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THE FLAT DIMENSION OF MIXED ABELIAN GROUPS AS E-MODULES

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1. Introduction. In 1989, Faticoni and Goeters [**6**] constructed a large variety of torsion-free abelian groups of finite rank which were flat as modules over their endomorphism rings, while Albrecht [**2**] gave examples of infinite rank groups with the same property in 1990. In his survey talk presented at the 1989 University of Connecticut abelian groups conference, R.S. Pierce proposed the general problem of E-flatness as a worthy study—particularly that of computing the flat dimension $(fd_{E(G)}(G))$ of an abelian group as a left module over $E = E(G)$. The only known major results along these lines were the theorem of Richman and Walker [**12**] proving that every reduced torsion group is E-flat and the theorem of Arnold [**3**] giving a criterion for a completely decomposable group to be E-flat.

In 1991, Vinsonhaler and Wickless [**15**] constructed finite rank, completely decomposable groups $\{G_n \mid 0 \leq n \leq \infty\}$ with $fd_E(G_n) = n$. Recently, Dugas and Faticoni [**7**] have announced a similar result employing a different method and constructing different kinds of torsion-free examples.

In this paper, we primarily study a class $\mathcal{G}$ of mixed abelian groups. The elements $G \in \mathcal{G}$ will be of finite torsion-free rank and embedded as a pure subgroup into ΠG_p, where G_p denotes the p-torsion subgroup of G. Furthermore, each $G \in \mathcal{G}$ will satisfy an additional requirement connected with the self-small property which was introduced in [**4**] by Arnold and Murley.

Our main result is that, for each $0 \leq n \leq \infty$, there exists $G \in \mathcal{G}$ with $fd_E(G) = n$. In the course of proving this, we show that, if $G \in \mathcal{G}$ has torsion-free rank n, then $fd_E(G) = fd_A(M)$. Here M is the algebra of $n \times n$ rational matrices and $A \subseteq M$ is a rational subalgebra associated with the group G. This latter result leads to a realization problem: Which subalgebras $A \subseteq M$ are associated with some $G \in \mathcal{G}$? We show

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that the class of realizable subalgebras is fairly large—in particular, large enough to obtain all possible flat dimensions. We give an example of a subalgebra of $M_4(\mathbf{Q})$ which cannot be realized.

We conclude the paper with some miscellaneous results connected with flatness of mixed groups.

2. Some classes of mixed groups. Throughout the paper G will be a mixed abelian group and will usually be reduced and of finite torsion-free rank. We let G_p be the p-torsion subgroup of G and $T = T(G) = \oplus G_p$. We always assume that G is an honest mixed group, that is, T is a proper subgroup of G. If G has torsion-free rank n, then a subset $X = \{x_1, \dots , x_n\} \subset G$ of independent elements of infinite order is called a maximal independent set. We first consider Γ, the class of (reduced) groups G such that G can be embedded as a pure subgroup of ΠG_p. We use the symbol $<$ to denote a pure subgroup, so if $G \in \Gamma$ we write $\oplus G_p < G < \Pi G_p$. The class Γ has occurred in [**9**], in the study of mixed groups with von Neumann regular endomorphism rings and in [**10**] in the study of mixed groups with principal projective endomorphism rings.

For each $G \in \Gamma$, it is easy to check that G/T is divisible. Thus, for $G \in \Gamma$, since $T = T(G)$ is reduced, the restriction map will be a (pure) embedding of the ring $E = E(G)$ into the ring $E(T) \cong \Pi E(G_p)$. Hence each $\lambda \in E$ can be regarded as a sequence (λ_p) where $\lambda_p \in E(G_p)$. In this way we can regard $\oplus E_p < E < \Pi E_p$ with $E_p = E(G_p)$. The next proposition records some useful results from [**10**] concerning E as a subring of the ring ΠE_p. We sketch the proofs for the reader's convenience.

Before stating the results we need some notation. Suppose $G \in \Gamma$ is of finite torsion-free rank. Fix a maximal independent set $X = \{x_1, \dots , x_n\} \subset G$. Then $\overline{X} = \{x_1 + T, \dots , x_n + T\}$ is a basis of the rational vector space $V = G/T$. Each $\lambda \in E$ induces a $\mathbf{Q}$-linear transformation $\bar{\lambda}$ on V. Define a ring homomorphism $\mu_X : E \to M_n(\mathbf{Q})$ by $\mu_X(\lambda) = \operatorname{mat}(\bar{\lambda})_{\overline{X}}$ where $\operatorname{mat}(\bar{\lambda})_{\overline{X}}$ denotes the matrix of the induced map $\bar{\lambda}$ with respect to the basis $\overline{X}$. Denote the set of p-components of the elements of X by $\{x_{1p}, \dots , x_{np}\}$, and let X_p be the subgroup of G_p generated by $\{x_{1p}, \dots , x_{np}\}$.

Proposition 2.1. *Let $G \in \Gamma$ be of finite torsion-free rank.*

a) *A sequence $\lambda = (\lambda_p) \in \Pi E_p$ is an endomorphism of G if and only if $\lambda(X) \subseteq G$ for some (any) maximal independent set $X \subseteq G$.*

b) *Let $\lambda = (\lambda_p) \in \Pi E_p$, and let $X = \{x_1, \dots, x_n\}$ be a maximal independent subset of G. Then $\lambda \in E$ if and only if it satisfies the following condition:*

(*)

There exists an $n \times n$ rational matrix α such that for almost all p we have the equation $\lambda_p(\sum c_i x_{ip}) = (x_{1p}, \dots, x_{np})\alpha(c_1, \dots, c_n)^t$. Here $(c_1, \dots, c_n)^t$ denotes an arbitrary column vector of integers $c_1, \dots, c_n$. (The matrix α will be simply $\mu_X(\lambda)$, the matrix of the map induced by λ on the rational vector space G/T).

c) i) $\ker \mu_X = \mathrm{Hom}\,(G, T)$ *and*

ii) $\ker \mu_X = \oplus E_p$ *if and only if $X_p = G_p$ for almost all primes p.*

Proof. (a) This is an easy exercise using the facts that X is a maximal independent torsion-free subset of G and that $\oplus G_P < G < \Pi G_p$.

(b) Note that for $\lambda \in E$, since $\mu_X(\lambda)$ is the matrix of $\bar{\lambda}$ with respect to $\overline{X}$, we have the rational matrix equation:

$$\lambda\left(\sum c_i \bar{x}_i\right) = (\bar{x}_1, \dots, \bar{x}_n)\mu_X(\lambda)(c_1, \dots, c_n)^t$$

for any choice of integers c_i. Here $\bar{g} = g + T$. This equation, viewed componentwise, gives the equation in $(*)$ for p greater than or equal to some suitably large k. In particular, k will be chosen large enough so that none of the rational entries in the matrix $\mu_X(\lambda)$ have denominators divisible by $p \geq k$. Conversely, if $\lambda = (\lambda_p) \in \Pi E(G_p)$ satisfies $(*)$, then it easily follows that $\lambda(X) \subset G$. By part (a), we obtain $\lambda \in E(G)$. Note that the matrix α will be precisely $\mu_X(\lambda)$.

