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TOPOLOGICAL NEARRINGS WHOSE ADDITIVE GROUPS ARE TORI

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1. Introduction. A nearring is a triple $(N, +, *)$ where $(N, +)$ is a group, $(N, *)$ is a semigroup and $(x + y) * z = (x * z) + (y * z)$ for all $x, y, z \in N$. For information about the algebraic theory of nearrings, one may consult [**4, 8** and **9**]. If the binary operations $+$ and $*$ are continuous, then $(N, +, *)$ is a topological nearring. This paper was motivated by the following question, "Given a topological group $(G, +)$, exactly what are the continuous multiplications $*$ on G such that $(G, +, *)$ is a topological nearring?" The answer, it turns out, involves knowing just what the continuous functions are from G into the space of endomorphisms of G under the compact-open topology. We apply this general result, which is a topological version of a theorem of J.R. Clay [**2**] to the n-dimensional torus T^n and we are able to completely describe those multiplications $*$ so that $(T^n, +, *)$ is a topological nearring. One reason that the case for T^n follows so quickly is that there are, in a certain sense, few continuous maps from T^n into its space of endomorphisms. The case is far different, however, for the Euclidean n-groups. There are many continuous functions from R^n into its space of endomorphisms and, consequently, the operations $*$ for which $(R^n, +, *)$ is a topological nearring are much more abundant and varied. We will begin our investigation of continuous nearring multiplications on R^n in a subsequent paper. In this paper, after we derive the general result, we focus entirely on applications to the n-dimensional torus. The main results of the paper are in Section 2 where we derive the general result and then apply it to the n-dimensional torus in order to explicitly describe all the continuous multiplications $*$ on T^n such that $(T^n, +, *)$ is a topological nearring. After we describe these multiplications in Section 2, we derive a few corollaries and then we determine the ideals of each such nearring. In Section 3 we determine all the homomorphisms from one such nearring into another, and we describe the endomorphism semigroups and the automorphism groups

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of these nearrings. Finally, we study the multiplicative semigroups of these nearrings in Section 4. Among other things, we completely determine Green's relations for these semigroups, and we characterize their regular elements which happen to coincide with the idempotents.

2. The main results. We begin this section by making precise some of the concepts discussed in the introduction. Let R^n denote the topological Euclidean n-group, and let Z^n denote the subgroup of R^n consisting of those vectors all of whose coordinates are integers. The quotient R^n/Z^n is a topological group which is referred to as an n-dimensional torus and is denoted by T^n. We will denote R^1, Z^1 and T^1 more simply by R, Z and T, respectively. Consider the mapping φ from R^n into the product of n copies of the complex plane which is defined by $\varphi(x_1, x_2, \dots, x_n) = (e^{2\pi x_1 i}, e^{2\pi x_2 i}, \dots, e^{2\pi x_n i})$. Evidently, $\operatorname{Ker}\varphi = Z^n$, and we see that $T^n = R^n/Z^n$ is isomorphic to the product of n copies of the topological group of complex numbers of modulus one. Here, an isomorphism between two topological algebraic structures will always mean homeomorphism as well and a homomorphism from one topological algebraic structure into another will also mean continuity.

In [**7**], we defined a *right distributive topological system* (hereafter denoted Rdts) as a triple $(S, \oplus, *)$ where $(S, \oplus)$ and $(S, *)$ are both topological groupoids (i.e., $\oplus$ and $*$ are continuous mappings from $S \times S$ into S) and $(a \oplus b) * c = a * c \oplus b * c$ for all $a, b, c \in S$. For any topological groupoid $(S, \oplus)$ we denote by $\operatorname{End}(S, \oplus)$ the space of all endomorphisms of $(S, \oplus)$ where the topology on $\operatorname{End}(S, \oplus)$ is the compact-open topology, hereafter referred to as the c-topology. We recall that the c-topology takes for a subbasis of the open sets all sets of the form $[K, G]$ where K is compact, G is open and $[K, G] = \{\varphi \in \operatorname{End}(S, \oplus) : \varphi[K] \subseteq G\}$. We note that by our convention, each element of $\operatorname{End}(S, \oplus)$ is continuous.

