G/G(\tau_3)$ with $G/G(\tau_i) \in B(T_{jk})$, $T_{jk} = \{\tau_0, \tau_j, \tau_k\}$ and $\{i, j, k\} = \{1, 2, 3\}$ [**5**, Proposition 1.4].

To show that G and H are nearly isomorphic, first observe that the τ_i-homogeneous completely decomposable groups $G(\tau_i)$ and $H(\tau_i)$ are isomorphic, since $\operatorname{rank} G(\tau_i) = \operatorname{rank} H(\tau_i)$. Also $T(G'') = T(G) = T(G/G(\tau_1)) \oplus T(G/G(\tau_2)) \oplus T(G/G(\tau_3)) \approx T(H) = T(H'')$ by hypothesis. For each prime p, $pA_i = A_i$ for some i, since $\tau_1 + \tau_2 + \tau_3 = \text{type}\, Q$. Thus, $T(G'')_p \approx T(G/G(\tau_i))_p$. This is because $pA_i = A_i$ and $G/G(\tau_j)$ quasi-isomorphic to $G(\tau_i) \oplus G(\tau_k)$ implies that $T(G/G(\tau_j))_p = 0$ for $\{i, j, k\} = \{1, 2, 3\}$. Since each $G/G(\tau_i)$ is in some $B(T_2)$, it follows that $T(G)$ is cyclic.

We next show that G'' is nearly isomorphic to H''. Examining p-components shows that $T(G/G(\tau_i)) \approx T(H/H(\tau_i))$ for each i. Moreover, $\operatorname{rank} G/G(\tau_i) = \operatorname{rank} H/H(\tau_i)$ and $\operatorname{rank}(G/G(\tau_i))(\tau_j) = \operatorname{rank}(H/H(\tau_i))(\tau_j)$ for each $1 \le i,\ j \le 3$, since G and H are almost completely decomposable. As a consequence of the classification in $B(T_2)$ mentioned in the introduction, $G/G(\tau_i) \in B(T_{jk})$ is nearly isomorphic to $H/H(\tau_i)$ for each i. Consequently, G'' is nearly isomorphic to H''.

Finally, G and H are nearly isomorphic. This is because $G'' = G \oplus C_G$ is nearly isomorphic to $H'' = H \oplus C_H$; $\operatorname{rank} G(\tau_i) = \operatorname{rank} H(\tau_i)$ for each i so that $C_G \approx C_H$; and cancellation holds for near isomorphism [**4**, Corollary 7.17].

(b) We know from the proof of (a) and [**24**] that, for an almost completely decomposable H with no rank-1 summands of type τ_0, $H'' = H \oplus H(\tau_1) \oplus H(\tau_2) \oplus H(\tau_3) \approx H/H(\tau_1) \oplus \cdots \oplus H/H(\tau_3)$, each $H/H(\tau_i) \in B(T_{jk})$ is a direct sum of indecomposables of rank ≤ 2, $T(H'') = T(H) = T(H/H(\tau_1)) \oplus T(H/H(\tau_2)) \oplus T(H/H(\tau_3))$, and that for each prime p, $T(H)_p = T(H/H(\tau_i))_p \ne 0$ implies that $pA_i \ne A_i$.

Now assume that H is indecomposable of rank ≥ 3. Then each $H(\tau_i) \ne 0$. There is some almost completely decomposable H' that is a direct sum of indecomposables of rank ≤ 3 and has the same invariants as H. This is because if G is indecomposable of rank 3 with G/C_G cyclic, then each $T(G/G(\tau_i))$ is cyclic, and G can be chosen so that each $T(G/G(\tau_i))$ simultaneously realizes a rank-1 summand of each $H(\tau_i)$ and a nonzero cyclic summand of each $T(H/H(\tau_i))$. By (a), H^1 is nearly isomorphic to H, whence H has rank 3, as desired. □

Let $A = (A_1, \dots, A_n)$ be an n-tuple of subgroups of Q with type $A_i = \tau_i$. Define $\mathbf{G}[\mathbf{A}] = A_1x_1 + \cdots + A_nx_n$, a subgroup of $Qx_1 \oplus \cdots \oplus Qx_{n-1}$, with $x_1 + \cdots + x_n = 0$. The group $G[A]$ is *cotrimmed strongly indecomposable* if $G[A](\tau_i) = A_ix_i$ for each i, see [**11** and references]. In this case $G[A] \in B(T_n)$ with $C_{G[A]} = G[A]$.

