ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 3, Summer 1995

A REFLEXIVE SPACE WITH NORMAL STRUCTURE THAT ADMITS NO UCED NORM

DENKA KUTZAROVA AND THOMAS LANDES

1. Introduction. A Banach space X is said to have normal structure if every bounded convex subset C of X with positive diameter $d = \sup\{||x-y|| : x, y \in C\}$ is contained in some ball with center in C and radius strictly smaller than d. This property was introduced by Brodskii and Milman [**2**] and happened to be important in the fixed point theory for nonexpansive mappings.

It was proved in [**6**] and [**3**] that uniform convexity in every direction implies normal structure. An example was constructed in [**4**] of a reflexive space Y without equivalent norm, uniformly convex in every direction, which answered a question in [**3**]. It is not difficult to see that the original norm of Y does not have normal structure. However, we shall prove here that Y admits an equivalent norm with normal structure. Since Y gives the only known pattern for constructing reflexive spaces without equivalent UCED norms, the problem if every reflexive space admits an equivalent norm with normal structure remains open (see [**1**]). In fact, the main result of the present paper was stated in [**5**], but the proof was not correct because of a misunderstanding of the construction in [**4**]. Since this article is to be considered as a correction to [**5**], we shall use almost the same notation.

2. Notation and results. A Banach space $(Y, ||\cdot||)$ is said to be uniformly convex in every direction if the conditions $x_n, y_n,\ z \in Y$, $||x_n|| \to 1$, $||y_n|| \to 1$, $||(x_n + y_n)/2|| \to 1$ and $x_n - y_n = \lambda_n z$, λ_n reals, imply that $||x_n - y_n|| \to 0$.

Following [**5**], for $Z = (\mathbf{R}^n, |\cdot|)$ with symmetric norm $|\cdot|$ the Z-direct sum of the normed spaces $X_1, \dots, X_n$ is its product space with norm $||(x_1, \dots, x_n)|| = |(||x_1||, \dots, ||x_n||)|$. A normed space X is said to have the sum-property if each Z-direct sum of finitely many copies of X has

The research of the first author is partially supported by the Bulgarian Ministry of Education and Science under the contract MM-213/92.
Received by the editors on June 1, 1993.

normal structure. The class of spaces having the sum-property is the largest subclass of spaces having normal structure which is closed under each finite Z-direct sum operation. We shall also need the following definition from [**5**]. Given a bounded sequence $\{x_m\} \subset X$, we consider the limit-functional

$$\Lambda(x) = \lim_{m\to\infty} ||x_m - x||$$

being defined for all $x \in X$ for which the limit exists.

We present now the construction from [**4**]. Let $\Gamma = \prod_{i=2}^{\infty}\{1, 2, \dots , i\}$. That is, Γ is the family of all sequences $\gamma = \{\gamma^i\}_{i=1}^{\infty}$ of positive integers such that $1 \le \gamma^i \le i+1$. Denote by Φ the family of all finite subsets of Γ which have the property that, if $\varphi \in \Phi$, then there is a positive integer m such that, if $\gamma_k = \{\gamma_k^i\}_{i=1}^{\infty}$ and $\gamma_j = \{\gamma_j^i\}_{i=1}^{\infty}$ are different members of φ, then $\gamma_k^m \ne \gamma_j^m$ and $\gamma_k^i = \gamma_j^i$ for $1 \le i \le m-1$. Let $\mathcal{F}$ be the set of all collections F of finitely many mutually disjoint elements of Φ. Define

$$X = \{x : \Gamma \to \mathbf{R} : \sup_{F\in\mathcal{F}} ||x||_F < \infty\};$$

where

$$||x||_F = \left[\sum_{\varphi\in F}\left(\sum_{\gamma\in\varphi}|x(\gamma)|\right)^2\right]^{1/2}, \quad F \in \mathcal{F};$$

and let

$$||x||_{12} = \sup_{F\in\mathcal{F}} ||x||_F, \quad x \in X.$$

It is proved in [**4**] that X is reflexive and the dual space $Y = X^*$ does not admit an equivalent UCED norm. Denote by $\{e_\gamma\}_{\gamma\in\Gamma}$ the natural unconditional basis of X. Since $||e_\gamma + e_\delta||_{12} = 2$ for all $\gamma, \delta \in \Gamma$, it is easy to prove the following.

Proposition. *The space $Y = X^*$, equipped with the dual original norm $||\cdot||_{12}$, lacks normal structure.*

However, define for $x \in X$, $||x||_2 = (\sum_{\gamma\in\Gamma}|x(\gamma)|^2)^{1/2} \le ||x||_{12}$, and let

$$||x|| = (||x||_2^2 + ||x||_{12}^2)^{1/2}.$$

Obviously, $||\cdot||$ is an equivalent norm on X.

Theorem. *The space $(X^*, ||\cdot||)$ has the sum-property and consequently it has normal structure.*

Proof. Assume that $(X^*, ||\cdot||)$ does not have the sum-property. Then, by [**5**], there is a sequence $\{x_n^*\}$ in X^* such that x_n^* is weakly null, $||x_n^*|| \to 1$, $\Lambda(x_n^*) = a_n \to 1$, where $\Lambda(x_n^*) = \lim_{i\to\infty} ||x_n^* - x_i^*||$.