In [**2**], J.R. Clay describes all nearring of order less than eight and in [**3**] he describes all nearrings of order eight whose additive groups are nonabelian and which satisfy certain additional natural conditions. In order to accomplish all this, he first describes, for any group $(G, +)$, all multiplications $*$ on G so that $(G, +, *)$ is a nearring. Our first result here, which is basic to our considerations in this paper, is a topological version of Clay's theorem.

Theorem 2.1. *Let* $(S, \oplus)$ *be a locally compact Hausdorff groupoid, and let* t *be a continuous map from the topological space* S *into the topological space* $\operatorname{End}(S, \oplus)$. *Let* $t(y) = \varphi_y$ *for each* $y \in S$ *and define* $x * y = \varphi_y(x)$ *for all* $x, y \in S$. *Then* $(S, \oplus, *)$ *is an* Rdts *and every multiplication for which* $(S, \oplus, *)$ *is an* Rdts *is obtained in this manner. Furthermore, the operation* $*$ *is associative so that* $(S, *)$ *is a semigroup if and only if* $\varphi_x \circ \varphi_y = \varphi_{\varphi_x(y)}$ *for all* $x, y \in S$.

Proof. We show first that if $*$ is defined as in the statement of the theorem, then $(S, \oplus, *)$ is an Rdts and our first step here will be to show that $*$ is continuous. Define a function F from $S \times S$ into S by $F(x, y) = x * y = \varphi_y(x)$. We must show that F is continuous. Suppose $F(x, y) = \varphi_y(x) \in G$ where G is an open subset of S. Since φ_y is continuous, there exists a neighborhood V of x such that $\varphi_y[V] \subseteq G$. Since S is locally compact, there exists a compact subset K of S and an open subset W of S such that

$$x \in W \subseteq K \subseteq V \tag{2.1.1}$$

and this implies that $t(y) = \varphi_y \in [K, G]$. Since t is continuous, it follows that there exists a neighborhood H of y such that $t[H] \subseteq [K, G]$, and this means simply that

$$\varphi_b[K] \subseteq G \quad \text{for all} \quad b \in H. \tag{2.1.2}$$

Now $W \times H$ is a neighborhood of (x, y), and we assert that $F[W \times H] \subseteq G$. Let $(a, b) \in W \times H$. We use both (2.1.1) and (2.1.2) to get

$$F(a, b) = a * b = \varphi_b(a) \in \varphi_b[W] \subseteq \varphi_b[K] \subseteq G,$$

and this verifies the fact that $*$ is a continuous binary operation. Furthermore,

$$(x \oplus y) * z = \varphi_z(x \oplus y) = \varphi_z(x) \oplus \varphi_z(y) = x * z \oplus y * z$$

for all $x, y, z \in S$, and we conclude that $(S, \oplus, *)$ is an Rdts.

Conversely, suppose $(S, \oplus, *)$ is an Rtds, and define $F(x, y) = x * y$ for all $x, y \in S$. For each $z \in S$, define a selfmap φ_z by $\varphi_z(x) = x * z =$

$F(x, z)$. Since F is continuous, it is immediate that φ_z is continuous for each $z \in S$. Moreover,

$$\varphi_z(x \oplus y) = (x \oplus y) * z = (x * z) \oplus (y * z) = \varphi_z(x) \oplus \varphi_z(y)$$

and we see that $\varphi_z \in \text{End}\,(S, \oplus)$ for all $z \in S$. Define a map t from S into $\text{End}\,(S, \oplus)$ by $t(x) = \varphi_x$. We must show that t is continuous. Suppose $t(y) = \varphi_y \in [K, G]$ where $[K, G]$ is a subbasic open subset of $\text{End}\,(S, \oplus)$. Then $\varphi_y[K] \subseteq G$ which means $F(x, y) = x * y = \varphi_y(x) \in G$ for every $x \in K$. Since F is continuous, there exists, for each $x \in K$ a neighborhood V_x of x and a neighborhood W_x of y such that $F[V_x \times W_x] \subseteq G$. The family $\{V_x\}_{x \in K}$ covers K and, since K is compact, some finite subcollection $\{V_{x_i}\}_{i=1}^n$ also covers K. Let $W = \cap\{W_{x_i}\}_{i=1}^n$. Then W is a neighborhood of y, and we assert that $t[W] \subseteq [K, G]$. Let $b \in W$. We must show that $t(b) = \varphi_b \in [K, G]$, that is, $\varphi_b[K] \subseteq G$. Let $a \in K$. Then $a \in V_{x_i}$ for some i, and we have

