A REFLLEXIVE SPACE WITH NORMAL STRUCTURE THAT ADMITS NO UCED NORM

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1. Introduction. A Banach space $X$ is said to have normal structure if every bounded convex subset $C$ of $X$ with positive diameter $d = \sup\{|x - y| : x, y \in C\}$ is contained in some ball with center in $C$ and radius strictly smaller than $d$. This property was introduced by Brodskii and Milman [2] and happened to be important in the fixed point theory for nonexpansive mappings.

It was proved in [6] and [3] that uniform convexity in every direction implies normal structure. An example was constructed in [4] of a reflexive space $Y$ without equivalent norm, uniformly convex in every direction, which answered a question in [3]. It is not difficult to see that the original norm of $Y$ does not have normal structure. However, we shall prove here that $Y$ admits an equivalent norm with normal structure. Since $Y$ gives the only known pattern for constructing reflexive spaces without equivalent UCED norms, the problem if every reflexive space admits an equivalent norm with normal structure remains open (see [1]). In fact, the main result of the present paper was stated in [5], but the proof was not correct because of a misunderstanding of the construction in [4]. Since this article is to be considered as a correction to [5], we shall use almost the same notation.

2. Notation and results. A Banach space $(Y, \| \cdot \|)$ is said to be uniformly convex in every direction if the conditions $x_n, y_n, z \in Y$, $\|x_n\| \to 1$, $\|y_n\| \to 1$, $\|(x_n + y_n)/2\| \to 1$ and $x_n - y_n = \lambda_n z$, $\lambda_n$ reals, imply that $\|x_n - y_n\| \to 0$.

Following [5], for $Z = (\mathbb{R}^n, \| \cdot \|)$ with symmetric norm $\| \cdot \|$ the $Z$-direct sum of the normed spaces $X_1, \ldots, X_n$ is its product space with norm $\|(x_1, \ldots, x_n)\| = \left(\|x_1\|, \ldots, \|x_n\|\right)$. A normed space $X$ is said to have the sum-property if each $Z$-direct sum of finitely many copies of $X$ has
normal structure. The class of spaces having the sum-property is the largest subclass of spaces having normal structure which is closed under each finite $Z$-direct sum operation. We shall also need the following definition from [5]. Given a bounded sequence $\{x_m\} \subset X$, we consider the limit-functional

$$
\Lambda(x) = \lim_{m \to \infty} ||x_m - x||
$$

being defined for all $x \in X$ for which the limit exists.

We present now the construction from [4]. Let $\Gamma = \prod_{i=1}^{\infty} \{1, 2, \ldots, i\}$. That is, $\Gamma$ is the family of all sequences $\gamma = \{\gamma_i\}_{i=1}^{\infty}$ of positive integers such that $1 \leq \gamma_i \leq i + 1$. Denote by $\Phi$ the family of all finite subsets of $\Gamma$ which have the property that, if $\varphi \in \Phi$, then there is a positive integer $m$ such that, if $\gamma_k = \{\gamma_{k_i}\}_{i=1}^{\infty}$ and $\gamma_j = \{\gamma_{j_i}\}_{i=1}^{\infty}$ are different members of $\varphi$, then $\gamma_k^{(m)} \neq \gamma_j^{(m)}$ and $\gamma_k^{(i)} = \gamma_j^{(i)}$ for $1 \leq i \leq m - 1$. Let $\mathcal{F}$ be the set of all collections $F$ of finitely many mutually disjoint elements of $\Phi$. Define

$$
X = \{x : \Gamma \to \mathbb{R} : \sup_{F \in \mathcal{F}} ||x||_F < \infty\};
$$

where

$$
||x||_F = \left[ \sum_{\varphi \in \mathcal{F}} \left( \sum_{\gamma \in \varphi} |x(\gamma)| \right)^2 \right]^{1/2}, \quad F \in \mathcal{F};
$$

and let

$$
||x||_{12} = \sup_{F \in \mathcal{F}} ||x||_F, \quad x \in X.
$$

It is proved in [4] that $X$ is reflexive and the dual space $Y = X^*$ does not admit an equivalent UCED norm. Denote by $\{\varepsilon_{\gamma}\}_{\gamma \in \Gamma}$ the natural unconditional basis of $X$. Since $||\varepsilon_{\gamma} + \varepsilon_{\delta}||_{12} = 2$ for all $\gamma, \delta \in \Gamma$, it is easy to prove the following.

**Proposition.** The space $Y = X^*$, equipped with the dual original norm $|| \cdot ||_{12}$, lacks normal structure.

However, define for $x \in X$, $||x||_2 = (\sum_{\gamma \in \Gamma} |x(\gamma)|^2)^{1/2} \leq ||x||_{12}$, and let

$$
||x|| = \left( ||x||_2^2 + ||x||_{12}^2 \right)^{1/2}.
$$
Obviously, $\| \cdot \|$ is an equivalent norm on $X$.

**Theorem.** The space $(X^*, \| \cdot \|)$ has the sum-property and consequently it has normal structure.

*Proof.* Assume that $(X^*, \| \cdot \|)$ does not have the sum-property. Then, by [5], there is a sequence \( \{x^n_\gamma \} \) in $X^*$ such that $x^n_\gamma$ is weakly null, $\|x^n_\gamma\| \to 1$, $\Lambda(x^n_\gamma) = a_n \to 1$, where $\Lambda(x^n_\gamma) = \lim_{n \to \infty} \|x^n_\gamma - x^n_\eta\|$.