(c) By definition of μ_X, we have $\mu_X(\lambda) = 0$ if and only if $\bar{\lambda}(\overline{X}) = 0$, that is, $\lambda(X) \subset T$. Since X is a $\mathbf{Q}$-basis for G/T the latter condition is equivalent to the requirement $\lambda(G) \subset T$. This proves part (i).

To prove (ii), let $\lambda = (\lambda_p) \in \ker \mu_X$. Then, in view of condition $(*)$, for almost all p, the map λ_p induces the zero endomorphism on X_p.

Thus, if $X_p = G_p$ for almost all p, then $\lambda_p = 0$ for almost all p, that is, $\lambda \in \oplus E_p$. On the other hand, suppose $I = \{p : X_p \neq G_p\}$ is infinite. For $p \in I$, define f_p to be the projection of G_p onto G_p/X_p followed by a nonzero map from G_p/X_p into G_p. Put $f_p = 0$ for $p \notin I$. Then the map $f = (f_p)$ is an element of infinite order in $\ker \mu_X$. Hence, in this case, $\ker \mu_X = \mathrm{Hom}\,(G,T) \supset \oplus E(G_p)$. □

The following theorem describes the groups in Γ:

Theorem 2.2. *The following conditions are equivalent for an abelian group G:*

a) $G \in \Gamma$.

b) i) *G_p is a direct summand of G for all primes p.*

ii) *G/T is divisible, but G is reduced.*

Proof. The implication a) implies b) is clear. For the converse, assume that b) holds, and let $\pi_p : G \to G_p$ be the projection map onto the direct summand G_p. Define a map $\delta : G \to \Pi G_p$ by $\delta(x) = (\pi_p(x))_p$ for $x \in G$. Since δ is the identity on T, it induces a map $\bar{\delta} : G/T \to [\Pi G_p]/T$. Since G/T is divisible, the image of $\bar{\delta}$ is pure in $[\Pi G_p]/T$. This implies that the image of δ is pure in ΠG_p.

Note that $\ker \delta \cap T = 0$, so $\ker \delta$ is a torsion-free subgroup of G. Let $x \in \ker \delta$, and write $G = G_p \oplus H$. Denote the restriction of δ to H by ε. The definition of δ ensures $x \in H$ and $\varepsilon(x) = 0$. Since H is p-divisible, there is a $y \in H$ with $x = py$. Then, $p\varepsilon(y) = \varepsilon(x) = 0$ implies $\varepsilon(y) \in G_p \cap \Pi_{q \neq p} G_q$ since $\pi_p(y) = 0$. This is only possible if $\delta(y) = 0$, that is, $y \in \ker \delta$. Therefore, $\ker \delta$ is a divisible torsion-free subgroup of G. Since $\mathbf{Q}$ is not contained in G, we have $\ker \delta = 0$.

Of particular interest for the remainder of the paper is the subclass of Γ whose elements are the self-small mixed groups such that G/T is divisible. Recall that an abelian group G is *self-small* if $\mathrm{Hom}\,(G, -)$ preserves direct sums of copies of G. Self-small groups have been of considerable importance in the torsion-free setting since their introduction in [**4**]. The next result connects the class Γ to the class of self-small abelian groups.

Theorem 2.3. *The following conditions are equivalent for a mixed abelian group G whose torsion-free rank is finite*:

a) *G is self-small.*

b) i) *G_p is finite for all primes p.*

ii) $\mathrm{Hom}\,(G,T) = \oplus_p E_p$.

c) *$E(G)$ is countable.*

Proof. a) $\Rightarrow$ b). By [**4**, Proposition 3.6], G_p has to be finite for all primes p. Suppose that $\mathrm{Hom}\,(G,T)$ is not torsion, and denote the canonical projection of T onto G_p by π_p. If $\phi : G \to T$ is not torsion, then $\pi_p\phi \neq 0$ for infinitely many primes p. Let S be the set of these primes, and define a map $\alpha : G \to \oplus_S G$ by $\alpha(g) = (\pi_p\phi(g))_{p\in S}$. The map α is well defined since $\phi(g) \in T$. In view of the choice of ϕ, $\phi(G)$ cannot be contained in $\oplus_n G$ for some $n < \omega$. This results in a contradiction to the fact that G is self-small.

b) $\Rightarrow$ c). The exact sequence $0 \to T \to G \to G/T \to 0$ induces a ring-homomorphism $\sigma : E \to E(G/T)$ by $[\sigma(\alpha)](g+T) = \alpha(g) + T$ for all $g \in G$ and $\alpha \in E$. We note that $\ker\sigma = \mathrm{Hom}\,(G,T)$. Since G_p is finite for all primes p and $\mathrm{Hom}\,(G,T) \cong \oplus_p E(G_p)$, we obtain that $\ker\sigma$ is countable. But since G/T is torsion-free of finite rank, $E(G/T)$ is also countable. Therefore, $E(G)$ is countable.

c) $\Rightarrow$ a) is established as in [**4**]. □

Corollary 2.4. *The following conditions are equivalent for a mixed abelian group G whose torsion free rank is finite*:

a) *G is reduced and self-small, and G/T is divisible.*

b) i) $G \in \Gamma$.

ii) *$X_p = G_p$ for almost all primes p.*

iii) *G_p is finite for all primes p.*

The corollary follows directly from the results of Proposition 2.1 and Theorems 2.2 and 2.3.

Throughout the remainder of the paper, we primarily consider the subclass $\mathcal{G}$ of Γ whose elements are of finite torsion-free rank and satisfy the additional conditions ii) and iii) of the last corollary.

3. The flat dimension of groups in G. The goal of this section is to study the flat and projective dimensions of groups in $\mathcal{G}$ as modules over their endomorphism ring. Let $G \in \mathcal{G}$ be of torsion-free rank n, and let $X \subset G$ be a maximal independent set. Let M denote the algebra of $n \times n$ rational matrices, and let $A = \mu_X(G)$. Then A is a subalgebra of M and, as noted in [**10**], is an invariant of G (independent up to isomorphism of the choice of X). Regard the rational vector space $V = G/T$ as an A-module in the natural way. For any ring R and R module K let $fd_R K$ ($pd_R K$) be the flat (projective) dimension of the module ${}_R K$.

Theorem 3.1. *With notation as above,*

a) $fd_E G = fd_A V$, *and*

b) $fd_A V \leq pd_E G \leq fd_A V + 1$.