$$\varphi_b(a) = a * b = F(a, b) \in F[V_{x_i} \times W_{x_i}] \subseteq G.$$

This verifies the continuity of the function t. Finally, one easily verifies that $z*(y*x) = (z*y)*x$ for all $x, y, z \in S$ if and only if $\varphi_x \circ \varphi_y = \varphi_{\varphi_x(y)}$ for all $x, y \in S$, and the proof is complete. $\square$

We are now in a position to describe all multiplications $*$ on T^n so that $(T^n, +, *)$ is a topological nearring. We will associate any endomorphism of the vector space R^n with the matrix which induces it. We recall that $T^n = R^n/Z^n$, and for each $x \in R^n$, we denote by $\bar{x}$ the equivalence class containing x. In other words, $\bar{x}$ is a typical point of T^n.

Theorem 2.2. *Let A be any $n \times n$ matrix whose elements are integers and define $\bar{x} * \bar{y} = \overline{Ax}$ for $\bar{x}, \bar{y} \in T^n$. Then $(T^n, +, *)$ is an* Rdts *and every multiplication $*$ for which $(T^n, +, *)$ is an* Rdts *is obtained in this manner. Now choose an idempotent matrix A whose elements are integers and again define $\bar{x} * \bar{y} = \overline{Ax}$ for $\bar{x}, \bar{y} \in T^n$. Then $(T^n, +, *)$ is a topological nearring and every multiplication $*$ for which $(T^n, +, *)$ is a topological nearring is obtained in this manner.*

Proof. We first determine $\text{End}\,(T^n, +)$. Suppose $\varphi \in \text{End}\,(T^n, +)$. According to the discussion on pages 82 and 83 of [**1**], there exists an

$n \times n$ matrix A such that $\varphi(\bar{x}) = \overline{Ax}$ and moreover $AZ^n \subseteq Z^n$. The latter means that the elements of A must be integers. Conversely, it follows easily that if φ is defined as above, then $\varphi \in \mathrm{End}\,(T^n, +)$. Thus, each element of $\mathrm{End}\,(T^n, +)$ is induced by an $n \times n$ matrix whose elements are integers. Among other things, this means that $\mathrm{End}\,(T^n, +)$ is countable. Now $\mathrm{End}\,(T^n, +)$, with the c-topology is completely regular and Hausdorff since T^n is completely regular and Hausdorff. Since $\mathrm{End}\,(T^n, +)$ is countable, it follows that $\mathrm{End}\,(T^n, +)$ must be totally disconnected and, since T^n is connected, this implies that the continuous functions from T^n into $\mathrm{End}\,(T^n, +)$ are precisely the constant functions. It follows from Theorem 2.1 that one gets all multiplications $*$ on T^n such that $(T^n, +, *)$ is an Rdts by choosing a matrix A whose elements are integers and defining $\bar{x} * \bar{y} = \overline{Ax}$.

Now suppose A is idempotent. Then

$$(\bar{x} * \bar{y}) * \bar{z} = (\overline{Ax}) * \bar{z} = \overline{AAx} = \overline{Ax} = \bar{x} * (\bar{y} * \bar{z}),$$

and we see that $(T^n, +, *)$ is a topological nearring. Suppose, conversely, that $(T^n, +, *)$ is a topological nearring. Then $*$ is associative which means $(\bar{x} * \bar{y}) * \bar{z} = \bar{x} * (\bar{y} * \bar{z})$. This in turn means that $\overline{AAx} = \overline{Ax}$ for all vectors $x \in R^n$. Thus, we have $(A^2 - A)x = A^2x - Ax \in Z^n$ for all $x \in R^n$, and this can happen only if $A^2 - A = 0$ which means A must be idempotent. This completes the proof. □

We will denote by $\mathrm{Ran}\,(A)$ the range of the linear transformation induced by a matrix A.

