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NORMING SETS AND COMPACTNESS

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ABSTRACT. Let $(X, \| \ \|)$ be a Banach space and B a norming subset of the closed unit ball B_{X^*} of the dual space X^*. It is proved that if $(B_{X^*}$,weak$^*)$ is sequentially compact then the convex hull of the norm bounded $\sigma(X,B)$-relatively compact subsets of X are $\sigma(X,B)$-relatively compact (Moreover, when $(B_{X^*}$,weak$^*)$ is angelic the norm bounded $\sigma(X,B)$- relatively countably compact subsets of X are $\sigma(X,B)$-relatively compact). As a consequence, if B is assumed to be a boundary of B_{X^*} (i.e. for every $x \in X$ there exists $e^* \in B$ such that $e^*(x) = \| x \|$) then the norm bounded $\sigma(X,B)$- relatively compact subsets of X are relatively weakly compact.

This note addresses the study of some aspects of the compact subsets of Banach spaces X endowed with topologies coarser than their weak topologies. It is well known that for a given Banach space X the classical theorems of Krein-Smulian (about the compactness of the closed convex hull of compact sets), Eberlein-Grothendieck (about the coincidence between relatively countably compact and relatively compact sets) and Eberlein-Smulian (about the coincidence between relatively countably compact, relatively compact and relatively sequentially compact sets) are true for any locally convex topology between the weak and the norm topology of X. Our aim here is to show that, under some general assumptions on the dual unit ball B_{X^*} of X^*, the previous theorems are still true for some topologies in X of the kind $\sigma(X,B)$, where B is any norming subset of B_{X^*}.

Our notation is standard: $(X, \| \ \|)$ will be a real Banach space, X^* its dual and B_X, respectively B_{X^*}, the unit ball of X, respectively of X^*. A subset B of the dual unit ball B_{X^*} is said to be norming, respectively a boundary for B_{X^*}, if $\| x \| = \sup\{|x^*(x)| : x^* \in B\}$ for

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every $x \in X$, respectively if for any $x \in X$ there exists $e^* \in B$ such that $e^*(x) = \| x \|$.

Given a compact Hausdorff space Y and a Radon probability μ on Y, we will write $M_\mu(Y)$ to denote the space of μ-measurable real-valued functions on Y. In the next theorem we shall use some of Talagrand's results concerning stable subsets of $M_\mu(Y)$, [**17**] (see also [**6**]). Stable subsets of $M_\mu(Y)$ are reasonable pointwise compact subsets which in a sense are "small." If $\mathcal{F}$ is a uniformly bounded stable subset of $M_\mu(Y)$, then the identity $i : (\mathcal{F}, t_p(Y)) \to (\mathcal{F}, \| \ \|_{L^1(\mu)})$ is continuous ($t_p(Y)$ denotes the topology of pointwise convergence on Y), [**17**, 9.5.3]. This last result is the key to prove the following theorem.

Theorem 1. *Let X be a Banach space and B a norming subset of B_{X^*}. If the closed dual unit ball (B_{X^*},weak*) is sequentially compact and H is a norm bounded $\sigma(X,B)$-relatively compact subset of X, then $\overline{\mathrm{co}\,(H)}^{\sigma(X,B)}$ is $\sigma(X,B)$-compact.*

Proof. Consider $Y := \overline{H}^{\sigma(X,B)}$ endowed with the topology induced by $\sigma(X,B)$. We have to prove that every Radon probability μ on the compact space Y has a barycenter x_μ in X. After doing this, standard arguments will allow us to conclude the proof.