Proof. (a) Since $G \in \mathcal{G}$, each G_p is finite. Thus, for each relevant prime p, there is a direct sum decomposition $G = G_p \oplus H^p$. Furthermore, since $H^p < \Pi_{q \neq p} G_q$, $pH^p = H^p$. Thus, both G_p and H^p are fully invariant summands of G. Since the finite group G_p is flat as a module over its own endomorphism ring [**12**], it follows that G_p also will be flat as an $E = E(G)$-module [**15**]. Hence, $T = \oplus G_p$ is a flat E-submodule of G. The exact sequence $0 \to T \to G \to V \to 0$ yields the information $fd_E G \leq \max\{fd_E T, fd_E V\}$ with the inequality being an equality except possibly in the case $fd_E V = fd_E T + 1$. Since $fd_E T = 0$ it follows that $fd_E G \leq fd_E V$ with equality except possibly in the case $fd_E G = 0$, $fd_E V = 1$. To see that this case cannot occur, suppose that $fd_E G = 0$ and let K be a right E-module. Then the sequence $0 = \operatorname{Tor}^1_E(K, G) \to \operatorname{Tor}^1_E(K, V) \to K \otimes T$ is exact. But, since V is divisible and each G_p is finite, $\operatorname{Tor}^1_E(K, V)$ is divisible and $K \otimes T$ is reduced. It follows that $\operatorname{Tor}^1_E(K, V) = 0$ for all K and, hence, $fd_E V = 0$. We have shown that $fd_E G = fd_E V$.

Finally, since $A \cong E/tE$, where tE denotes the torsion subgroup of E, which is a two-sided ideal of E, and A is torsion-free divisible, it is easy to check that any flat A-module remains so when regarded as an E-module. Hence, $fd_E V \leq fd_A V$. Conversely, any flat resolution of

V as an E-module can be tensored with $\mathbf{Q}$ to obtain a flat resolution of V as an A-module. Thus, $fd_E V = fd_A V$ and the proof of (a) is complete.

(b) Plainly $pd_E G \geq fd_E G = fd_A V$, the last equality by part (a). Suppose $fd_A V = t$. Choose a resolution $0 \to K_{t-1} \to F_{t-1} \to \cdots \to F_0 \to V \to 0$ where the F_i's are free A-modules and K_{t-1} is a flat A-module. The module $F_{t-1} \cong A^s \cong [E/tE]^s$ has projective dimension one as an E-module. Since K_{t-1} is a finitely generated flat module over the Artinian algebra A, K_{t-1} is a projective A-module. It follows that the projective dimension of K_{t-1} as an E-module is also one. Since $0 \to K_{t-1} \to F_{t-1} \to K_{t-2} \to 0$ is an exact sequence of E-modules, ($K_{t-2} = \ker F_{t-2} \to F_{t-3}$) we have $pd_E K_{t-2} \leq 2$. Working up through the resolution of V, we see that $pd_E V \leq t+1$. Finally note that, since each G_p is finite, T is a projective E-module [**12**]. The exact sequence $0 \to T \to G \to V \to 0$ thus implies $pd_E G \leq pd_E V$. □

Corollary 3.2. *Regard $M = M_n(\mathbf{Q})$ as a left module over its subalgebra A. Then ${}_E G$ is flat if and only if ${}_A M$ is projective.*

Proof. Since $M \cong V^n$ as A-modules, the module ${}_A M$ is projective if and only if the module ${}_A V$ is projective. By the theorem, ${}_E G$ is flat if and only if ${}_A V$ is flat. As observed above, ${}_A V$ is flat if and only if ${}_A V$ is projective. □

In addition, we obtain the following description when M is a projective left A-module.

Theorem 3.3. *The following conditions are equivalent*:

i) *M is projective as a left A-module.*

ii) *Any left M-module is projective as a natural left A-module.*

iii) *Any right M-module is injective as a natural right A-module.*

iv) *M is an injective right A-module.*

Proof. i) $\Rightarrow$ ii). Since M is semi-simple Artinian, every left M-module X is isomorphic to a direct summand of $\oplus_I M$ for some index-set I. By i), X is projective.

ii) $\Rightarrow$ iii). Let X be any right M-module, and consider the right A-module

$$\hat{X} = \operatorname{Hom}_M({}_AM_M, X_M).$$

We observe that X and $\hat{X}$ are naturally isomorphic as right A-modules. An exact sequence $0 \to U \to L \to W \to 0$ of right A-modules induces the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Hom}_A(W,\hat{X}) & \longrightarrow & \operatorname{Hom}_A(L,\hat{X}) & \xrightarrow{\alpha} & \operatorname{Hom}_A(U,\hat{X}) & & \\ & & \downarrow \iota & & \downarrow \iota & & \downarrow \iota & & \\ 0 & \longrightarrow & \operatorname{Hom}_M(W\otimes_A M, X) & \longrightarrow & \operatorname{Hom}_M(L\otimes_A M, X) & \longrightarrow & \operatorname{Hom}_M(U\otimes_A M, X) & \longrightarrow & 0 \end{array}$$

in which the vertical isomorphisms are given by the adjoint functor theorem. The bottom row is exact since M is a flat left A-module and X is an injective right M-module. The commutativity of the diagram shows that α is onto. Therefore, $X \cong \hat{X}$ is an injective right A-module.

iii) $\Rightarrow$ i). Let $0 \to X \xrightarrow{f} Y$ be an exact sequence of right A-modules. We wish to show that the induced map $f \otimes \mathrm{id}_M$ is a monomorphism. We do so by standard means observing that $(-\otimes_A M, \operatorname{Hom}_M(M_M, -))$ is an adjoint pair of functors [**8**].

We have natural maps $\phi : X \to \operatorname{Hom}_M(M, X \otimes_A M)$ given by $[\phi(x)](m) = x \otimes m$ and $\chi : \operatorname{Hom}_M(M, X \otimes_A M) \otimes_A M \to X \otimes_A M$ given by $\chi(\alpha \otimes m) = \alpha(m)$. A simple computation shows $\chi(\phi \otimes \mathrm{id}_M) = \mathrm{id}_{X\otimes M}$. Since $\operatorname{Hom}_M(M, X \otimes_A M)$ is a right M-module, it is injective as a right A-module. Noting that ϕ is a map of right A-modules, we obtain a map $g : Y \to \operatorname{Hom}_M(M, X \otimes_A M)$ with $gf = \phi$. This, in turn, yields $(g \otimes \mathrm{id}_M)(f \otimes \mathrm{id}_M) = \phi \otimes \mathrm{id}_M$. If $t \in \ker(f \otimes \mathrm{id}_M)$, then

$$0 = \chi(g \otimes \mathrm{id}_M)(f \otimes \mathrm{id}_M)(t) = \chi(\phi \otimes \mathrm{id}_M)(t) = t,$$

which shows that $f \otimes \mathrm{id}_M$ is a monomorphism, and M is flat as an A-module.

