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WEIGHTED L^2-MULTIPLIERS

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ABSTRACT. In this paper we give a simple proof of the characterization of an L^2 multiplier with weight $|x|^{2k}$, $k \in \mathbf{N}$.

1. Introduction. Let $f \mapsto \hat{f}$ be the Fourier transform, $f \mapsto \check{f}$ the inverse Fourier transform, and m a bounded measurable function on $\mathbf{R}$. We say that m is a *multiplier* for $L^p(\mathbf{R})$, $1 \leq p \leq \infty$, if $f \in L^2 \cap L^p$ implies $(m\hat{f})\check{}$ is in L^p and satisfies

$$||(m\hat{f})\check{}\,||_p \leq C_p ||f||_p \quad \text{with } C_p \text{ independent of } f.$$

For $\alpha \geq 0$, we express $L^2(x^{2\alpha})$ the collection of f with $||f||^2_{L^2(x^{2\alpha})} = \int_{-\infty}^{\infty} |x^\alpha f(x)|^2 \, dx < \infty$, and $\mathcal{S}_{00}(\mathbf{R}) = \{f$ in the Schwartz class $\mathcal{S}(\mathbf{R})$: $\hat{f}$ has compact support not including the origin$\}$. For $f \in \mathcal{S}_{00}$, it is easy to check that $xf \in \mathcal{S}_{00}$ and $\hat{f}^{(k)}$ vanishes in a neighborhood of the origin for all $k \in \mathbf{N}$. Thus we have $\hat{f}^{(k)}(x) = \int_0^x \hat{f}^{(k+1)}(t)\, dt$ for all $k \in \mathbf{N}$. Furthermore, it is well known that $\mathcal{S}_{00}$ is dense in $L^2(x^{2\alpha})$ (see [**4**]).

Hörmander [**1**] gave a sufficient condition for multipliers in 1960. Kurtz and Wheeden [**2**] proved a weighted version of the Hörmander multiplier theorem. Both gave sufficient conditions for multipliers, but not necessary conditions. Muckenhoupt, Wheeden, and Young [**4**] provided sufficient and necessary conditions for L^2 multipliers with power weight $|x|^{2\alpha}$, $\alpha \in \mathbf{R}$. In this paper we use the principle of mathematical induction to give a simple proof of the characterization of an L^2 multiplier with weight $|x|^{2k}$, $k \in \mathbf{N}$. Finally, we mention that C will be used to denote a constant which may vary from line to line.

We recall that Hardy's inequality with weights is stated as following.

Theorem 1 [**3**]. *If $1 \leq p \leq \infty$, there is a finite constant C for which*

$$(1.1)\quad \bigg(\int_0^\infty \bigg|U(x)\int_0^x f(t)\,dt\bigg|^p dx\bigg)^{1/p} \leq C\bigg(\int_0^\infty |V(x)f(x)|^p\,dx\bigg)^{1/p}$$

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is true if and only if

$$\sup_{r>0}\left(\int_r^{\infty}|U(x)|^p\,dx\right)^{1/p}\left(\int_0^r|V(x)|^{-p'}\,dx\right)^{1/p'}<\infty, \tag{1.2}$$

where $1/p+1/p'=1$.

We say that m satisfies a *Hörmander condition of order* k, $k\in\mathbf{N}$, denoted by $m\in H(2,k)$, if $m\in L^{\infty}(\mathbf{R})\cap C^k(\mathbf{R}\backslash\{0\})$ and

$$\sup_{R>0}R^{2\alpha-n}\int_{R<|x|\le 2R}|D^{\alpha}m(x)|^2\,dx<+\infty \quad \text{for } \alpha=0,1,2,\dots,k. \tag{1.3}$$

2. Weighted L^2 multipliers. In this section we would like to get sufficient and necessary conditions of a weighted L^2-multiplier. We start from estimates of $||x^{1-a}\hat{f}'||_2$ and $||m^{(k)}\hat{f}^{(a-k)}||_2$.

Lemma 2. *For* $a\in\mathbf{N}$, *there exists a constant* C, *depending on* a *only, such that*

$$\int_0^{\infty}|x^{1-a}\hat{f}'(x)|^2\,dx\le C\int_0^{\infty}|\hat{f}^{(a)}(x)|^2\,dx \tag{2.1}$$

$$\int_{-\infty}^{0}|x^{1-a}\hat{f}'(x)|^2\,dx\le C\int_{-\infty}^{0}|\hat{f}^{(a)}(x)|^2\,dx \tag{2.2}$$

for all $f\in\mathcal{S}_{00}$.