Corollary 2.3. *Let $(T^n, +, *)$ be a topological nearring. Then $T^n * T^n = T^n$ if and only if $\bar{x} * \bar{y} = \bar{x}$ for all $\bar{x}, \bar{y} \in T^n$.*

Proof. It is immediate that $T^n * T^n = T^n$ if $\bar{x} * \bar{y} = \bar{x}$ for all $\bar{x}, \bar{y} \in T^n$. Suppose, conversely, that $T^n * T^n = T^n$. Now Theorem (2.2) tells us that there exists an idempotent matrix A whose elements are integers such that $\bar{x} * \bar{y} = \overline{Ax}$ for all $\bar{x}, \bar{y} \in T^n$. But $T^n * T^n = T^n$ means that for each $\bar{y} \in T^n$, there exists $\bar{x}, \bar{z} \in T^n$ such that $\overline{Ax} = \bar{x} * \bar{z} = \bar{y}$ and this means that for each $y \in R^n$, there exists an $\bar{x}$ such that $Ax - y = s$ for some $s \in Z^n$. Consequently,

$$R^n = \bigcup_{s \in Z^n} \mathrm{Ran}\,(A) + s.$$

Let $m = \dim(\operatorname{Ran}(A))$. Now $\operatorname{Ran}(A) + s$ is closed for each $s \in Z^n$ and $\dim(\operatorname{Ran}(A) + s) = m$ for each $s \in Z^n$. Thus, we see that R^n is the union of a countable number of closed subspaces, each of dimension m and it now follows from Theorem III 2 page 30 of [**6**] that $m = n$. This means that $\operatorname{Ran}(A) = R^n$ and, since A is idempotent, it must be the identity matrix. Consequently, $\bar{x} * \bar{y} = \overline{Ax} = \bar{x}$ for all $\bar{x}, \bar{y} \in T^n$. □

Given any topological group $(G, +)$ with more than one element, there are always at least two multiplications one can define which will result in topological nearrings. One of these is to define $x * y = 0$ for all $x, y \in G$ and the second is to define $x * y = x$ for all $x, y \in G$. Our next corollary provides us with a topological group where these are the only two multiplications which result in topological nearrings. Recall that $T = R/Z$ denotes the one-dimensional torus and that a *nonring* is a nearring which is not a ring.

Corollary 2.4. *For each integer n define a multiplication $*_n$ on T by $\bar{x} *_n \bar{y} = \overline{nx}$ for $\bar{x}, \bar{y} \in T$. Then $(T, +, *_n)$ is an* Rdts *and every* Rdts *for which $(T, +)$ is the additive topological group is obtained in this manner. Furthermore, $(T, +, *_0)$ and $(T, +, *_1)$ are topological nearrings and these are the only topological nearrings for which $(T, +)$ is the additive topological group. In particular, the topological group $(T, +)$ admits exactly one nonring structure which is $(T, +, *_1)$ and it is evidently not zero-symmetric.*

Proof. According to Theorem 2.1, we find all multiplications $*$ on T such that $(T, +, *)$ will be an Rdts by taking all 1×1 matrices A of integers and defining $\bar{x} * \bar{y} = \overline{Ax}$. This, of course, simply means choosing an integer n and defining $\bar{x} * \bar{y} = \overline{nx}$. By that same theorem, one gets a topological nearring precisely when the matrix is idempotent. In the one-dimensional case only the matrices (0) and (1) are idempotent, and these evidently correspond to the binary operations $*_0$ and $*_1$. □

We observed in the previous corollary that the only zero-symmetric topological nearring with additive topological group $(T, +)$ is the zero ring. The next result says the same for all T^n and is an immediate consequence of Theorem (2.1).

Corollary 2.5. *Suppose* $(T^n, +, *)$ *is a zero-symmetric topological nearring. Then* $\bar{x} * \bar{y} = \bar{0}$ *for all* $\bar{x}, \bar{y} \in T^n$.

In what follows, when a multiplication $*$ is defined on T^n by $\bar{x}*\bar{y} = \overline{Ax}$ where A is any matrix whose elements are integers, we will refer to $*$ as the *multiplication which is induced by* A and we will refer to $(T^n, +, *)$ as the *nearring which is induced by* A. For any subset $H \subseteq R^n$, we let $\overline{H} = \{\bar{x} : x \in H\}$.