Since A is Artinian, and M is a finitely generated A-module, M is finitely related. But, finitely related flat modules are projective.

Since iii) obviously implies iv), it remains to show the converse. The proof of iv) $\Rightarrow$ iii) is a routine dualization of the proof of i) $\Rightarrow$ ii). $\square$

The last result leads us naturally to the task of describing those A's for which M is a projective left A-module. This is a very delicate issue as the next example shows.

Example 3.4. Suppose that A is a subalgebra of $M = M_n(\mathbf{Q})$ such that $m = \dim_{\mathbf{Q}} A$ divides n. Then there is a subalgebra A_0 of M with $A_0 \cong A$ for which M is projective as an A_0-module.

Proof. Set $R = A \cap M_n(\mathbf{Z})$. The rank of R as an abelian group is m. Since we may assume $m > 1$, Corollary 3.10 of [**6**] yields that there is a torsion-free abelian group G of rank n with $E(G) = R$ which is flat as a left R-module. Then, $\mathbf{Q}G$ is flat, hence, projective, over the ring $A = \mathbf{Q}R$. Using arguments similar to the one used in the proof of the previous theorem, the embedding of A into M which is induced by the action of A on $\mathbf{Q}G$ produces a subalgebra A_0 over which M is projective. □

4. Realizing algebras. Let A be a subalgebra of M, the full $n \times n$ rational matrix algebra. We call A *realizable* if there exists a $G \in \mathcal{G}$ and a maximal independent set $X \subset G$ such that $\mu_X[E(G)] = A$. Note that, in our set-up, the torsion-free rank of G will be n. In this section we investigate the class of realizable algebras and show that it is fairly large. Our first theorem gives some closure properties for this class.

Theorem 4.1. a) *If $A \subset M_n(\mathbf{Q})$ is realizable and $\beta \in M_n(\mathbf{Q})$ is invertible, then the algebra $\beta^{-1}A\beta \subset M$ is realizable.*

b) *If $A \subset M$ is realizable and B is a direct summand of A, then $B \subset M$ is realizable.*

c) *If $A \subset M_n(\mathbf{Q})$ and $A' \subset M_{n'}(\mathbf{Q})$ are realizable, then $A \oplus A' \subset M_{n+n'}(\mathbf{Q})$ is realizable.*

d) *If $A \subset M_n(\mathbf{Q})$ is realizable, then $M_t(A) \subset M_{nt}(\mathbf{Q})$ is realizable.*

Proof. (a) Without loss of generality we can assume that the matrix β has integral coefficients. Let $G \in \mathcal{G}$ and $X \subset G$ with $\mu_X[E(G)] = A$. Write the elements of X as a row vector $(x_1, \dots, x_n)$ and define a row vector $(x'_1, \dots, x'_n)$ by the matrix equation $(x'_1, \dots, x'_n) = X\beta$.

Then $X' = \{x'_1, \dots, x'_n\}$ is a maximal independent subset of G with $\mu_{X'}[E(G)] = \beta^{-1}A\beta$.

(b) As above, suppose $\mu_X[E(G)] = A$. If B is a direct summand of A, then $B = \bar{e}A$ where $\bar{e}$ is an idempotent in A. Say $\overline{\mu_X}[a + tE] = \bar{e}$ where $\overline{\mu_X}$ is the induced isomorphism $\overline{\mu_X} : E/tE \cong A$. Recalling that $\oplus E_p < E < \Pi E_p$ we can modify the coset representative a of $a+tE$ to obtain a new representative $a + tE = e + tE$ with $e^2 = e$. If $H = eG$ and $Y = eX = \{ex_1, \dots, ex_n\}$ it is not hard to check that $H \in \mathcal{G}$ and that $\mu_Y[E(H)] = B \subset M$.

(c) Let G, $G' \in \mathcal{G}$ with independent sets X, X' be such that $\mu_X[E(G)] = A \subset M_n(\mathbf{Q})$ and $\mu_{X'}[E(G')] = A' \subset M_{n'}(\mathbf{Q})$. Regard $A \oplus A' \subset M_n(\mathbf{Q}) \oplus M_{n'}(\mathbf{Q}) \subset M_{n+n'}(\mathbf{Q})$, the last inclusion via

$$A \oplus A' \rightarrow \begin{bmatrix} A & 0 \\ 0 & A' \end{bmatrix}.$$

We will construct groups H, $H' \in \mathcal{G}$ with independent sets Y, Y' which have the property that $\mu_{Y \cup Y'}[E(H \oplus H')] = A \oplus A' \subset M_{n+n'}(\mathbf{Q})$. Here $Y \cup Y'$ will be regarded as a maximal torsion-free independent subset of $H \oplus H'$ in the natural way.

List the elements in $M_n(\mathbf{Q})\backslash A$, say $M_n(\mathbf{Q})\backslash A = \{\beta_1, \beta_2, \dots, \beta_i, \dots\}$. For each $\beta_i \notin A$, condition $(*)$ of Proposition 2.1 must fail for the matrix β_i. That is, there exists an infinite set of primes P_i such that for $p \in P_i$ the equation $\sum c_j x_{jp} \rightarrow (x_{1p}, \dots, x_{np})\beta_i(c_1, \dots, c_n)^t$ does not define a legitimate endomorphism of $X_p = G_p$. Take P to be an infinite set of primes such that $P\backslash P_i$ is infinite for each i. Let $\nu : G \rightarrow \Pi_{p \notin P} G_p$ be the natural projection map, and let $H_0 = \nu(G)$, $Y_0 = \nu(X)$. Then $H_0 \in \mathcal{G}$ with maximal independent set Y_0. Clearly the matrices in the subalgebra A still satisfy condition $(*)$ with respect to H_0 and Y_0. Furthermore, since an infinite subset of each P_i still remains in $\{p : (H_0)_p \neq 0\}$, no new matrices outside of A will be associated with endomorphisms of H_0. Thus, $\mu_{Y_0}[E(H_0)] = A$. Now redo the construction, taking into account both $M_n(\mathbf{Q})\backslash A$ and $M_{n'}(\mathbf{Q})\backslash A'$, to construct H_0, Y_0 and H'_0, Y'_0 and an infinite set of primes P such that $\mu_{Y_0}[E(H_0)] = A$, $\mu_{Y'_0}[E(H'_0)] = A'$ with H_0, H'_0 having no p-torsion for $p \in P$. Divide the set P into two infinite disjoint subsets P_1 and P_2. Modify the group H_0, first by adding $\oplus_{p \in P_1} V_p$ to the torsion subgroup of H_0, where each V_p is an n-dimensional vector space over $\mathbf{Z}/p\mathbf{Z}$ with basis $\{v_{1p}, \dots, v_{np}\}$. Then augment each of the independent torsion-free elements in $Y_0 = \{y_1, \dots, y_n\}$ by setting $y_{ip} = v_{ip}$ for all $p \in P_1$. Call the new group and independent set H and Y. Using the set of primes P_2 do an analogous augmentation to H'_0 and Y'_0 to obtain H' and Y'. Since we have added independent components to the torsion-free elements of Y_0 (respectively, Y'_0) we have not destroyed condition $(*)$ for any of the matrices in A (respectively, A') on any of the new H_p's (respectively $(H')_p$'s). Thus, we still have $\mu_Y[E(H)] = A$ (respectively, $\mu_{Y'}[E(H')] = A'$). Moreover, in view of condition $(*)$ on the set $P_1 \cup P_2$, any map λ from H to H' or from H' to H (viewed as an endomorphism of $H \oplus H'$) must have $\mu_{Y\cup Y'}(\lambda) = 0$. It follows that

$$\mu_{Y\cup Y'}[E(H \oplus H')] = \begin{bmatrix} A & 0 \\ 0 & A' \end{bmatrix}$$

as desired.