Each group H in $B(T_n)$ is quasi-isomorphic to a finite direct sum of strongly indecomposables in $B(T_n)$. Finite rank strongly indecomposables in $B(T_3)$ are of the form A_0, A_1, A_2, A_3, or $G[A_1, A_2, A_3]$ [**6**]. Moreover, if type $A_i' = \tau_i$ for each i, then $G[A_1, A_2, A_3] \approx G[A_1', A_2', A_3']$ if and only if there is $0 \ne q \in Q$ with $q(A_1, A_2, A_3) = (A_1', A_2', A_3')$ [**11**].

The following example is a counterexample to [**5**, Corollary 1.2]. It remains unknown as to whether or not $\tau_1 + \tau_2 + \tau_3 = \text{type}\, Q$ implies that each indecomposable in $B(T_3)$ has rank ≤ 3.

Example 3.2. There is a rank 3 indecomposable $H \in B(T_3)$ such that $\tau_1 + \tau_2 + \tau_3 = \text{type}\, Q$ and $C_H \approx G \oplus A_3$ for $G = G[A_1, A_2, A_3]$ cotrimmed strongly indecomposable.

Proof. Choose $T_3 = \{\tau_1, \tau_2, \tau_3\}$ such that $\tau_1 + \tau_2 + \tau_3 = \text{type}\, Q$, subgroups A_i of Q containing 1 of type τ_i and a prime p with $1/p \notin A_2$ and $1/p \notin A_3$. Define $H = A_1x_1 + A_2x_2 + A_3x_3 + A_3y + Z(x_2 + y)/p$, a subgroup of $Qx_1 \oplus Qx_2 \oplus Qy$ with $x_1 + x_2 + x_3 = 0$. A routine computation of the $H(\tau_i)$'s shows that $C_H \approx A_1x_1 + A_2x_2 + A_3x_3 + A_3y \approx G[A_1, A_2, A_3] \oplus A_3$. To see that H is indecomposable, choose an endomorphism f of H. Since $H(\tau_i) = A_ix_i$ for $i = 1, 2$ and $x_3 = -x_1 - x_2$, it follows that $f(x_i) = ax_i$ for some $a \in Q$ and $1 \leq i \leq 3$. Moreover, $H(\tau_3) = A_3x_3 \oplus A_3y$ and $H/C_H \approx Z/pZ$ imply that $f(y) = cx_3 + by$ and $ax_2 + cx_3 + by = f(x_2 + y) \equiv t(x_2 + y) \pmod p$ for some $t \in Z$. Hence, $c \equiv 0 \pmod p$ and $a \equiv t \equiv b \pmod p$, whence f is congruent to $a \pmod p$. This shows that if f is idempotent, then $f = 0$ or 1 and H is indecomposable, as desired. □

Each $H \in B(T_3)$ with no rank-1 summands of type τ_0 is quasi-isomorphic to $G[A]^r \oplus A_1^{r(1)} \oplus A_2^{r(2)} \oplus A_3^{r(3)}$ for $A = (A_1, A_2, A_3)$, $G[A]$ strongly indecomposable, and each A_i a subgroup of Q of type τ_i. Thus, rank $H = 2r + r(1) + r(2) + r(3)$ and each $H(\tau_i)$ has rank $r + r(i)$.

If $H = C_H \in B(T_3, p)$, then $F(H) \in \text{Rep}_3(Z/pZ)$ as the last coordinate of $F(H)$ is 0. If $F(H) \in \text{Rep}_3(Z/pZ)$ is indecomposable, then $\dim F(H) = \text{rank}\, H \leq 2$ and $F(H)$ is of the form $F(A_i)$ or $F(G[A])$ (see [**6**]). Thus, we restrict attention to those indecomposable H's with $H \neq C_H$.

The following theorem shows that there are no indecomposable *acd* groups in $B(T_3, p)$ with rank ≥ 4. In particular, by Theorem 2.3c, if H is an *acd* group, then H is indecomposable if and only if $F(H)$ is indecomposable. More generally, for an indecomposable H, the $r(i)$'s are bounded by 1 while r is unbounded.