Proof. It suffices to prove inequality (2.1) since inequality (2.2) is obtained from (2.1) by changing variables x into $-x$. Obviously, (2.1) holds for $a=1$. For $a=2$, we set $U(x)=x^{-1}$ and $V(x)=1$ in (1.2) and get

$$\begin{aligned}\int_0^{\infty}|x^{-1}\hat{f}'(x)|^2\,dx&=\int_0^{\infty}\left|x^{-1}\int_0^x\hat{f}''(t)\,dt\right|^2dx\\&\le C\int_0^{\infty}|\hat{f}''(x)|^2\,dx.\end{aligned}$$

For $a \geq 3$, let $U(x) = x^{1-a}$ and $V(x) = x^{2-a}$. Then

$$\int_r^\infty |U(x)|^2\,dx \cdot \int_0^r |V(x)|^{-2}\,dx = \frac{4}{(2a-3)^2} \qquad \text{for all } r > 0.$$

We apply Theorem 1 again to get

$$\begin{aligned}\int_0^\infty |x^{1-a}\hat{f}'(x)|^2\,dx &= \int_0^\infty \left| x^{1-a}\int_0^x \hat{f}''(t)\,dt\right|^2 dx \\ &\leq C\int_0^\infty |x^{2-a}\hat{f}''(x)|^2\,dx.\end{aligned}$$

Repeating the same process, we have

$$\begin{aligned}\int_0^\infty |x^{1-a}\hat{f}'(x)|^2\,dx &\leq C\int_0^\infty |x^{2-a}\hat{f}''(x)|^2\,dx \\ &\leq C\int_0^\infty |x^{3-a}\hat{f}^{(3)}(x)|^2\,dx \\ &\;\vdots \\ &\leq C\int_0^\infty |x^{-1}\hat{f}^{(a-1)}(x)|^2\,dx \\ &\leq C\int_0^\infty |\hat{f}^{(a)}(x)|^2\,dx,\end{aligned}$$

which completes the proof. □

Lemma 3. *If $m \in H(2,a)$, then there exists a constant C such that $||m^{(k)}\hat{f}^{(a-k)}||_2 \leq C||\hat{f}^{(a)}||_2$ for all $f \in \mathcal{S}_{00}$ and $0 \leq k \leq a$.*

Proof. Given $f \in \mathcal{S}_{00}$, for $a = 1$ and $k = 0$, it follows from $m \in L^\infty(\mathbf{R})$ that $||m\hat{f}'||_2 \leq C||\hat{f}'||_2$. For $a = 1$ and $k = 1$, we set $U(x) = m'(x)$ and $V(x) = 1$. Since $m \in H(2,1)$, we have

$$\begin{aligned}\int_r^\infty |m'(x)|^2\,dx \cdot \int_0^r dx &= r\sum_{k=0}^\infty \int_{2^k r}^{2^{k+1}r} |m'(x)|^2\,dx \\ &\leq r\sum_{k=0}^\infty C(2^k r)^{-1} = C \quad \text{for all } r > 0.\end{aligned}$$

It follows from Theorem 1 that

$$\begin{aligned}\int_0^\infty |m'(x)\hat{f}(x)|^2\,dx &= \int_0^\infty \left|m'(x)\int_0^x \hat{f}'(t)\,dt\right|^2 dx\\ &\le C\int_0^\infty |\hat{f}'(x)|^2\,dx\\ &\le C||\hat{f}'||_2^2.\end{aligned}$$

Similarly, we have $\int_{-\infty}^0 |m'(x)\hat{f}(x)|^2\,dx \le C||\hat{f}'||_2^2$, and hence $||m'\hat{f}||_2 \le C||\hat{f}'||_2$.

Assume that $||m^{(k)}\hat{f}^{(a-k)}||_2 \le C||f^{(a)}||_2$ is true for $a = n$, $0 \le k \le a$. If $a = n+1$, since $f \in \mathcal{S}_{00}$ implies $xf \in \mathcal{S}_{00}$, we have $||m^{(k)}\hat{f}^{(a-k)}||_2 = ||m^{(k)}(\hat{f}')^{(a-1-k)}||_2 = ||m^{(k)}\widehat{xf}^{(a-1-k)}||_2 \le C||\widehat{xf}^{(a-1)}||_2 = C||(\hat{f}')^{(a-1)}||_2 = C||\hat{f}^{(a)}||_2$ for $0 \le k < a$. For the case of $k = a$, we set $U(x) = m^{(a)}(x)$, $V(x) = x^{1-a}$. It follows from $m \in H(2,a)$ that

$$\begin{aligned}&\int_r^\infty |m^{(a)}(x)|^2\,dx \cdot \int_0^r x^{-2+2a}\,dx\\ &\qquad = \frac{r^{2a-1}}{2a-1}\sum_{k=0}^\infty \int_{2^k r}^{2^{k+1}r} |m^{(a)}(x)|^2\,dx\\ &\qquad \le C\frac{2^{2a}}{(2^{2a}-2)(2a-1)} \quad \text{for all } r>0.\end{aligned}$$