(d) Regard $M_t(A) \subset M_{nt}(\mathbf{Q})$ via the usual block representation. Let G^t be the direct sum of t copies of G. If $\mu_X[E(G)] = A$ it follows directly from the definitions that $\mu_{X^t}[E(G^t)] = M_t(A)$. Here if $X = \{x_1, \dots, x_n\}$ then X^t denotes the natural set of nt independent elements in G^t obtained from the elements of X. □

The next series of results provides us with examples of realizable algebras.

Theorem 4.2. *Let $\bar{n} = \{1, 2, \dots, n\}$, and let N be a subset of the Cartesian product $\bar{n} \times \bar{n}$. If the rational subspace of matrices*

$$A = \{(a_{ij} \mid a_{ij} = 0 \text{ for } i, j \in N,\ a_{ij} \text{ is arbitrary otherwise}\}$$

is a subalgebra of $M_n(\mathbf{Q})$, then A is realizable.

Proof. Assume that the subspace A, as defined above, is a subalgebra of $M_n(\mathbf{Q})$. For $1 \le j \le n$, let $N_j = \{i \mid (i, j) \in N\}$. Note that N does not contain any pair (j, j) since, by assumption, $1 \in A$. Therefore, $j \notin N_j$.

Write the set Π of all primes as a disjoint union of n countable infinite subsets, say $\Pi = \cup_{j=0}^{n} \Pi_j$. If $p \in \Pi_0$, then let G_p be the free $\mathbf{Z}/p\mathbf{Z}$-module with basis $\{v_1 \dots v_{np}\}$. Now assume $j > 0$. For $p \in \Pi_j$, let G_p

be the free $\mathbf{Z}/p\mathbf{Z}$-module on the symbols $\{v_{ip} \mid i \in N_j\}$. If some N_j should be empty, then set $G_p = 0$ for each $p \in \Pi_j$. We construct $G \in \mathcal{G}$ as the pure subgroup of ΠG_p generated by $\oplus G_p$ and a torsion-free independent set $X = \{x_1, \dots, x_n\}$. Fix i with $1 \leq i \leq n$. As before, each $x_i = (x_{ip}) \in \Pi G_p$ will be defined by specifying its p-components. Let p be a prime in Π_j. Define $x_{ip} = 0$ if $i \notin N_j$ and $x_{ip} = v_{ip}$ otherwise.

We claim that, for a group G constructed as above, $\mu_X(E(G)) = A$. First suppose $\alpha \in \mu_X(E(G))$ and $(i,j) \in N$. Since $i \in N_j$, we have $x_{ip} = v_{ip}$ for all primes $p \in \Pi_j$. But, since $j \notin N_j$, we have set $x_{jp} = 0$ for all $p \in \Pi_j$. Thus, for all $p \in \Pi_j$, the matrix α cannot induce an endomorphism of G_p sending x_{jp} to x_{ip}. Since Π_j is an infinite set of primes, condition $(*)$ of Proposition 2.1 implies that α must have 0 in its (i,j)-component. It follows that $\mu_X(E(G)) \subseteq A$.

To see that $A \subseteq \mu_X(E(G))$, suppose that $(i,j) \notin N$. It will be enough to show that the matrix (e_{ij}) with 1 in the (i,j)-component and 0's elsewhere is in $\mu_X(E(G))$. Then the $\mathbf{Q}$-algebra $\mu_X(E(G))$ contains the rational vector space spanned by $\{(e_{ij}) \mid (i,j) \notin N\}$. By definition, this latter vector space is precisely A.

To show that $(e_{ij}) \in \mu_X(E(G))$, we verify that the assignment $x_{jp} \to x_{ip}$ defines a legitimate endomorphism of G_p for all p, and then refer to condition $(*)$. By our construction of G_p, the map $x_{jp} \to x_{ip}$ will be legitimate except in the case that $x_{jp} = 0$ and $x_{ip} = v_{ip}$. We claim that this case cannot occur.

By way of contradiction, suppose that $x_{jp} = 0$ and $x_{ip} = v_{ip}$ for some prime $p \in \Pi_t$. Since $x_{jp} = 0$, we have $j \notin N_t$. On the other hand, $x_{ip} = v_{ip}$ yields $i \in N_t$. Thus, $(j,t) \notin N$ and $(i,t) \in N$. Recall that $(i,j) \notin N$ by assumption. Hence, both matrices (e_{jt}) and (e_{ij}) are elements of A, but the matrix $(e_{it}) = (e_{ij})(e_{jt})$ is not. The resulting contradiction proves our claim and completes the proof of the theorem. □

Theorem 4.3. *Any algebraic number field is realizable.*

Proof. Let $K = \mathbf{Q}(d)$ be an algebraic number field. Choose an irreducible polynomial $m(x)$ in $\mathbf{Z}[x]$ with $m(d) = 0$ whose coefficients are relatively prime. Then the set $S = \{\textit{primes}\ p : m(x)\ \textit{has a root in}\ \mathbf{Z}/p\mathbf{Z}\}$ is infinite [**5**]. Let R be the pure subring (with unit) of $\Pi_{p\in S}\mathbf{Z}/p\mathbf{Z}$ generated by $T = \oplus_{p\in S}\mathbf{Z}/p\mathbf{Z}$ and $c = (c_p)$ where $c_p \in \mathbf{Z}/p\mathbf{Z}$ is a root of $m(x)$. It is not hard to see that $c+T$ is an algebraic element in the rational algebra $\Pi_{p\in S}\mathbf{Z}/p\mathbf{Z}/\oplus_{p\in S}\mathbf{Z}/p\mathbf{Z}$ with minimal polynomial $m(x)$ over $\mathbf{Q}(1+T)$. Thus, $T = \oplus_{p\in S}\mathbf{Z}/p\mathbf{Z} < R < \Pi_{p\in S}\mathbf{Z}/p\mathbf{Z}$ with $R/T \cong K$. Since R is pure in the direct product, p-rank $R = 1$ for all $p \in S$. It follows that R is an E-ring, that is, $R \cong E(R,+)$ via $r \to$ left multiplication by r (see [**13**] for details). If $X = \{1+T, c+T, c^2+T, \dots, c^{n-1}+T\}$ where degree $m(x) = n$, then $\mu_X(R) = K \subset M_n(\mathbf{Q})$, the last containment via the regular representation of K with respect to its $\mathbf{Q}$-basis X.