Theorem 2.6. *Let* A *be any idempotent matrix whose elements are integers, and let* $(T^n, +, *)$ *be the topological nearring which is induced by* A. *Choose any subgroup* G *of* R^n *such that* $Z^n \subseteq G$ *and* $AG \subseteq G$. *Then* $\overline{G}$ *is an ideal of* $(T^n, +, *)$ *and every ideal of* $(T^n, +, *)$ *is obtained in this manner. Moreover,* $\overline{G}$ *is a closed ideal of* $(T^n, +, *)$ *if and only if* G *is a closed subgroup of* R^n.

Proof. Suppose first that $Z^n \subseteq G$ and that $AG \subseteq G$. Then $\overline{G}$ is a subgroup of T^n which is normal since T^n is abelian. For any $\bar{x} \in \overline{G}$ and any $\bar{y} \in T^n$, we have $\bar{x} * \bar{y} = \overline{Ax} \in \overline{G}$. Finally, for all $\bar{x}, \bar{y} \in T^n$ and $\bar{z} \in \overline{G}$, we also have

$$\bar{x} * (\bar{y} + \bar{z}) - \bar{x} * \bar{y} = \overline{Ax} - \overline{Ax} = \bar{0} \in \overline{G}$$

and we see that $\overline{G}$ is an ideal of $(T^n, +, *)$.

Now suppose that $\overline{G}$ is an ideal of $(T^n, +, *)$. It is immediate that $Z^n \subseteq G$. Moreover, we have $\overline{G} * T^n \subseteq \overline{G}$. This means that, for any $\bar{x} \in \overline{G}$ and any $\bar{y} \in T^n$, we have $\bar{x} * \bar{y} \in \overline{G}$ which implies that there exists a $\bar{z} \in \overline{G}$ such that $\overline{Ax} = \bar{z}$. Consequently, $Ax - z = v$ where $v \in Z^n$, and this implies that $Ax = v + z \in G$. The last assertion of the theorem is precisely one of the assertions on page 77 of [**1**]. □

The previous result indicates that ideals are abundant in these nearrings. We discuss some of them. Let A be any idempotent $n \times n$ matrix whose elements are integers, and let $(T^n, +, *)$ be the nearring which is induced by A. Let G be all those vectors in R^n with rational coordinates. Evidently $AG \subseteq G$ and so it follows from the previous result that $\overline{G}$ is an ideal of $(T^n, +, *)$. Since G is dense in R^n, $\overline{G}$ is dense in T^n. For a slightly different type of example, let

$$G = \{(x_1, x_2, \dots, x_n) : x_i = a_i + b_i\sqrt{2} \text{ where } a_i, b_i \in Z\}.$$

One easily checks that, in this case also, $AG \subseteq G$ and, hence, $\overline{G}$ is an ideal of $(T^n, +, *)$. Of course, $A\mathrm{Ran}\,(A) = \mathrm{Ran}\,(A)$ but $Z^n \not\subseteq \mathrm{Ran}\,(A)$ unless A is the identity matrix so that $\overline{\mathrm{Ran}\,(A)}$ is not an ideal of $(T^n, +, *)$ whenever A is not the identity matrix.

3. Homomorphisms. Let A be an idempotent $n \times n$ matrix whose elements are integers, and let B be an idempotent $m \times m$ matrix whose elements are also integers. Let $(T^n, +, *)$ be the nearring induced by A, and let $(T^m, +, \diamond)$ be the nearring which is induced by B. Our next result describes all the homomorphisms from $(T^n, +, *)$ into $(T^m, +, \diamond)$. The symbol $\bar{x}$ will be used to denote a typical element in both T^n and T^m since we think no confusion will result. In the former case, $x \in R^n$ while in the latter case $x \in R^m$, of course.