Take support functionals $x_n \in X$ such that $||x_n|| = ||x_n^*||^{-1}$, $x_n^*(x_n) = 1$. Clearly, $||x_n|| \to 1$. It is easy to see that x_n tends weakly to zero. Indeed, assume the contrary. Since the basis $\{e_\gamma\}$ is shrinking, then for some $\gamma \in \Gamma$, $e_\gamma^*(x_m) \geq b > 0$ for infinitely many m. As $||x+y||^2 \geq ||x||^2 + ||y||^2$ whenever $x, y \in X$ have disjoint supports, then for the elements $z_m = x_{m|\Gamma\setminus\{\gamma\}}$ we have $||z_m|| \leq 1 - \eta$ for some $\eta > 0$ and infinitely many m. On the other hand, since x_n^* is weakly null, we get $x_n^*(z_n) \to 1$, which is a contradiction.

For each i, $(x_i^* - x_n^*)(x_i - x_n) = 2 - x_i^*(x_n) - x_n^*(x_i) \to 2$ as $n \to \infty$. Hence, for each i,

$$\liminf_n ||x_i - x_n|| \geq 2a_i^{-1}.$$

Since $||\cdot||$ is a lattice norm and x_n is weakly null, we obtain

$$(*) \qquad \liminf_n ||x_i + x_n|| \geq 2a_i^{-1}, \quad \text{for every } i.$$

It is not hard to check that $(*)$, the uniform convexity of l_2 and $x_n \to 0$ weakly, imply $||x_n||_2 \to 0$.

Fix $0 < \varepsilon < 1/7$. Choose $0 < \delta < \varepsilon$ such that whenever $u, v \in l_2$, $||u||_2$, $||v||_2 \leq 1 + \delta$, $||u + v||_2 \geq 2(1-\delta)$, we get $||u - v||_2 < \varepsilon$.

Fix i such that $||x_n|| < 1 + \delta$, $||x_n||_2 < \delta/4$ and $a_n < (1-\delta/4)^{-1}$ for all $n \geq i$. Since $\{e_\gamma\}$ is a basis, there exists a finite subset $A \subset \Gamma$, such that $||x_{i|_{\Gamma\setminus A}}|| < \varepsilon$.

It follows from the definition of Φ that, for every choice of $\gamma_1, \gamma_2 \in \Gamma$, with $\gamma_1 \neq \gamma_2$,

$$k(\gamma_1, \gamma_2) = \max\{|\varphi| : \varphi \in \Phi, \{\gamma_1, \gamma_2\} \subset \varphi\} < \infty,$$

where $|\varphi|$ denotes the number of elements of φ. Let

$$k = |A| \max\{k(\gamma_1, \gamma_2) : \gamma_1, \gamma_2 \in A, \gamma_1 \neq \gamma_2\}.$$

Clearly, $k < \infty$. Since $||x_n||_2 \to 0$, there exists $j \geq i$ such that, for $n \geq j$,

$$\max_{\gamma \in \Gamma} |e_\gamma^*(x_n)| \leq \varepsilon/k.$$

By $(*)$ and $a_i < (1-\delta/4)^{-1}$, choose $n \geq j$ so that $||x_i + x_n|| > 2 - \delta$. Thus, it follows from $||x_i + x_n||_2 < \delta/2$ that $||x_i + x_n||_{12} > 2 - 3\delta/2$. Therefore, there exists an $F = \{\varphi_j\} \in \mathcal{F}$ such that $||x_i + x_n||_F > 2(1-\delta)$. We have $||x_i + x_n||_F \leq ||x_i||_F + ||x_n||_F \leq 1 + \delta + ||x_n||_F$, whence

$$||x_n||_F > 1 - 3\delta.$$

Let $F_2 = \{\varphi_j \in F : |\varphi_j \cap A| \geq 2\}$ and $F_1 = F \backslash F_2$. Consider

$$u = \bigg(\sum_{\gamma \in \varphi_j} |e_\gamma^*(x_i)| \bigg)_{j=1}^{\infty},$$
$$v = \bigg(\sum_{\gamma \in \varphi_j} |e_\gamma^*(x_n)| \bigg)_{j=1}^{\infty}$$

like elements of l_2. By $||u+v||_2 \geq ||x_i + x_n||_F$ and the choice of δ, we obtain $||u - v||_2 < \varepsilon$. Since $||x_i||_2 < \delta/4$ and $||x_{i|_{\Gamma \backslash A}}|| < \varepsilon$, then $||x_i||_{F_1} < 2\varepsilon$. Hence, $||x_n||_{F_1} < 3\varepsilon$. Moreover, $||x_n||_{F_2} < k(\varepsilon/k) = \varepsilon$. Thus,

$$||x_n||_F < 4\varepsilon,$$

which is in contradiction with $||x_n||_F > 1 - 3\varepsilon$.

Therefore, $(X^*, ||\cdot||)$ has the sum-property. □

REFERENCES

1. A. Aksoy and M. Khamsi, *Nonstandard methods in fixed point theory*, Springer, New York, 1990.

2. M. Brodskii and D. Milman, *On the center of a convex set*, Dokl. Akad. Nauk SSSR **59** (1948), 837–840.

3. M. Day, R.C. James and S. Swaminathan, *Normed linear spaces that are uniformly convex in every direction*, Canad. J. Math. **23** (1971), 1051–1059.

4. D.N. Kutzarova and S.L. Troyanski, *Reflexive Banach spaces without equivalent norms which are uniformly convex or uniformly differentiable in every direction*, Studia Math. **72** (1982), 91–95.

5. T. Landes, *Normal structure and the sum-property*, Pacific J. Math. **123** (1986), 127–147.

6. V. Zizler, *On some rotundity and smoothness properties of Banach spaces*, Dissertationes Math. (Rozprawy Mat.) **87** (1971), 5–33.

Institute of Mathematics, Bulgarian Academy of Sciences, 1090 Sofia, Bulgaria; e-mail: denka@bgearn.acad.bg

University of Paderborn, D-4790 Paderborn, Germany