Example 4.4. The algebra of rational quaternions is realizable.

Proof. Let H_4 be the algebra of rational quaternions. Regard $H_4 \subset M_4(\mathbf{Q})$ via the usual left regular representation, specifically H_4 is the subalgebra of $M_4(\mathbf{Q})$ consisting of all matrices of the form

$$\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}$$

with a, b, c, d in $\mathbf{Q}$. We show that H_4 is realizable. Let P be the set of all primes of the form $4n+1$. For $p \in P$ choose $c_p \in \mathbf{Z}/p\mathbf{Z}$ such that $c_p^2 \equiv -1(p)$. Write P as the disjoint union of two infinite subsets $P = P_1 \cup P_2$. For each $p \in P$, let G_p be a two-dimensional $\mathbf{Z}/p\mathbf{Z}$ vector space with basis $\{x_p, y_p\}$, $G_p = \mathbf{Z}/p\mathbf{Z}x_p \oplus \mathbf{Z}/p\mathbf{Z}y_p$. We construct G as the pure subgroup of ΠG_p generated by $\oplus_{p\in P} G_p$ and $X = \{x, y, z, w\}$. Here $x = (x_p)$, $y = (y_p)$, $z = (z_p)$, and $w = (w_p)$ where, for $p \in P_1$, we set $z_p = c_p x_p$ and $w_p = c_p y_p$ and, for $p \in P_2$ we define $z_p = c_p y_p$ and $w_p = -c_p x_p$. Suppose $\lambda \in E(G)$ with $\mu_X(\lambda) = \alpha \in M_4(\mathbf{Q})$. A short computation, using Proposition 2.1(b) and the facts that, for $p \in P_1$, we have $\lambda(z_p) = c_p\lambda(x_p)$ and $\lambda(w_p) = c_p\lambda(y_p)$, shows that the matrix α must be of the form

$$\alpha = \begin{bmatrix} \alpha & -b' & -c & -d' \\ b & a' & -d & c' \\ c & d' & a & -b' \\ d & -c' & b & a' \end{bmatrix}.$$

Also, for $p \in P_2$, we have $\lambda(z_p) = c_p\lambda(y_p)$. Using this and looking at the action of α on the collection of G'_ps for $p \in P_2$ forces the relations $a = a'$, $b = b'$, $c = c'$ and $d = d'$. This is precisely what we need to conclude that $\mu_X[E(G)] = H_4$. $\square$

Example 4.5. There are groups G, $H \in \mathcal{G}$, which realize the isomorphic subalgebras

$$A = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & a \end{bmatrix} \middle| \, a, b, c \in \mathbf{Q} \right\}$$

and

$$B = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ c & b & a \end{bmatrix} \middle| \, a, b, c \in \mathbf{Q} \right\}$$

of $M = M_3(\mathbf{Q})$ respectively such that G is a flat $E(G)$-module while H has infinite flat dimension over $E(H)$. This is due to the fact that M is flat as a left A-module but has infinite projective dimension as a left B-module.

Proof. It is easily checked that

$$\begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & a \end{bmatrix} \to \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ c & b & a \end{bmatrix}$$

defines an isomorphism between A and B. We construct $G \in \mathcal{G}$ and a maximal independent subset X of G with $\mu_X(G) = A$.

Partition the set of primes into two infinite sets Π_1 and Π_2. A slight modification of the construction in Theorem 4.2 yields a group $G_1 \in \mathcal{G}$ and an independent set $X_1 = \{x, y, z\}$ in G_1 which realizes

$$\left\{ \begin{bmatrix} a_1 & 0 & 0 \\ b & a_2 & 0 \\ c & 0 & a_3 \end{bmatrix} \middle| \, a_1, a_2, a_3, b, c \in \mathbf{Q} \right\}$$

using only the primes from Π_1. Thus $(G_1)_p = 0$ for all $p \in \Pi_2$. We shall augment G_1 by adding direct summands G_p to T for each $p \in \Pi_2$

to construct a group $G \in \mathcal{G}$ which satisfies the additional requirement that $a_1 = a_2 = a_3$.

For $p \in \Pi_2$, we take $G_p = (\mathbf{Z}/p^2\mathbf{Z})x_p$ and define $y_p = z_p = px_p$. Let G be the group obtained from G_1 via this augmentation, and consider

$$\begin{bmatrix} a_1 & 0 & 0 \\ b & a_2 & 0 \\ c & 0 & a_3 \end{bmatrix} \in \mu_X(G).$$

Then, for almost all $p \in \Pi_2$, we have $\alpha(x_p) = a_1x_p + by_p + cz_p$, $\alpha(y_p) = a_2y_p$ and $\alpha(z_p) = a_3z_p$. Multiplying the first equation by p yields that a_1 is congruent a_2 modulo p. A comparison of the last two equations gives that a_2 is congruent a_3 modulo p. Since Π_2 is infinite, we must have $a_1 = a_2 = a_3$.

We observe that $V = \mathbf{Q}^3$ is isomorphic to

$$A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

as left A-modules. This implies that M is projective as an A-module.

We can construct a group $H \in \mathcal{G}$ which realizes B in a similar manner. To show that H has infinite flat dimension over its endomorphism ring, we first observe that $J = J(B)$ has infinite projective dimension as a left B-module. To see this, we note that

$$J \cong B \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \oplus B \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

If we consider the epimorphism of $B \oplus B$ onto J which is defined by

$$(\alpha, \beta) \to \alpha \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \oplus B \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

then its kernel is $J \oplus J$. Hence, J has infinite projective dimension.

Define a map from $B \oplus B$ onto V by

$$(\alpha, \beta) = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \oplus \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The kernel of this map is

$$K = B\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \oplus B\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + B\left[\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\right].$$

Observe that K maps onto J by projection onto the first component. Hence, the projective dimension of K, and therefore of V, is infinite. Thus, $pd_B(M) = \infty$.

In connection with (or in contrast to) Example 4.5, we want to point out the following situation: Suppose that A and B are subalgebras of a full rational matrix algebra M that are isomorphic via conjugation by an invertible element β of M. If A is realizable by $G \in \mathcal{G}$ and a maximal independent set X, then Theorem 3.1 shows that B is realizable by G and $X\beta$. Hence, $fd_A(M) = fd_A(V) = fd_E(G) = fd_B(V) = fd_B(M)$ in this case.