Take a Radon probability μ on Y and define $Z := \{x^*|_Y : x^* \in \mathrm{co}\,(B)\}$ and $\mathcal{F} := \{x^*|_Y : x^* \in B_{X^*}\}$. Z is a uniformly bounded subset of the space of continuous functions $C(Y)$ on Y and has the property that every sequence in it has a pointwise convergent subsequence. The last implies that Z is a topologically stable subset of $C(Y)$, [**17**, 14.1.7] and so it is a stable subset of $M_\mu(Y)$, [**17**, 14.1.7]. On the other hand, since B norming, the convex hull $\mathrm{co}\,(B)$ of B is weak* dense in B_{X^*} and so we obtain that $\overline{Z}^{t_p(Y)} = \{x^*|_Y : x^* \in B_{X^*}\} (= \mathcal{F})$. Since the pointwise closure of stable subsets of $M_\mu(Y)$ is stable, we get that $\mathcal{F}$ is a stable subset of $M_\mu(Y)$. Using the above mentioned Talagrand's theorem, [**17**, 9.5.3] we obtain the continuity of the map

$$\begin{aligned} (B_{X^*}, \mathrm{weak}^*) &\longrightarrow (\mathcal{F}, \| \ \|_{L^1(\mu)}) \\ x^* &\longrightarrow x^*|_Y \end{aligned}$$

Therefore the restriction to B_{X^*} of the linear functional $T_\mu : X^* \to$

$\mathbf{R}$ given by $T_\mu(x^*) := \int_Y x^*|_Y d\mu$ is weak* continuous. Now the Grothendieck completeness theorem, [**12**, Section 21.9.4], applies to conclude the existence of an element x_μ in X such that $T_\mu(x^*) = x^*(x_\mu)$ for every x^* in X^*. This x_μ is the barycenter of μ that we are looking for. The map $\mu \to x_\mu$ from the $\sigma(C(Y)^*, C(Y))$ compact convex set $P(Y)$ of all Radon probabilities on Y into X is $\sigma(C(Y)^*, C(Y)) - \sigma(X, B)$- continuous and its range is a $\sigma(X, B)$-compact convex set which contains Y, so the proof is concluded. □

If (B_{X^*},weak*) is assumed to be angelic (a topological space Y is said to be angelic, [**7**], if the closure of every relatively countably compact subset A of Y is compact and consists precisely of the limits of sequences from A) the proof of the previous theorem can be simplified and provides a stronger result. For doing this we will need some results about measures on topological spaces. Given a topological space Y, $C_b(Y)$ is the Banach space of bounded continuous real valued functions on Y endowed with the supremum norm $\| \ \|_\infty$ and $\mathcal{M}(Y)$ is the dual space $(C_b(Y), \| \ \|_\infty)^*$, for which we adopt the Alexandroff representation as the space of finite, finitely-additive zero-set regular Baire measures on Y, [**18**].

Theorem 2. *Let X be a Banach space and B a norming subset of B_{X^*}. If the closed dual unit ball (B_{X^*},weak*) is angelic and H is a norm bounded $\sigma(X, B)$-relatively countably compact subset of X, then $\overline{\operatorname{co}(H)}^{\sigma(X,B)}$ is $\sigma(X, B)$-compact. Therefore, the norm bounded $\sigma(X, B)$-relatively countably compact subsets of X are $\sigma(X, B)$-relatively compact.*

Proof. Consider $Y := \overline{H}^{\sigma(X,B)}$ endowed with the topology induced by $\sigma(X, B)$. Now we will state that every Baire probability μ on Y has a barycenter x_μ in X.

Since H is $\sigma(X, B)$-relatively countably compact, every $\sigma(X, B)$-continuous real function on Y is bounded, which means that Y is a pseudocompact space. For pseudocompact spaces Y, the space $\mathcal{M}(Y)$ is made up of countably additive measures defined on the Baire σ-field of Y, [**8**]. Take a Baire probability μ on Y. Since B

norming, the convex hull $\mathrm{co}\,(B)$ of B is weak* dense in B_{X^*} and so the angelicity of $(B_{X^*}$,weak*) allows us to ensure that for every $x^* \in B_{X^*}$ there is a sequence in $\mathrm{co}\,(B)$ that converges to x^* for the weak* topology. Therefore, for every $x^* \in X^*$ the function $x^*|_Y$ is μ-integrable and we can consider the linear functional $T_\mu : X^* \to \mathbf{R}$ given by $T_\mu(x^*) := \int_Y x^*|_Y \, d\mu$. The Lebesgue convergence theorem gives us that the restriction $T_\mu|_{B_{X^*}}$ is weak*-sequentially continuous which implies that it is weak* continuous since $(B_{X^*}$,weak*) is an angelic compact space. Now we follow the lines of the proof of the previous theorem but considering the map $\mu \to x_\mu$ from the $\sigma(\mathcal{M}(Y), C_b(Y))$-compact convex subset $\mathcal{P}(Y)$ of all Baire probabilities on Y into X. □