Theorem 3.1. *Let D be an $m \times m$ matrix whose elements are integers, and suppose further that $DA = BD$ and define $\varphi(\bar{x}) = \overline{Dx}$. Then φ is a homomorphism from $(T^n, +, *)$ into $(T^m, +, \diamond)$ and every homomorphism from $(T^n, +, *)$ into $(T^m, +, \diamond)$ is obtained in this manner.*

Proof. We first verify that $\varphi(\bar{x}) = \overline{Dx}$ defines a homomorphism. Since $AZ^n \subseteq Z^m$, it follows from the remarks on pages 82 and 83 of [**1**] that φ is a homomorphism from the topological group $(T^n, +)$ into the topological group $(T^m, +)$. In addition to this, since $DA = BD$, we have

$$\begin{aligned}\varphi(\bar{x} * \bar{y}) &= \varphi\overline{Ax} = \overline{DAx} = \overline{BDx} \\ &= \overline{Dx} \diamond \overline{Dy} = \varphi(\bar{x}) \diamond \varphi(\bar{y})\end{aligned}$$

and we conclude that φ is a homomorphism from $(T^n, +, *)$ into $(T^m, +, \diamond)$.

Suppose, conversely, that φ is a homomorphism from $(T^n, +, *)$ into $(T^m, +, \diamond)$. It follows from the remarks on pages 82 and 83 that there exists an $m \times n$ matrix D such that $\varphi(\bar{x}) = \overline{Dx}$ and $DZ^n \subseteq Z^m$. The latter condition means that the elements of D must all be integers. Finally, for all $\bar{x}, \bar{y} \in T^n$, we have

$$\begin{aligned}\overline{DAx} &= \varphi(\overline{Ax}) = \varphi(\bar{x} * \bar{y}) = \varphi(\bar{x}) \diamond \varphi(\bar{y}) \\ &= \overline{Dx} \diamond \overline{Dy} = \overline{BDx}.\end{aligned}$$

This means that $(DA - BD)x = DAx - BDx \in Z^m$ for all $x \in R^n$ which readily implies that $DA - BD$ must be the zero matrix. Thus, $DA = BD$ and the proof is complete. □

We use the symbol End $(T^n, +, *)$ to denote the endomorphism semigroup of $(T^n, +, *)$ and the symbol Aut $(T^n, +, *)$ to denote the automorphism group of $(T^n, +, *)$. Denote by $C(A)$ the centralizer of an $n \times n$ matrix in the semigroup of all $n \times n$ matrices of integers. That is, $C(A)$ consists of all $n \times n$ matrices B whose elements are integers such that $AB = BA$. Finally, denote by $G(A)$ all those matrices B in $C(A)$ such that $\det A = \pm 1$.

Theorem 3.2. *Let A be an $n \times n$ idempotent matrix of integers, and let $(T^n, +, *)$ be the topological nearring induced by A. Then* End $(T^n, +, *)$ *is isomorphic to $C(A)$ and* Aut $(T^n, +, *)$ *is isomorphic to $G(A)$.*

Proof. By Theorem 3.1, every endomorphism φ of $(T^n, +, *)$ is induced by an element B of $C(A)$. That is, $\varphi(\bar{x}) = \overline{Bx}$ for each $\bar{x} \in T^n$. Define a map Φ from End $(T^n, +, *)$ into $C(A)$ by $\Phi(\varphi) = B$. Let ψ be another endomorphism of $(T^n, +, *)$ which is induced by a matrix C. Then we have $\varphi \circ \psi(\bar{x}) = \varphi(\overline{Cx}) = \overline{BCx}$ for all $\bar{x} \in T^n$, and we see that $\Phi(\varphi \circ \psi) = BC = \Phi(\varphi)\Phi(\psi)$. It is evident that Φ is surjective and it is immediate that Φ is injective as well. Consequently, Φ is an isomorphism from End $(T^n, +, *)$ onto $C(A)$.

Suppose we now determine when an element $\varphi \in$ End $(T^n, +, *)$ actually belongs to Aut $(T^n, +, *)$. Again, let A be the matrix which induces φ. According to Proposition 5, page 83 of [**1**], φ is an automorphism of $(T^n, +, *)$ if and only if A is an automorphism of R^n whose restriction to Z^n is also an automorphism of Z^n. This means that not only must A have integral elements but A^{-1} must have integral elements as well, and this can happen if and only if $\det A = \pm 1$. It now follows in a routine manner that the mapping Φ defined by $\Phi(\varphi) = A$ for $\varphi \in \mathrm{Aut}\,(T^n, +, *)$ is an isomorphism from $\mathrm{Aut}\,(T^n, +, *)$ onto $G(A)$.