We thank Professor C. Vinsonhaler for suggesting that we investigate the following example as a good candidate for a nonrealizable algebra.

Example 4.6. The algebra

$$A = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ c & c & a & 0 \\ d & e & 0 & a \end{bmatrix} \middle| \, a, b, c, d \in \mathbf{Q}; e = b - c + 2d \right\}$$

is not realizable.

Proof. It is easy to check that A is a subalgebra of $M_4(\mathbf{Q})$. Suppose that $A = \mu_X(E(G))$ with $G \in \mathcal{G}$ and $X = \{x, y, z, w\}$ a maximal independent torsion-free subset of G. Then, for almost all p, the maps associated with the matrices

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \alpha_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$\alpha_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$$

via condition $(*)$ of Proposition 2.1b will be legitimate endomorphisms on G_p.

We claim that if α_1, α_2 and α_3 induce legitimate endomorphisms on G_p, then so does

$$\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Hence, $\alpha \in \mu_X(E(G))$ by Proposition 2.1b. But $\alpha \notin A$ results in a contradiction.

To prove the claim, let $H_p = \langle z_p, w_p \rangle \subseteq G_p$. If order$(z_p - w_p) \geq$ order(w_p), then $w_p - z_p$ is an element of maximal order in H_p. Hence, $\langle w_p - z_p \rangle$ is a direct summand of H_p; and there exists a map $\beta : H_p \to H_p$ such that $\beta(z_p - w_p) = w_p$. In this case $\bar{\alpha} = \beta\bar{\alpha}_2$ is a legitimate endomorphism of G_p, where the bar denotes the map induced by a matrix. If order$(z_p - w_p) <$ order (w_p), then z_p and w_p are both elements of maximal order in H_p. Arguing as above, there exists $\gamma : H_p \to H_p$ with $\gamma(z_p) = w_p$. In this case $\bar{\alpha} = \bar{\alpha}_3 - \gamma(\bar{\alpha}_1 + \bar{\alpha}_2)$ is also legitimate.

We conclude this section with a result that establishes the existence of groups in $\mathcal{G}$ of arbitrary flat dimension.

Theorem 4.7. *For every $0 \leq k \leq \infty$ there exist $G \in \mathcal{G}$ with $fd_E G = k$.*

Proof. In view of Theorem 3.1, it suffices to produce, for each $0 \leq k \leq \infty$, a group $G \in \mathcal{G}$ with maximal independent set X such that $\mu_X[E(G)] = A$ where $A \subset M$ is a subalgebra with $fd_A V = k$. Here V and M will be a rational vector space and a matrix algebra of appropriate size; and A will act on V in the natural way. For $k = \infty$ this has been done as part of Example 4.5. For finite k we appeal to Theorem 5 of [**15**] which proves that for every finite k there exists a finite rank completely decomposable group H with $fd_{\mathbf{Q}E(H)}\mathbf{Q}H = k$. Thus, $A = \mathbf{Q}E(H)$, the quasi-endomorphism algebra of H, is a subalgebra of $M = M_r(\mathbf{Q})$, where r is the rank of H, such that $fd_A V = k$, where $V = \mathbf{Q}H$. To complete the proof, we note that if H is finite rank completely decomposable then $A = \mathbf{Q}E(H)$ is a subalgebra of the type considered in Theorem 4.2. Hence, A is realizable. $\square$

5. Related topics. Closely related to the flatness of a module is the question when it is faithful in the sense of [**14**]. Recall that a left module M over a ring R is *faithful* if $IM = M$ implies $I = R$ for all right ideals I of R. The module is *faithfully flat* if it is flat and faithful.

Theorem 5.1. *The following conditions are equivalent for a mixed abelian group* G:

a) *G is faithful as a left $E(G)$-module.*

b) i) *G_p is finite and homogeneous for all primes p.*

ii) *G/G_p is p-divisible for all primes p with $G_p \neq 0$.*

iii) *G/T is a faithful E/tE-module.*

Proof. a) $\Rightarrow$ b). We first show that G_p is reduced for all primes p. If this is not the case, then let D be the largest divisible subgroup of G_p. Write $G = D \oplus B$ for some subgroup B of G, and define a map $\phi : G \to G$ by $\phi(x + b) = px + b$ for all $x \in D$ and $b \in B$. Since ϕ is onto, the right ideal $I = \phi E(G)$ of $E(G)$ satisfies $IG = G$. There is an $\alpha \in E(G)$ with $\phi\alpha = 1_G$. Since D is fully invariant in G, we have $\alpha(D) \subseteq D$. If $x \in D[p]$, then $x = \phi\alpha(x) = p\alpha(x) = 0$. The resulting contradiction shows that G_p has to be reduced. If G_p is not homogeneous, then neither is its p-basic subgroup X_p. Therefore, $G = U_1 \oplus U_2 \oplus V$ where U_1 and U_2 are nonisomorphic cyclic p-groups. Without loss of generality, we may assume that there is an epimorphism

$\sigma : U_2 \to U_1$. Let $H = U_2 \oplus U_2 \oplus V$ and define $\tau : H \to G$ by $\tau(x, y, z) = \sigma(x) + y + v$ for all $x, y \in U_2$ and $v \in V$. Since H is G-projective in the sense of [**1**], the right ideal $I = \{\tau\alpha \mid \alpha \in \mathrm{Hom}\,(H, G)\}$ of $E(G)$ satisfies $IG = G$. Hence, there is a map $\alpha : G \to H$ with $\tau\alpha = 1_G$. If $x \in U_1$, then there are $u_1, u_2 \in U_2$ and $v \in V$ with $\alpha(x) = u_1 + u_2 + v$. Thus, $x = \tau\alpha(x) = \sigma(u_1) + u_2 + v$. This shows that $u_2 = v = 0$ and α induces $\tilde{\alpha} : U_1 \to U_2$ with $\sigma\tilde{\alpha} = 1_G$. This is not possible however. This shows that G_p is homogeneous. Write $G_p = \oplus_J \mathbf{Z}/p^n\mathbf{Z}$ for some index-set J and $n < \omega$, and consider a decomposition $G = G_p \oplus W$. Let $\pi_i : G \to \mathbf{Z}/p^n\mathbf{Z}$ be the projection on the ith coordinate and $\delta : G \to W$ any projection with kernel G_p. We consider the right ideal I of $E(G)$ which is generated by δ and the π's. Since $IG = G$, we have $I = E(G)$ and there are $i_1, \dots, i_n \in I$ and $r_0, \dots, r_n \in E(G)$ with $1_G = \delta r_0 + \sum_{i=1}^n \pi_{i_j} r_j$. This yields $G \subseteq \pi_{i_j}(G) \oplus \cdots \oplus \pi_{i_n}(G) \oplus V$ which is not possible unless J is finite.