Corollary 3.3. *Let p be a prime with $(\tau_i)_p$ finite for each i, $k = Z/pZ$, and $H \in B(T_{3,p})$ with* $\operatorname{rank} H = 2r + r(1) + r(2) + r(3)$ *and* $\operatorname{rank} H(\tau_i) = r + r(i)$.

The only indecomposable H's in $B(T_{3,p})$ with $F(H)$ indecomposable and $H \neq C_H$ are:

(i) *$r(1) = r(2) = r(3) = 0$, $H/C_H \approx k^r$ for $r \geq 1$, and $pH/pC_H = (1 + M)k^r$ for some $r \times r$ indecomposable matrix M such that M is invertible and 1 is not an eigenvalue of M.*

(ii) *$r(i) = 0$ for some i, $r(j) = 1$ for $j \neq i$, and $H/C_H \approx k^{r+1}$ for $r \geq 0$.*

(iii) *$r(1) = r(2) = r(3) = 1$, and $H/C_H \approx k^{r+2}$ for $r \geq 0$ or else some $r(i) = 1$, $r(j) = 0$ for $j \neq i$, and $H/C_H \approx k^r$ for $r \geq 1$.*

(iv) *$r(1) = r(2) = r(3) = 1$ and $H/C_H \approx k^{r+1}$ for $r \geq 0$.*

Proof. Assume that $F(H) = U = (U, U_1, U_2, U_3, U_4) \in \operatorname{Rep}_4(k)$ is indecomposable. By Theorem 2.3a, $U = U_1 + U_2 + U_3$ is a vector space with $\dim U \geq \dim U_i + \dim U_j$ for each $i \neq j \leq 4$ and $U_i \cap U_4 = 0$ for each $i \leq 3$. Define $\mathbf{dim\,U} = (d, d_1, d_2, d_3, d_4)$, where $d = \dim U = 2r + r(1) + r(2) + r(3) = \operatorname{rank} H$, $d_i = \dim U_i = r + r(i) = \operatorname{rank} H(\tau_i)$ for $1 \leq i \leq 3$, and $d_4 = \dim H/C_H$, recalling that H is p-locally free. As a consequence of the conditions on the U_i's we have $d \leq d_1 + d_2 + d_3$ and $d_i + d_j \leq d$ for each $i \neq j \leq 4$.

Define the defect of $U \in \operatorname{Rep}_4(Z/pZ)$ to be $\rho(\mathbf{U}) = d_1 + d_2 + d_3 + d_4 - 2d$. For an indecomposable representation $U \in \operatorname{Rep}_r(R)$, it is known that $\rho(U) = -2, -1, 0, 1$ or 2 [**19** or **13**]. There is a complete list of indecomposable representations, up to isomorphism, given in [**13**] in terms of their defects and dimension vectors up to allowable permutations (these permutations are not listed here):

	$\rho(U)$	$\dim U$		
1. [**13**, (i)]	0	$(2m, m, m, m, m)$	$U_4 = (1+M)U_1$,	M^{-1} exists
1′. [**13**, (ii)]	0	$(2m, m, m, m, m)$	$U_4 = (1+M)U_1$,	$M^m = 0$
2. [**13**, (iii)]	0	$(2m+1, m+1, m+1, m, m)$		
3. [**13**, (v)]	1	$(2m, m, m, m, m+1)$		
4. [**13**, (vii)]	1	$(2m+1, m+1, m, m+1, m+1)$		

5. [**13**, (ix)]	2	$(2m+1, m+1, m+1, m+1, m+1)$
6. [**13**, (iv)]	-1	$(2m+2, m+1, m+1, m, m+1)$
7. [**13**, (vi)]	-1	$(2m+1, m+1, m, m, m)$
8. [**13**, (viii)]	-2	$(2m+1, m, m, m, m)$

Cases 2, 3, 4 and 5 do not occur for representations of the form $F(H) = U$ since $d_i + d_j \leq d$ for $i \neq j \leq 4$. Case $1'$ also does not occur, since $U_4 \cap U_j \neq 0$ for some $j \leq 3$ (see [**13**, Lemma 1.b]). For case 1, $\rho(U) = 0$, $d = 2r + r(1) + r(2) + r(3) = 2m$, $d_i = r + r(i) = m$ for $i \leq 3$ and $d_i + d_j \leq d$ yield $r = m = d_4$ and $r(i) = 0$ for $i \leq 3$. In case 6, $r = m$, $d = 2r + 2$, $(r(1), r(2), r(3))$ is a permutation of (1,1,0), and $d_4 = r + 1$. As for case 8, if $m = 0$, then $U = (k, 0, 0, 0, 0)$. If $m \neq 0$, then $m = r + 1$, $d = 2r + 3$, $(r(1), r(2), r(3)) = (1, 1, 1)$ and $d_4 = r + 1$.