Example 3.3. Let

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and let $(T^2, +, *)$ be the nearring induced by A. One can verify that $B \in C(A)$, (that is, $AB = BA$) if and only if $b = 0$ and $d = a + c$ so that, according to Theorem 3.2, $\mathrm{End}\,(T^2, +, *)$ is isomorphic to

$$C(A) = \left\{ \begin{pmatrix} a & 0 \\ c & a+c \end{pmatrix} : a, c \in Z \right\}.$$

Now $\mathrm{Aut}\,(T^2, +, *)$ is isomorphic to the group of units of $\mathrm{End}\,(T^2, +, *)$ which, according to Theorem 3.2 consists of all those matrices $B \in C(A)$ such that $\det B = \pm 1$. One easily verifies that there are exactly four such matrices and they are as follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}.$$

Since each of these elements is an involution, we see that $\mathrm{Aut}\,(T^2, +, *)$ is isomorphic to the Klein four-group. We make one further observation. Suppose we choose $B \in C(A)$ but $0 \neq \det B \neq \pm 1$. Then B is an automorphism of the vector space R^2 but, nevertheless, the endomorphism it induces on $(T^2, +, *)$ cannot be an automorphism. For example, choose

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}.$$

Then $\varphi\overline{(1, 1/2)} = \overline{B(1, 1/2)} = \overline{(1, 2)} = \overline{(0, 0)}$ but $\overline{(1, 1/2)} \neq \overline{(0, 0)}$.

4. The multiplicative semigroups. Let A be an $n \times n$ idempotent matrix of integers, and let $(T^n, +, *)$ be the topological nearring induced by A. The multiplicative semigroup, $(T^n, *)$ of $(T^n, +, *)$ will be denoted more simply by $S(A)$. In this section we determine some of the properties of $S(A)$. For any terms regarding semigroups which are not defined, one may consult [**5**]. The symbols $\mathcal{H}, \mathcal{L}, \mathcal{R}, \mathcal{D}$ and $\mathcal{J}$ will denote the usual Green's relations, and for any set X, we let $\Delta(X) = \{(x, x) : x \in X\}$. Finally, $S(A)^1$ denotes the semigroup formed by adjoining 1 to $S(A)$.

Theorem 4.1. *The following statements are valid for all* $\bar{x}, \bar{y} \in S(A)$.

$$\bar{x}\mathcal{L}\bar{y} \quad \textit{if and only if} \quad \bar{x} = \bar{y} \quad \textit{or} \quad \bar{x}, \bar{y} \in \overline{\mathrm{Ran}\,(A)}. \tag{4.1.1}$$

$$\bar{x}\mathcal{R}\bar{y} \quad \textit{if and only if} \quad \bar{x} = \bar{y}. \quad \textit{That is,} \quad \mathcal{R} = \Delta(S(A)). \tag{4.1.2}$$

$$\mathcal{L} = \mathcal{D} = \mathcal{J} \quad \textit{and} \quad \mathcal{H} = \mathcal{R}. \tag{4.1.3}$$

Proof. By definition, $\bar{x}\mathcal{L}\bar{y}$ if and only if $S(A)^1 * \bar{x} = S(A)^1 * \bar{y}$, that is to say, if and only if $\{\bar{x}\} \cup \overline{\mathrm{Ran}\,(A)} = \{\bar{y}\} \cup \overline{\mathrm{Ran}\,(A)}$. Evidently, the latter equality holds if and only if $\bar{x} = \bar{y}$ or $\bar{x}, \bar{y} \in \overline{\mathrm{Ran}\,(A)}$. This verifies (4.1.1). Now suppose $\bar{x}\mathcal{R}\bar{y}$ but that $\bar{x} \neq \bar{y}$. Then we have

$$\{\bar{x}, \overline{Ax}\} = \bar{x} * S(A)^1 = \bar{y} * S(A)^1 = \{\bar{y}, \overline{Ay}\}.$$

Since $\bar{x} \neq \bar{y}$, we must have

$$\bar{x} = \overline{Ay} \quad \text{and} \quad \bar{y} = \overline{Ax}. \tag{4.1.4}$$