To show ii), assume that $p(G/G_p) \neq G/G_p$ for some prime p with $G_p \neq 0$. Since G_p is finite, there are $n < \omega$ and an epimorphism $\phi : \oplus_n(G/G_p) \to G_p$. Therefore, ϕ induces a map $\tilde{\phi} : \oplus_{n+1} W \to G$ by $\tilde{\phi}(x, y) = \phi(x) + y$ for all $x \in W^n$ and $y \in W$. Since W is a direct summand of G, we have that $I = \{\tilde{\phi}\alpha \mid \alpha \in \mathrm{Hom}\,(G, W^{n+1})\}$ is a right ideal of $E(G)$ with $IG = G$. Thus, $I = E(G)$ and there exists $\alpha : G \to W^{n+1}$ with $\tilde{\phi}\alpha = 1_G$. The existence of such an α contradicts that fact that $\mathrm{Hom}\,(G_p, W) = 0$.

Finally, let J be a right ideal of E/tE with $J(G/T) = G/T$. If I is a right ideal of $E(G)$ which contains tE and satisfies $I/tE = J$, then $IG + T = G$. Since each G_p is a finite direct summand of G, it follows that $T = (tE)G$. Hence, $T \subseteq IG$ and $IG = G$. Then $I = E(G)$, and G/T is a faithful E/tE-module.

b) $\Rightarrow$ a). Let I be a right ideal of $E(G)$ with $G = IG$. Consider a prime p with $G_p \neq 0$. Since G_p is finite, write $G = G_p \oplus W$ for some subgroup W of G. Observe that condition ii) guarantees that W is fully invariant in G. Therefore, $E(G) = E_p \oplus E(W)$ and $I = I_p \oplus J$ for right ideals I_p of E_p and J of $E(W)$, respectively. In particular, we obtain $G_p = I_p G_p$. Suppose that we have shown that G_p is faithful as an E_p-module. Then $I_p = E_p$ and $tE \subseteq I$. We have $(I/tE)(G/T) = (IG + T)/T = G/T$. Since G/T is a faithful E/tE-

module, $I/tE = E/tE$ and G is a faithful $E(G)$-module.

To complete the proof, observe that G_p is a free $\mathbf{Z}/p^n\mathbf{Z}$-module, and an abelian group H satisfies $S_{G_p}(H) = H$ if and only if $p^n H = 0$. Thus, every exact sequence $H \to G_p \to 0$ with $S_{G_p}(H) = H$ splits. By [**1**], G_p is a faithful E_p-module. □

Corollary 5.2. *Let $G \in \mathcal{G}$. Then G is faithfully flat as an $E(G)$-module if and only if*

i) *G_p is homogeneous.*

ii) *G/T is faithfully flat as an E/tE-module.*

Proof. Observe that $A \cong E/tE$ since $G \in \mathcal{G}$. By Theorem 3.1, G is flat as an $E(G)$-module if and only if G/T is flat as an A-module. □

Finally, we obtain an extension of a theorem in [**10**].

Theorem 5.3. *Let G be an abelian group such that G_p is finite elementary abelian for all primes p and such that $\oplus G_p < G < \Pi G_p$. (No restriction is imposed on the torsion-free rank of G.)*

a) $J(E(G)) = 0$.

b) *If $E(G)$ is left (semi-) hereditary, then E/tE is left (semi-) hereditary.*

c) *If E/tE is von Neumann regular, then $E(G)$ is right and left semi-hereditary.*

Proof. a) As in the discussion at the beginning of Section 2, we can regard $\oplus E_p < E < \Pi E_p$. Since each G_p is a finite dimensional $\mathbf{Z}/p\mathbf{Z}$-vector space, each $E_p = E(G_p)$ is semisimple. Hence, E is semi-simple as a subdirect product of semisimple rings.

b) Let I be a projective left ideal of $E(G)$, and choose a dual basis $\{(x_j, \phi_j) \mid j \in J\}$ of I where $x_j \in I$ and $\phi_j : I \to E(G)$. We show that $K = (I + tE)/tE$ is a projective left ideal of E/tE. Once this has been established, part b) of the theorem follows immediately. Define a map $\psi_j : K \to E/tE$ by $\psi_j(x+t+tE) = \phi_j(x)+tE$ for all $x \in I$ and $t \in tE$. Suppose that $x_1 + t_1 + tE = x_2 + t_2 + tE$. Then, $x_1 - x_2 \in I \cap tE$. We observe that $\phi_j(x_1 - x_2) \in tE$. Thus, ψ_j is well-defined. Clearly, ψ_j is an E/tE-morphism, and $\psi_j(x_j + tE) = 0$ for almost all j. If $x \in K$, then there is a $y \in I$ with

$$\begin{aligned} x = y + tE &= \left(\sum_{j \in J} \phi_j(y)x_j \right) + tE \\ &= \sum_{j \in J} \psi_j(y + tE)(x_j + tE). \end{aligned}$$

This shows that K has a dual basis, and is projective.

c) Let I be a finitely generated left ideal of $E(G)$. Since E/tE is von Neumann regular, there is $a \in E(G)$ with $(I + tE)/tE = (E(G)a + tE)/tE$. Suppose that I is generated by $y_1, \dots, y_n$. For each i, there are $r_i \in E(G)$ and $z_i \in tE$ with $y_i = r_i a + z_i$. We can find primes $p_1, \dots, p_m$ with $z_1, \dots, z_m \in E_{p_1} \oplus \dots \oplus E_{p_m}$. Consequently, $I + (E_{p_1} \oplus \dots \oplus E_{p_m}) = E(G)a + (E_{p_1} \oplus \dots \oplus E_{p_m})$. Since $G = G_{p_1} \oplus \cdots G_{p_m} \oplus (G \cap \Pi_{j>m} G_{p_j})$ is a decomposition of G into fully invariant direct summands, we have $E(G) = E_{p_1} \oplus \dots \oplus E_{p_m} \oplus S$ with $S = E(G \cap \Pi_{j>m}|; G_{p_j})$. We obtain $I = (I \cap S) \oplus \oplus_{i=1}^m (I \cap E_{p_i})$ and $E(G)a = (E(G) \cap S) \oplus \oplus_{i=1}^m (E(G)a \cap E_{p_i})$. Since E_{p_i} is a semi-simple Artinian ring each $I \cap E_{p_i}$ is a projective $E(G)$-module. Moreover, $E(G)a \cap S$ is cyclic as an $E(G)$-module. By [**10**, Theorem 3.8], $E(G)a \cap S$ is projective. Thus, $(I \cap S) \oplus E_{p_1} \oplus \dots \oplus E_{p_m} = (E(G)a \cap S) \oplus E_{p_1} \oplus \dots \oplus E_{p_m}$ yields that $I \cap S$ is a projective $E(G)$-module. Therefore, I is projective.

The right case is discussed similarly. □

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