Finally, for case 7, $d = 2m + 1 = 2r + r(1) + r(2) + r(3) \geq d_i + d_4 = r + r(i) + d_4 = 2m$ or $2m + 1$ for each $i \leq 3$. If $d_4 = m + 1$, then $r + r(i) = m$, $r(i) = 1$ for each $i \leq 3$ and $r = m - 1$. If $d_4 = m$,then $r = m$ and $(r(1), r(2), r(3))$ is a permutation of (1,0,0).

It remains to show that the representations in cases (1), (6), (7), and (8) are of the given form. For (1),

$$U = (k^{2r}, k^r e_1, k^r e_2, k^r(e_1 + e_2), (1 + M)k^r e_1) \in \text{image } F,$$

since M invertible and 1 not an eigenvalue of M implies that $U_i \cap U_j = 0$ for each $1 \leq i \neq j \leq 4$. Cases (6), (7) and (8) are constructed using nilpotent matrices in [**13**]. An examination of these constructions, which are not repeated here, shows that these representations are as given. Consequently, the correspondences (1) $\to$ (i), (6) $\to$ (ii), (7) $\to$ (iii), and (8) $\to$ (iv) show that the list in (a) is complete. □

Example 3.4. There are indecomposable *acd* groups of ranks 4, 5 and 6 in $B(T_3, p^2)$.

Proof. In view of Theorem 2.3c, it is sufficient to find indecomposable U's in $\text{Rep}_4(R)$ with ranks 4, 5 and 6, where $R = Z/p^2Z$.

For the first example, define $U = (R^4, Re_1 \oplus Re_2, Re_3, Re_4, V)$, where $V = R(e_1 + e_3 + e_4) \oplus R(pe_2 + e_3 + \alpha e_4)$ and both α and $1 - \alpha$ are units of R. Let f be an endomorphism of U and write $f = (A, b, c)$, where

$A = (a_{ij})$ is a 2×2 R-matrix and $b, c \in R$. An arbitrary element of V has the form $xe_1 + pye_2 + (x+y)e_3 + (x+\alpha y)e_4$. Apply f to each given basis element of V and equate coefficients to see that f is congruent modulo p to the matrix sequence (A, b, c) with

$$A = \begin{pmatrix} \alpha_{11} & 0 \\ * & \alpha_{11} \end{pmatrix}, \qquad b = (\alpha_{11}), \quad \text{and} \quad c = (\alpha_{11}).$$

Thus, if f is idempotent, it follows that $f = 0$ or 1 so that U is indecomposable.

For the second one, define $U = (R^5, Re_1 \oplus Re_2, Re_3 \oplus Re_4, Re_5, V)$, where $V = R(pe_1+e_3+e_5) \oplus R(e_2+pe_4+e_5)$. Assume that $f = (A, B, c)$ is an endomorphism of U with A and B 2×2 R-matrices and $c \in R$. Applying f to each of the given basis elements of V and observing that an arbitrary element of V is $pxe_1 + ye_2 + xe_3 + pye_4 + (x+y)e_5$ shows that $f \pmod p \equiv (A, B, c)$ with

$$A = \begin{pmatrix} \alpha_{11} & * \\ 0 & \alpha_{11} \end{pmatrix}, \qquad B = \begin{pmatrix} \alpha_{11} & 0 \\ * & \alpha_{11} \end{pmatrix}, \quad \text{and} \quad c = (\alpha_{11}).$$

Once again, if $f^2 = f$, then $f = 0$ or 1, whence U is indecomposable, as desired.

The third one is given by $U = (R^6, Re_1 \oplus Re_2, Re_3 \oplus Re_4, Re_5 \oplus Re_6, V)$, where $V = R(e_1 + pe_3 + pe_5 + e_6) \oplus R(pe_2 + e_4 + e_6)$. Computations analogous to those above show that U is indecomposable.

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Baylor University, Waco, TX 76798-7328