Applying Theorem 1 and (2.1), we get

$$\begin{aligned}\int_0^\infty |m^{(a)}(x)\hat{f}(x)|^2 dx &= \int_0^\infty \left|m^{(a)}(x)\int_0^x \hat{f}'(t)\,dt\right|^2 dx\\ &\le C\int_0^\infty |x^{1-a}\hat{f}'(x)|^2\,dx\\ &\le C\int_0^\infty |\hat{f}^{(a)}(x)|^2\,dx.\end{aligned}$$

Similarly, we have

$$\int_{-\infty}^0 |m^{(a)}(x)\hat{f}(x)|^2\,dx \le C\int_{-\infty}^0 |\hat{f}^{(a)}(x)|^2\,dx.$$

Both inequalities imply $||m^{(a)}\hat{f}||_2 \le C||\hat{f}^{(a)}||_2$ which proves the inequality for $k = a$. By the principle of induction, $||m^{(k)}\hat{f}^{(a-k)}||_2 \le C||\hat{f}^{(a)}||_2$ for all $a \in \mathbf{N}$, and $0 \le k \le a$. $\square$

We are ready to prove a sufficient condition of a weighted L^2-multiplier.

Theorem 4. *If $m \in H(2,a)$, there exists a finite constant C such that $||(m\hat{f})\check{}\,||_{L^2(x^{2k})} \le C||f||_{L^2(x^{2k})}$ for all $f \in \mathcal{S}_{00}$, $0 \le k \le a$.*

Proof. Obviously, the inequality holds for $k = 0$ by the boundedness of m. We notice that $m \in H(2,a)$ implies $m \in H(2,k)$ for all $1 \le k \le a$. By Lemma 3, we get

$$\begin{aligned} ||(m\hat{f})\check{}\,||_{L^2(x^{2k})} &= C||D^k(m\hat{f})||_2 \\ &\le C\sum_{j=0}^{k}\binom{k}{j}||m^{(j)}(\hat{f})^{(k-j)}||_2 \\ &\le C||\hat{f}^{(k)}||_2 = C||\widehat{x^k f}||_2 \\ &= C||f||_{L^2(x^{2k})} \quad \text{for all } f \in \mathcal{S}_{00}, \quad 1 \le k \le a. \qquad \square \end{aligned}$$

The next theorem states that $m \in H(2,a)$ is also a necessary condition. The first part of the following proof is the same as the one in [**4**]. For reasons of complement, we write the detailed proof as follows.

Theorem 5. *Let $m \in C^a(\mathbf{R}\backslash\{0\})$. If $||(m\hat{f})\check{}\,||_{L^2(x^{2k})} \le C||f||_{L^2(x^{2k})}$ for all $f \in \mathcal{S}_{00}$, $1 \le k \le a$, then $m \in H(2,a)$.*

Proof. First we show $m \in L^\infty(\mathbf{R})$. Since m is locally integrable, almost every point of $\mathbf{R}$ is a Lebesgue point of m and $|m|$. Let $x_0 \ne 0$ be an arbitrary Lebesgue point of m and $|m|$ with $m(x_0) \ne 0$. It suffices to show $|m(x_0)| \le C$. Let $\phi \in C_0^\infty(\mathbf{R})$ satisfy

(i) $\operatorname{supp}\phi \subseteq \{x : 1/2 \le |x| \le 4\}$,

(ii) $\phi(x) = 1$ for $1 \le |x| \le 2$,

(iii) $0 \le \phi \le 1$,

(iv) $\phi(x) = \phi(-x)$,

(v) $\int_{\mathbf{R}} \phi\, dx = 4$.