Take support functionals $x_n \in X$ such that $\|x_n\| = \|x^n_\gamma\|^{-1}$, $x^n_\gamma(x_n) = 1$. Clearly, $\|x_n\| \to 1$. It is easy to see that $x_n$ tends weakly to zero. Indeed, assume the contrary. Since the basis $\{e^\gamma \}$ is shrinking, then for some $\gamma \in \Gamma$, $e^\gamma(x_m) \geq b > 0$ for infinitely many $m$. As $\|x + y\|^2 \geq \|x\|^2 + \|y\|^2$ whenever $x, y \in X$ have disjoint supports, then for the elements $z_m = x_m|\Gamma \setminus \{\gamma\}$ we have $\|z_m\| \leq 1 - \eta$ for some $\eta > 0$ and infinitely many $m$. On the other hand, since $x^n_\gamma$ is weakly null, we get $x^n_\gamma(z_m) \to 1$, which is a contradiction.

For each $i$, $(x^n_i - x^n_\gamma)(x_i - x_n) = 2 - x^n_\gamma(x_n) - x^n_\gamma(x_i) \to 2$ as $n \to \infty$. Hence, for each $i$,

$$\lim_{n \to \infty} \|x_i - x_n\| \geq 2a_i^{-1}.$$

Since $\| \cdot \|$ is a lattice norm and $x_n$ is weakly null, we obtain

\[(\ast) \quad \lim_{n \to \infty} \inf \|x_i + x_n\| \geq 2a_i^{-1}, \text{ for every } i.\]

It is not hard to check that $(\ast)$, the uniform convexity of $l_2$ and $x_n \to 0$ weakly, imply $\|x_i\|_2 \to 0$.

Fix $0 < \varepsilon < 1/7$. Choose $0 < \delta < \varepsilon$ such that whenever $u, v \in l_2$, $\|u\|_2, \|v\|_2 \leq 1 + \delta$, $\|u + v\|_2 \geq 2(1 - \delta)$, we get $\|u - v\|_2 < \varepsilon$.

Fix $i$ such that $\|x_i\| < 1 + \delta$, $\|x_n\|_2 < \delta/4$ and $a_n < (1 - \delta/4)^{-1}$ for all $n \geq i$. Since $\{e^\gamma \}$ is a basis, there exists a finite subset $A \subset \Gamma$, such that $\|x_{n_{i+1}}\| < \varepsilon$.

It follows from the definition of $\Phi$ that, for every choice of $\gamma_1, \gamma_2 \in \Gamma$, with $\gamma_1 \neq \gamma_2$,

$$k(\gamma_1, \gamma_2) = \max\{|\varphi| : \varphi \in \Phi, \{\gamma_1, \gamma_2\} \subset \varphi\} < \infty,$$

where $|\varphi|$ denotes the number of elements of $\varphi$. Let

$$k = |A| \max\{k(\gamma_1, \gamma_2) : \gamma_1, \gamma_2 \in A, \gamma_1 \neq \gamma_2\}.$$
Clearly, \( k < \infty \). Since \( \|x_n\|_2 \to 0 \), there exists \( j \geq i \) such that, for
\( n \geq j \),
\[ \max_{\gamma \in \Gamma} |e^*_\gamma(x_n)| \leq \varepsilon/k. \]

By \((*)\) and \( a_i < (1 - \delta/4)^{-1} \), choose \( n \geq j \) so that \( \|x_i + x_n\| > 2 - \delta \).
Thus, it follows from \( \|x_i + x_n\|_2 < \delta/2 \) that \( \|x_i + x_n\|_2 > 2 - 3\delta/2 \).
Therefore, there exists an \( F = \{ \varphi_j \} \in \mathcal{F} \) such that \( \|x_i + x_n\|_F > 2(1 - \delta) \). We have
\[ \|x_i + x_n\|_F \leq \|x_i\|_F + \|x_n\|_F \leq 1 + \delta + \|x_n\|_F, \]
whence
\[ \|x_n\|_F > 1 - 3\delta. \]

Let \( F_2 = \{ \varphi_j \in \mathcal{F} : |\varphi_j \cap A| \geq 2 \} \) and \( F_1 = F \setminus F_2 \). Consider
\[
\begin{align*}
  u &= \left( \sum_{\gamma \in \varphi_j} |e^*_\gamma(x_i)| \right)_{j=1}^\infty, \\
  v &= \left( \sum_{\gamma \in \varphi_j} |e^*_\gamma(x_n)| \right)_{j=1}^\infty
\end{align*}
\]
like elements of \( l_2 \). By \( \|u + v\|_2 \geq \|x_i + x_n\|_F \) and the choice of \( \delta \),
we obtain \( \|u - v\|_2 < \epsilon \). Since \( \|x_i\|_2 < \delta/4 \) and \( \|x_{i\setminus A}\| < \epsilon \), then
\( \|x_i\|_{F_1} < 2\epsilon \). Hence, \( \|x_n\|_{F_1} < 3\epsilon \). Moreover, \( \|x_n\|_{F_2} < k(\varepsilon/k) = \varepsilon \).
Thus,
\[ \|x_n\|_F < 4\epsilon, \]
which is in contradiction with \( \|x_n\|_F > 1 - 3\epsilon \).

Therefore, \((X^*, \|\cdot\|)\) has the sum-property. \( \square \)

REFERENCES


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