A particular class of angelic compact spaces are the Corson compact: a compact space K is said to be Corson compact if it is (homeomorphic to) a compact subset of $\mathbf{R}^\Gamma$ (for some set Γ) such that for every $x = (x(\gamma))$ in K the set $\{\gamma : x(\gamma) \neq 0\}$ is countable, [**3**]. Assuming that $(B_{X^*}$,weak*) is Corson we can complete the previous theorem in the following

Corollary 2.1. *Let X be a Banach space and B a norming subset of B_{X^*}. If the closed dual unit ball $(B_{X^*}$,weak*) is Corson compact and H is any norm bounded subset of X, then the following are equivalent*:

(i) *H is $\sigma(X,B)$-relatively countably compact in X.*

(ii) *H is $\sigma(X,B)$-relatively sequentially compact in X.*

(iii) *H is $\sigma(X,B)$-relatively compact in X.*

Proof. The equivalence between (i) and (iii) follows from Theorem 2. Since (ii) $\Rightarrow$ i) is obvious, it remains to prove that iii) $\Rightarrow$ ii).

Assume that H is $\sigma(X,B)$-relatively compact, and consider $K = \overline{H}^{\sigma(X,B)}$ as a compact subset of the space of continuous functions on $(B_{X^*}$,weak*), $C(B_{X^*}$,weak*), provided with the topology of pointwise convergence on B. The separable subsets of $(B_{X^*}$,weak*) are metrizable, because $(B_{X^*}$,weak*) is Corson, and so an application of Theorem 4.3 of [**14**] gives us that K is a Radon-Nikodym compact space (see [**14**] for the definition). Now Corollary 5.4 of [**14**] can be applied to obtain that K is sequentially compact and the proof is concluded. □

Let us remark that many Banach spaces enjoy the properties required in the previous theorems. The class of Banach spaces having weak* sequentially compact dual unit ball contains the weakly countably determined Banach spaces, [**16**] and the weak Asplund Banach spaces, [**5**, p. 239]. In fact, the weakly countably determined Banach spaces have Corson dual unit ball.

The previous results can be used to study the following question formulated by Godefroy in [**9**]. Let X be a Banach space, B a boundary for B_{X^*} and H a norm bounded $\sigma(X,B)$-compact subset of X. Is H weakly compact? That this question has a positive answer when X does not contain l^1 or when B is the set formed by the extreme points $\mathrm{Ext} B_{X^*}$ of B_{X^*} has been stated by G. Godefroy [**9**], and J. Bourgain and M. Talagrand [**1**], respectively. We also refer to [**4**, Problem I.1.2] where this open question is recalled and annotated. Theorems 1 and 2 enable us to prove the following:

Corollary 2.2. *Let X be a Banach space and B a boundary for B_{X^*}.*

(i) *If $(B_{X^*}$,weak*$)$ is sequentially compact, then the norm bounded $\sigma(X,B)$-relatively compact subsets of X are relatively weakly compact.*

(ii) *If $(B_{X^*}$,weak*$)$ is angelic, then the norm bounded $\sigma(X,B)$-relatively countably compact subsets of X are relatively weakly compact.*

Proof. i) If H is a norm bounded $\sigma(X,B)$-relatively compact subset of X, the closed convex hull of H, $\overline{\mathrm{co}\,(H)}^{\sigma(X,B)}$, is $\sigma(X,B)$-compact after Theorem 1. Now the theorem in [**7**, p. 99] tells us that $\overline{\mathrm{co}\,(H)}^{\sigma(X,B)}$ is weakly compact and the proof is done. The proof of ii) can be done in the same way as the proof of i) but using Theorem 2 instead of Theorem 1. □