Set $\phi_r(x) \equiv (1/r)\phi(x/r)$. Then

$$\begin{aligned}
&|m * \phi_r(x_0) - 4m(x_0)| \\
&\qquad = \left| \int_{\mathbf{R}} m(y)\phi_r(x_0 - y)\, dy - m(x_0) \int_{\mathbf{R}} \phi(y)\, dy \right| \\
&\qquad = \left| \frac{1}{r} \int_{r/2 \le |y - x_0| \le 4r} \{m(y) - m(x_0)\} \phi\left(\frac{x_0 - y}{r} \right) dy \right| \\
&\qquad \le \frac{1}{r} \int_{|y - x_0| \le 4r} |m(y) - m(x_0)|\, dy \\
&\qquad \to 0 \quad \text{as } r \to 0.
\end{aligned}$$

Hence, $\lim_{r\to 0} m * \phi_r(x) = 4m(x_0)$. Similarly, $\lim_{r\to 0} |m| * \phi_r(x_0) = 4|m(x_0)|$. Let $0 < r < |x_0|/8$ be chosen such that $|m * \phi_r(x_0)| > 2|m(x_0)|$ and $|m| * \phi_r(x_0) < 8|m(x_0)|$. Define $g \in \mathcal{S}_{00}(\mathbf{R})$ by $\hat{g}(x) = \phi_r(x_0 - x) = (1/r)\phi((x_0 - x)/r)$. Since $|e^{ixt} - e^{ixx_0}| \le |x(t - x_0)| \le 1/8$ for $|x| \le 1/(32r)$ and $|t - x_0| \le 4r$,

$$\begin{aligned}
2\pi |(m\hat{g})\check{}\,(x)| &= \left| \int_{\mathbf{R}} m(t)\hat{g}(t) e^{ixx_0}\, dt + \int_{\mathbf{R}} m(t)\hat{g}(t)(e^{ixt} - e^{ixx_0})\, dt \right| \\
&\ge \left| \int_{\mathbf{R}} m(t)\hat{g}(t)\, dt \right| - \frac{1}{8} \int_{|t - x_0| \le 4r} |m(t)\hat{g}(t)|\, dt \\
&= |m * \phi_r(x_0)| - \frac{1}{8} |m| * \phi_r(x_0) \\
&> |m(x_0)| \quad \text{for } |x| \le \frac{1}{32r}.
\end{aligned}$$

Hence, $|m(x_0)|^2 \le 4\pi^2 |(m\hat{g})\check{}\,(x)|^2$ for $|x| \le 1/(32r)$. Multiplying both sides by x^2 and taking integration on $|x| \le 1/(32r)$, we obtain

$$|m(x_0)|^2 \le C r^3 \int_{|x| \le 1/(32r)} |(m\hat{g})\check{}\,(x)|^2 |x|^2\, dx.$$

It follows from the assumption and a change of variables that

$$\begin{aligned}|m(x_0)|^2 &\leq Cr^3 \int_{\mathbf{R}} |g(x)|^2 |x|^2 \, dx \\ &= Cr^3 \int_{\mathbf{R}} |\hat{\phi}(rx)|^2 |x|^2 \, dx \\ &= C \int_{\mathbf{R}} |\hat{\phi}(x)|^2 |x|^2 \, dx \leq C.\end{aligned}$$

Thus $m \in L^\infty(\mathbf{R})$, and hence there exists a constant C such that $||(m\hat{f})\check{\ }||_2 \leq C||f||_2$ for all $f \in L^2(\mathbf{R})$.

To prove $m \in H(2,a)$, we still have to show (1.3). Define f_r, $r > 0$, by $\hat{f}_r(x) = \phi(x/r)$. Then $\hat{f}_r \in C_0^\infty(\mathbf{R})$, $f_r(x) = rf_1(rx)$, and $\hat{f}_r(x) = 1$ for $r \leq |x| \leq 2r$, which imply

$$\begin{aligned}\int_{\mathbf{R}} |f_r(x)x^k|^2 \, dx &= \int_{\mathbf{R}} |rf_1(rx)x^k|^2 \, dx \\ &= r^{1-2k} \int_{\mathbf{R}} |f_1(x)|^2 |x|^{2k} \, dx \leq Cr^{1-2k}\end{aligned}$$

for all $k \geq 0$ and all $r > 0$. The Plancherel theorem and the assumption give

$$\begin{aligned}\int_{r<|x|\leq 2r} |m^{(k)}(x)|^2 \, dx &= \int_{r<|x|\leq 2r} |D^k(m\hat{f}_r)(x)|^2 \, dx \\ &\leq C \int_{\mathbf{R}} |x^k (m\hat{f}_r)\check{\ }(x)|^2 \, dx \\ &\leq C \int_{\mathbf{R}} |x^k f_r(x)|^2 \, dx \\ &\leq Cr^{1-2k}\end{aligned}$$

for all $0 \leq k \leq a$ and all $r > 0$. □

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