Thus, $x - Ay = s \in Z^n$ and hence $Ax - Ay = Ax - A^2y = A(x - Ay) = As \in Z^n$ which means that $\overline{Ax} = \overline{Ay}$. This, together with (4.1.4) implies that $\bar{x} = \overline{Ay} = \overline{Ax} = \bar{y}$ which is a contradiction. Consequently, we see that $\bar{x} = \bar{y}$ whenever $\bar{x}\mathcal{R}\bar{y}$ and we have verified (4.1.2). It now follows that $\mathcal{H} = \mathcal{L} \cap \mathcal{R} = \mathcal{L} \cap \Delta(S(A)) = \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{L} \circ \Delta(S(A)) = \mathcal{L}$.

Finally, for any $\bar{x} \in S(A)$, we have

$$\begin{aligned} S(A)^1 * \bar{x} * S(A)^1 &= \{\bar{x}\} \cup (S(A) * \bar{x}) \\ &\quad \cup (\bar{x} * S(A)) \cup (S(A) * \bar{x} * S(A)) \\ &= \{\bar{x}\} \cup \overline{\mathrm{Ran}\,(A)} \cup \{\overline{Ax}\} \cup \overline{\mathrm{Ran}\,(A)} \\ &= \{\bar{x}\} \cup \overline{\mathrm{Ran}\,(A)}. \end{aligned}$$

Consequently, $\bar{x}\mathcal{J}\bar{y}$ if and only if $\{\bar{x}\} \cup \overline{\mathrm{Ran}\,(A)} = \{\bar{y}\} \cup \overline{\mathrm{Ran}\,(A)}$ and we have already seen that the latter holds if and only if $\bar{x}\mathcal{L}\bar{y}$. Thus, (4.1.3) has been verified and the proof is complete. □

The symbol I will be used to denote the identity matrix, that is, the matrix with 1's down the diagonal and 0's elsewhere.

Theorem 4.2. *The following statements are equivalent.*

(4.2.1) $\bar{x}$ *is a regular element of* $S(A)$.

(4.2.2) $\bar{x}$ *is an idempotent element of* $S(A)$.

(4.2.3) $(A - I)x \in Z^n$.

Proof. Suppose $\bar{x}$ is regular. Then $\bar{x} * \bar{y} * \bar{x} = \bar{x}$ for some $\bar{y} \in T^n$. This implies that $\overline{Ax} = \bar{x}$ which, in turn, implies that $(A - I)x = Ax - x \in Z^n$. This verifies that (4.2.1) implies (4.2.3). Now suppose (4.2.3) holds. Then $Ax - x \in Z^n$ which implies $\overline{Ax} = \bar{x}$ and it readily follows from this that $\bar{x}$ is idempotent. Thus, (4.2.3) implies (4.2.2) and since it is immediate that (4.2.2) implies (4.1.1), the proof is complete. □

Example 4.3. Let

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

Then let $B = A - I$ and get

$$B = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}.$$

According to Theorem 4.2, $\bar{x} \in S(A)$ is idempotent if and only if $Bx \in Z^n$. One readily verifies that $Bx = (-x_1, 2x_1, 3x_1)$ and we see that $\bar{x}$ is idempotent if and only if x_1 is an integer.

Example 4.4. This time, let

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{pmatrix}.$$

Again, let $B = A - I$ and get

$$B = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & -1 \end{pmatrix}.$$

One verifies that, in this case, $Bx = (2x_2 - x_1, 0, x_2 - x_3)$. Consequently, $\bar{x}$ is idempotent if and only if $2x_2 - x_1$ and $3x_2 - x_3$ are both integers. For example, this is satisfied by the vector $(1, 3/2, 1/2)$ and we see that $\overline{(1, 3/2, 1/2)}^2 = \overline{A(1, 3/2, 1/2)} = \overline{(3, 3/2, 9/2)} = \overline{(1, 3/2, 1/2)}$.

Theorem 4.2 tells us that the regular elements and the idempotents coincide for any semigroup of the form $S(A)$. We also note that, for each of these semigroups, the collection of idempotents is a left zero subsemigroup of $S(A)$.

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