Remark 1. When $B = \mathrm{Ext}\, B_{X^*}$, the above mentioned Bourgain-Talagrand's result, [**1**], about the weak compactness of the norm bounded $\sigma(X,B)$)-relatively countably compact subsets of a Banach space X, has been used by C. Stegall, [**15**], to give the following extension of a well known result of Namioka:

Theorem (C. Stegall, [**15**]). *Suppose T is a Čech-complete space and $f : T \to X$ is a function into the Banach space X such that for any x^* in B we have that $x^* \circ f$ is continuous. Then f is norm-continuous at each point of a dense G_δ subset T_0 of T.*

Let us finish pointing out that using Corollary 2.2 instead of Bourgain-Talagrand's theorem, Stegall's proof of the previous theorem works for the case of a general boundary B when the dual unit ball B_{X^*} is assumed to be angelic.

Remark 2. Take a fixed norm bounded $\sigma(X, B)$-relatively compact subset H of a Banach space X. In order to prove that $\overline{\text{co}\,(H)}^{\sigma(X,B)}$ is $\sigma(X, B)$-compact it is enough to assume that H satisfies the following condition:

$P(B)$: For every sequence (x_n^*) in B there exists a subsequence $(x_{n_k}^*)$ such that $(x_{n_k}^*(h))$ converges for every h in H.

This local property of H can be used to extend Theorem 1 without assuming that B_{X^*} is sequentially compact. In the paper [**2**] we give several sufficient conditions to ensure that a given H has property $P(B)$ and show that subsets having this property have a behavior analogous to the weak* compact subsets of dual Banach spaces Z^* for which Z does not contain l^1. Applications to spaces of vector-valued Bochner integrable functions as well as to spaces of countably additive measures are also included in [**2**].

REFERENCES

1. J. Bourgain and M. Talagrand, *Compacité extrémale*, Proc. Amer. Math. Soc. **80** (1980), 68–70.

2. B. Cascales and G. Vera, *Topologies weaker than the weak topology of a Banach space*, J. Math. Anal. Appl. **182** (1994), 41–68.

3. H.H. Corson, *Normality in subsets of product spaces*, Amer. J. Math. **81** (1959), 137–141.

4. R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs Surveys Pure Appl. Math. **64**, 1993.

5. J. Diestel, *Sequences and series in Banach spaces*, Springer-Verlag, New York, 1984.

6. M. van Dulst, *Characterizations of Banach spaces not containing* ℓ^1, Centrum voor Wiskunde en Informatica, Amsterdam, 1989.

7. K. Floret, *Weakly Compact Sets*, LNM **801**, Springer-Verlag, New York, 1980.

8. I. Glicksberg, *The representation of functionals by integrals*, Duke Math. J. **19** (1980), 253–261.

9. G. Godefroy, *Boundaries of a convex set and interpolation sets*, Math. Ann. **277** (1987), 173–184.

10. J.E. Jayne, I. Namioka and C.A. Rogers, *Topological properties of Banach spaces*, Proc. London Math. Soc **66** (1993), 651–672.

11. J.E. Jayne, J. Orihuela, A.J. Pallarés and G. Vera, *σ-fragmentability of multivalued maps and selection theorems*, J. Funct. Anal. **117** (1993), 243–273.

12. G. Köthe, *Topological vector spaces* I, Springer-Verlag, New York, 1969.

13. I. Namioka, *Separate continuity and joint continuity*, Pacific. J. Math. **51** (1974), 515–531.

14. ———, *Radon-Nikodým compact spaces and fragmentability*, Mathematika **34** (1989), 258–281.

15. C. Stegall, *Generalizations of a theorem of Namioka*, Proc. Amer. Math. Soc. **102** (1974), 515–531.

16. M. Talagrand, *Espaces de Banach faiblement K-analytiques*, Ann. Math. **110** (1979), 407–438.

17. ———, *Pettis integral and measure theory*, Mem. Amer. Math. Soc. **307**, 1984.

18. S. Varadarajan, *Measures on topological spaces*, Amer. Math. Soc. Transl. **48** (1965), 161–228.

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